Graded Rings Corresponding to Polarised K3 Surfaces and $\mathbb{Q}$-Fano 3 folds

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Contents

List of Tables

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Declaration

I declare that the work contained in this thesis is original, to the best of my knowledge, except where otherwise stated.

Summary

In this thesis we study polarised varieties (X, D), especially K3 surfaces and $\mathbb{Q}$-Fano 3 folds, and the corresponding graded rings $R(X, D)$. The general aim is to describe graded rings explicitly by generators and relations which in good cases can be used to get results for X and its moduli. A polarised variety (X, D) is a projective variety and a polarisation is given by a $\mathbb{Q}$-ample Weil divisor (that is, some multiple of D is Cartier and ample). Usually, our varieties are singular and the singularities are canonical singularities. We start with the Hilbert function $P_n = \dim H^0(X, nD)$ given by an appropriate form of the Riemann–Roch formula and write the Hilbert series of P_n as a rational function which tells us the possible shape of the ring $R(X, D)$. Then we study varieties and their graded rings by a number of methods such as complete intersection, Pfaffian and projection methods.

We produce lists of families of polarised K3 surfaces (X, D) with a given $\mathbb{Q}$-ample Weil divisor and at worst Du Val singularities for which the corresponding graded ring $R(X, D)$ has embedding codimensions 3 and 4, and similarly, $\mathbb{Q}$-Fano 3 folds in codimensions 3 and 4 by using the Hilbert function.

Chapter 1 gives basic notation, definitions and results for nonsingular varieties and varieties with canonical singularities and discusses the forms of the Riemann–Roch formula we use later on.

In Chapter 2 we introduce numerical data on a polarised K3 surface. Then we show that numerical data and the corresponding Hilbert function P_n can be given by a simple formula by studying the resolution of singularities as in the case of a nonsingular K3 surface.

In Chapter 3 we give a strategy of the search and an outline of the Hilbert function method.

Chapter 4 is an application of Chapter 3. We show that the argument with P_1 and P_2 based on $2 \leq P_1 \leq 3 + \operatorname{codim} X$ for codimension ≤ 3 to cut down the whole universe to a few hundred possibilities.

In Chapter 5 we explain a relation between "projection" and the resolution of singularities and how this under some mild conditions leads to the existence of K3 surfaces in codimensions 3 and 4. We work on model examples and give explicit constructions of K3 surfaces by using birational geometry.

The remaining problems that we hope to settle in the near future are the existence results for $\mathbb{Q}$-Fano 3 folds including three new candidates with $P_1 = \dim H^0(-K_X) = 0$, some K3 surfaces in codimension 4 and how these examples relate to a known structure theory for codimension 4 Gorenstein ideals.

Introduction

The study of graded rings is very important for the classification of surfaces of general type, del Pezzo surfaces, canonical 3 folds and Fano 3 folds etc. The graded ring R associated to a variety X and a divisor D (typically ample) is defined by

$$R(X, D) = \bigoplus_{m \in \mathbb{Z}} H^0(X, mD), \quad \text{with} \quad X = \operatorname{Proj} R(X, D).$$

There is a two-way correspondence between (X, D) and its graded ring $R = R(X, D)$. In traditional literature, D is replaced by the canonical divisor K to study canonical models. There are still many open problems of classifying surfaces of general type and 3 folds for given numbers K^2 and p_g to which $R(X, D)$ applies.

We mainly study K3 surfaces X with Du Val singularities and a $\mathbb{Q}$-ample Weil divisor D embedded in weighted projective spaces, especially the cases where the ring R is "small". "Small" means that R can conveniently be expressed in terms of generators and relations, such as hypersurfaces, codimension 2 complete intersections, codimension 3 Pfaffians and known models for codimension 4 rings. Studying varieties in weighted projective spaces come naturally in the sense that taking Proj of a graded ring automatically gives a variety embedded in a weighted projective space. Many our varieties are locally toric. This makes easy to work with singularities by using toric methods (see Danilov [6]). We study singularities of X, especially cyclic quotient singularities, in terms of a nonsingular K3 surface S obtained by resolving them. In this case, the resolution on S is an "almost star-shaped" weighted graph which gives rise to a lattice L. One can go beyond that to describe K3 surfaces analytically. Reid conjectured that the primitive sublattice of L can easily be determined in terms of submultiples of D. If the ring $R(X, D)$ is simple such as hypersurfaces he gave a sketch of proof by using Lefschetz theory (see Reid [22]). Before that, Belcastro proved this by using toric descriptions of "famous" 95 hypersurfaces in her Ph.D. thesis (see Belcastro [3]).

Hypersurfaces and codimension 2 complete intersection K3 surfaces were studied by Reid, Iano-Fletcher and independently by Yonemura. Fletcher gave necessary and sufficient conditions that make codimensions 1 and 2 K3 surfaces quasismooth and he also did similar works on canonical 3 folds and $\mathbb{Q}$-Fano 3 folds in codimensions 1 and 2. He showed that there is a $\mathbb{Q}$-Fano 3 fold $X(12, 14) \subset \mathbb{P}(2, 3, 4, 5, 6, 7)$ with empty anticanonical linear system $|-K|$ (see Fletcher [9]).

It is very interesting to know a complete structure of Gorenstein ideals I in a regular local ring A and their minimal resolution, in other words, their relations and syzygies of A/I. A ring is Gorenstein if and only if it is Cohen–Macaulay and the canonical divisor of the corresponding variety is trivial. A Gorenstein (Cohen–Macaulay) ideal means that A/I is Gorenstein (Cohen–Macaulay).

The Hilbert–Burch theorem states that any ideal I of codimension 2 in a regular local ring A with $n + 1$ generators is Cohen–Macaulay if and only if it is the ideal generated by the $n \times n$ minors of an $n \times (n + 1)$ matrix. In 1890, Hilbert proved this for a polynomial ring and, later on, in 1968, Burch proved it for a regular local ring. Serre showed that if an ideal I has codimension 2, then I is Gorenstein if and only if the corresponding variety $V(I)$ is a complete intersection.

In 1974, Buchsbaum–Eisenbud [4] gave a structure theorem for a Gorenstein ideal of codimension 3 to obtain a complete determinantal description of such an ideal. Their main result for a Gorenstein ideal I of codimension 3 in a regular local ring A is the following: if an ideal I of codimension 3 is Gorenstein then it is the ideal generated by the Pfaffians of the $2m \times 2m$ submatrices obtained by deleting the ith row and the corresponding column of some $(2m + 1) \times (2m + 1)$ skew symmetric matrix M of rank $2m$ (or simply the submaximal Pfaffians of M). Conversely, every Gorenstein ideal of codimension 3 is given by the submaximal Pfaffians of some skew symmetric matrix.

In 1982 there was an attempt done by Kustin–Miller to give a complete structure theorem on codimension 4 Gorenstein ideals. In 1983 they gave a construction of "big Gorenstein ideals from small ones" [15]. The proof is based on constructing a resolution of big Gorenstein ideals. Their main theorem is the following: let $(b) = (b_1, \dots, b_n) \subset (a) = (a_1, \dots, a_m)$ be Gorenstein ideals of codimensions $c + 1, c$ respectively, and ϕ a map of complexes from the minimal resolution $\mathbb{F}$ of $A/(b)$ to the minimal resolution $\mathbb{E}$

of $A/(a)$:

$$\begin{array}{lccccccccccc}
\mathbb{F}: & & & 0 & \longrightarrow & F_c & \longrightarrow & \cdots & \longrightarrow & F_1 & \xrightarrow{b} & F_0 \\
& & & & & \downarrow{\scriptstyle \phi_c} & & & & \downarrow{\scriptstyle \phi_1} & & \downarrow{\scriptstyle \phi_0} \\
\mathbb{E}: & 0 & \longrightarrow & E_{c+1} & \xrightarrow{a^t} & E_c & \longrightarrow & \cdots & \longrightarrow & E_1 & \xrightarrow{a} & E_0
\end{array}$$

where the F_i and the E_i are isomorphic to a direct sum of A such that

$$\phi_0 = \mathrm{id}\colon A \longrightarrow A, \quad \phi_1\colon A^n \longrightarrow A^m, \dots, \phi_c\colon A \longrightarrow A^m$$

then the ideal generated by the b_i and $c_1 + va_1, \dots, c_m + va_m$ is a Gorenstein ideal of codimension $c+1$ in $A[v]$ where v is an indeterminate and $\phi_c^t = (c_1, \dots, c_m)$. Common known examples are hypersurface sections of codimension 3 Pfaffians, ideals generated by the $(n-1) \times (n-1)$ minors of a generic $n \times n$ matrix, especially, $\mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$ and $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^7$, both given by nine equations and sixteen syzygies.

By using the resolution structure of ideals together with the Hilbert function we develop a method, called the Hilbert function method, which extends and improves Fletcher and Reid's finite tree search method to produce lists of 70 candidates for K3 surfaces in codimension 3 and at least 138 candidates for K3 surfaces in codimension 4. What the method does is a reflection in combinatorics of the Hilbert series $P(t)$ of the following idea of geometry and algebra: to find a regular sequence y_0, y_1, y_2 with weights a_i which cuts down the dimension of the ring R three times to give a new ring with zero dimension, in other words multiplying $P(t)$ by $(1-t^{a_1})(1-t^{a_2})(1-t^{a_2})$ will give the Hilbert series of the new ring, and then to find the generators of the new ring. This can be interpreted geometrically by cutting out transversally the cone of the corresponding variety via these hyperplanes $(y_i = 0)$.

In 1997–1998, Reid independently stated the theorem of Kustin–Miller as above by using birational geometry, called a new construction of varieties "unprojection". He replaces a map of complexes by the idea of adjunction (see Remark 5.10). Unprojection is an application of birational geometry to commutative algebra and provides new examples, particularly K3 surfaces in codimensions 3 and 4 (see Chapter 5). We follow Reid's approach to prove the existence of K3 surface candidates. In here any construction of new varieties we get by lifting from any codimension we call unprojection. This can be done by adding more than one variable. For example, it can be from codimension 1 to 3 or codimension 2 to 4 etc. Therefore we have different types of unprojection. The first type of unprojection corresponds to Kustin-Miller and Reid's construction, namely starting codimension c Gorenstein ideals to get codimension $c+1$ Gorenstein ideals. This construction works for any c. The second type of unprojection is a special technique works from codimension 1 to 3 and the ideal (a) is not Gorenstein. Unprojection proves the existence of almost all candidates in codimension 3 (see Theorem 5.2) and also many in codimension 4. In general, it is almost impossible to check the quasismoothness of a variety (that is, the variety has only at worst cyclic quotient singularities, see Section 1.4 for details) on each strata of weighted projective space by hand. The main point of using unprojection is to get rid of this problem by using Bertini's theorem in codimensions 1 and 2 to prove the quasismoothness of candidates in codimensions 3 and 4, although it needs special treatment in some cases (see Chapter 5 for examples).

We use similar techniques to study $\mathbb{Q}$-Fano 3 folds. The results for a polarised K3 surface can be extended to a $\mathbb{Q}$-Fano 3 fold in Chapter 3. I prove that all $\mathbb{Q}$-Fano 3 folds Pfaffian in codimension 3 have general hyperplane sections which are K3 surfaces. In case of codimension 4 there are three candidates. Their existence is still an open question.

Chapter 1

Preliminaries

This chapter introduces some notation, basic concepts and results for an algebraic variety of dimension n over an algebraically closed field k of characteristic zero. Standard references for this material are [11], [18] and [19].

1.1 Notation

k	algebraically closed field of characteristic zero
k^*	multiplicative group of k
V	projective variety of dim n over k
S	nonsingular surface
$K(V)$	function field of V
V^0	nonsingular locus of V
ω_V	sheaf of regular canonical differentials on V^0
K_V	canonical divisor corresponding to ω_V
$\mathbb{A}^n$	affine n-space over k
$\mathbb{P}^n$	projective n-space over k
μ_r	cyclic group of rth roots of unity
A_{r-1}	cyclic Du Val singularity $\frac{1}{r}(1,-1)$
$\frac{1}{r}(a,-a,1)$	terminal cyclic quotient singularity
$\mathcal{B}$	basket of cyclic quotient singularities
$\mathrm{wt}(x_i)$	weighted homogeneous degree of x_i
$k[x_0,\dots,x_n]$	polynomial ring
$\mathbb{P}=\mathbb{P}(a_0,\dots,a_n)$	weighted projective space of weights $a_0,\dots,a_n$
$\mathbb{P}(x_0,\dots,x_n)$	$\mathbb{P}$ with homogeneous coordinates x_i of weights a_i
$\mathrm{Cone}(X)$	affine cone over a weighted projective variety X
$\mathrm{Cone}^*(X)$	punctured affine cone over X
$P_n=\dim H^0(X,nD)$	Hilbert function of a graded ring R for each n

$P(t) = \sum_{n=0}^{\infty} P_n t^n$	Hilbert series of P_n
Pf_i	defining equations of codimension 3 Pfaffians X

1.2 Basic notions for a nonsingular variety

All varieties are nonsingular in this section.

1.2.1 The Riemann–Roch formula

For a divisor D on an algebraic curve C the Riemann–Roch formula takes the form

$$\chi(C, \mathcal{O}_C(D)) = \chi(\mathcal{O}_C) + \deg D.$$

For a divisor D on an algebraic surface S it takes the form

$$\chi(S, \mathcal{O}_S(D)) = \chi(\mathcal{O}_S) + \frac{1}{2}D(D - K_S).$$

If S is a K3 surface, that is $K_S = \mathcal{O}_S$ and $H^1(S, \mathcal{O}_S) = 0$, we have $\chi(\mathcal{O}_S) = 2$, therefore

$$\chi(S, \mathcal{O}_S(D)) = 2 + \frac{1}{2}D^2.$$

1.2.2 The adjunction formula

Let Y be a codimension 1 subvariety in an algebraic variety V of dimension n. Then the adjunction formula takes the form

$$K_Y = (K_V + Y)_{|Y}.$$

It follows that for a curve C on a surface S we have the genus formula

$$2g(C) - 2 = (K_S + C)C$$

and on a K3 surface $C^2 = 2g(C) - 2$ and then $C^2 \geq -2$. For an irreducible curve C on a surface S the arithmetic genus p_aC is given by

$$2p_aC - 2 = (K_S + C)C.$$

By the adjunction formula

$$g(C) = p_aC.$$

1.3 Basic notions for a singular variety

We are interested in studying K3 surfaces with at worst Du Val singularities and a $\mathbb{Q}$-ample Weil divisor and $\mathbb{Q}$-Fano 3 folds. We first define these varieties and then explain the singularities involved.

A K3 *surface* is a projective surface X with $K_X = \mathcal{O}_X$ and $q(X) = H^1(X, \mathcal{O}_X) = 0$ which if nonsingular, we denote by S. By a *polarised* K3 *surface* (X, D) we mean a K3 surface X having at worst Du Val singularities and a $\mathbb{Q}$-ample Weil divisor D.

A Weil divisor D is a $\mathbb{Q}$-ample divisor if some multiple of D is Cartier and ample. For a Weil divisor D on a projective variety V we can associate a sheaf $\mathcal{O}_V(D)$ of rank one, called a divisorial sheaf, given by

$$\Gamma(U, \mathcal{O}_V(D)) = \left\{ f \in K(V) \mid \operatorname{div} f + D \geq 0 \right\}$$

for an open subset U in V.

Let V be a projective variety with at worst canonical singularities. Then V is called a $\mathbb{Q}$-Fano variety if $-K_V$ is ample.

1.3.1 Singularities

Canonical singularities

A projective variety V has at worst *canonical (respectively terminal) singularities* if it satisfies the following two conditions:

i. there exists an integer $r \geq 1$ such that rK_V is a Cartier divisor;

ii. if $f \colon W \to V$ is a resolution of V and $\{E_i\}$ the family of exceptional prime divisors, then

$$rK_W = f^*(rK_V) + \sum a_i E_i$$

where $a_i \geq 0$ (respectively $a_i > 0$).

The smallest r for which rK_V is Cartier in a neighbourhood of a singularity $P \in V$ is called the *index* of P.

In the case of a surface, the canonical singularities are the nonsingular points together with the *Du Val singularities*.

Du Val singularities

Du Val singularities can be characterised in many different ways [20].

A surface X has a *Du Val singularity* P if the following two conditions hold:

i. ω_X is invertible at P;

ii. if $f\colon S \to X$ is a minimal resolution of X at P then $f^*\omega_X = \omega_S$.

It follows immediately that a Du Val singularity is a canonical singularity of index 1.

An alternative characterisation is the following: $P \in X$ is locally analytically isomorphic to a quotient singularity $\mathbb{A}^2/G$, where $G \subset \mathrm{SL}(2,k)$ is a finite group. Another is: Du Val singularities are hypersurface singularities with one of the following normal forms

$$\begin{aligned}
&A_n\colon x^2 + y^2 + z^{n+1} = 0 \quad \text{for } n \geq 1,\\
&D_n\colon x^2 + y^2 z + z^{n-1} = 0 \quad \text{for } n \geq 4,\\
&E_6\colon x^2 + y^3 + z^4 = 0,\\
&E_7\colon x^2 + y^3 + yz^3 = 0,\\
&E_8\colon x^2 + y^3 + z^5 = 0.
\end{aligned}$$

For example, consider the group $G = \mathbb{Z}_2$ acting on the coordinates x_1, x_2 of $\mathbb{A}^2$ by $(x_1, x_2) \mapsto (-x_1, -x_2)$. Then X is locally analytically isomorphic to $\operatorname{Spec} k[x_1^2, x_2^2, x_1x_2]$, which corresponds to the singularity A_1 in the above list.

For our purposes, a non cyclic Du Val singularity can usually be replaced by a basket of cyclic quotient singularities. A *basket of cyclic quotient singularities*, $\mathcal{B}$, introduced by Reid (Section 8, [19]), is a set of cyclic quotient singularities $\frac{1}{r}(a,-a)$ with a coprime to r. These are not points of X but they make the same contribution as the singularities of X to the Riemann–Roch formula. Similarly, any terminal 3-fold singularity can be considered as a basket $\mathcal{B}$ of terminal cyclic quotient singularities $\frac{1}{r}(a,-a,1)$ where a is coprime to r. Later on, we restrict ourself to K3 surfaces with cyclic quotient singularities and $\mathbb{Q}$-Fano 3 folds with terminal cyclic quotient singularities.

Cyclic quotient singularities

Let $x_1, \ldots, x_n$ be affine coordinates on $\mathbb{A}^n$ and assign positive integer weights a_i to the x_i. A singular point $P \in V$ is called a cyclic quotient singularity if it has a neighbourhood which is analytically isomorphic to $(\mathbb{A}^n, 0)/\mu_r$, for some r, where the group action $\mu_r \times \mathbb{A}^n \to \mathbb{A}^n$ is given by

$$\varepsilon(x_1, \dots, x_n) = (\varepsilon^{a_1} x_1, \dots, \varepsilon^{a_n} x_n).$$

We denote such a singularity by

$$\frac{1}{r}(a_1, \dots, a_n).$$

The following results are taken from Sections 4 and 5 of [19].

Theorem 1.1 *A necessary and sufficient condition for a cyclic quotient singularity $\frac{1}{r}(a_1, \dots, a_n)$ to be canonical (respectively terminal) is that*

$$\frac{1}{r}\sum_i \overline{la_i} \geq 1 \quad (\textit{respectively} > 1)$$

for $l = 1, \dots, r-1$ where $^{-}$ denotes the smallest residue modulo r.

By the terminal lemma (see Section 5, [19]) terminal cyclic quotient 3-fold singularities are of the form $\frac{1}{r}(a, -a, 1)$ where a is coprime to r, or equivalently, of the form $\frac{1}{r}(1, -1, b)$ where $ab \equiv 1 \mod r$. Similarly, canonical cyclic quotient surface singularities are of the form $\frac{1}{r}(a, -a)$ with a coprime to r, or equivalently, of the form $\frac{1}{r}(1, -1)$. Canonical cyclic quotient surface singularities are A_{r-1}, Du Val singularities. Indeed, the group μ_r acts on the affine coordinates x, y of $\mathbb{A}^2$ by

$$\varepsilon\colon \begin{cases} x \longmapsto \varepsilon x \\ y \longmapsto \varepsilon^{-1} y \end{cases}$$

where ε is an element of μ_r.

Let π be the quotient map $\mathbb{A}^n \to \mathbb{A}^n/\mu_r$. Then the group μ_r acts on $\pi_*\mathcal{O}_{\mathbb{A}^n}$ and induces a splitting $\pi_*\mathcal{O}_{\mathbb{A}^n} = \mathcal{L}_0 \oplus \cdots \oplus \mathcal{L}_{r-1}$ into r eigensheaves where

$$\mathcal{L}_i = \{f \in \pi_*\mathcal{O}_{\mathbb{A}^n} \mid \varepsilon(f) = \varepsilon^i f \quad \text{for all } \varepsilon \in \mu_r\}.$$

A singularity (V, P) with a Weil divisor D is a *cyclic quotient singularity of type* $_i(\frac{1}{r}(a_1, \dots, a_n))$ if (V, P) is locally isomorphic to $\frac{1}{r}(a_1, \dots, a_n)$ and $\mathcal{O}_V(D) \cong \mathcal{L}_i$. In the case of a surface with a Weil divisor D, $\frac{1}{r}(a, -a)$ is of the form $\frac{1}{r}(1, -1)$ and $\mathcal{O}_V(D) \cong \mathcal{L}_b$ where $1 \leq b \leq r-1$ and $ab \equiv 1 \mod r$, equivalently, a cyclic quotient singularity of type $_b(\frac{1}{r}(1, -1))$. For convenience we will denote this $\frac{1}{r}(a, -a)$ of *type* b.

1.3.2 The Riemann–Roch formula for a singular surface

The Riemann–Roch formula for a given surface X with a finite number of Du Val singularities $\{Q_i\}$ and a $\mathbb{Q}$-Cartier divisor D reads

$$\chi(X, \mathcal{O}_X(D)) = \chi(X, \mathcal{O}_X) + \frac{1}{2}(D^2 - DK_X) + \sum_i C_{Q_i}(D) \tag{1.1}$$

where $C_{Q_i}(D)$ is a contribution coming from the singularity of $\mathcal{O}_X(D)$ at Q_i and $C_{Q_i}(D) \in \mathbb{Q}$ (Section 9, [19]). In the case of a cyclic quotient singularity $Q = \frac{1}{r}(a, -a)$ the contribution is

$$C_Q(D) = -\frac{b(r-b)}{2r}$$

where $ab \equiv 1 \mod r$.

The Riemann–Roch formula takes the following form

$$P_n = \chi(X, \mathcal{O}_X(nD)) = \chi(X, \mathcal{O}_X) + \frac{1}{2}((nD)^2 - (nD)K_X) + \sum_i C_{Q_i}(nD) \tag{1.2}$$

for $n \geq 1$ and $P_0 = 1$. For every $n \geq 1$, the contribution from a singularity $Q = \frac{1}{r}(a, -a)$ can be calculated by

$$C_Q(nD) = -\frac{\overline{nb}(r - \overline{nb})}{2r}$$

where $^-$ denotes the smallest residue modulo r.

1.3.3 The Riemann–Roch formula for a $\mathbb{Q}$-Fano 3 fold

Let V be a $\mathbb{Q}$-Fano 3 fold with at worst canonical singularities Q. We can assume that Q is of the form $\frac{1}{r}(1, -1, b)$, where r is coprime to b, so that the Riemann–Roch formula of V can be written in the following form:

$$\begin{aligned} P_n = h^0(V, \mathcal{O}_V(-nK_V)) = {} & \frac{(2n+1)n(n+1)}{12}(-K_V)^3 \\ & + (1+2n)\chi(\mathcal{O}_V) - \sum_Q l(Q, n+1) \end{aligned} \tag{1.3}$$

for all $n \in \mathbb{N}$ where

$$l(Q, n+1) = \sum_{k=1}^{n} \frac{\overline{bk}(r - \overline{bk})}{2r}$$

and where $^-$ denotes the smallest residue modulo r (see Chapter 2 in Fletcher [9]).

1.4 Weighted projective spaces

Our main aim is to study K3 surfaces and $\mathbb{Q}$-Fano with singularities, embedded in a weighted projective space, via their graded rings. We give some definitions and simple properties of weighted projective space. For more details, see Fletcher [9], Dolgachev [7].

Let $a_0, \dots, a_n$ be positive integers and consider the graded polynomial ring $A = k[x_0, \dots, x_n]$, graded by $\mathrm{wt}(x_i) = a_i$. The *weighted projective space* associated to the ring A is defined by

$$\mathbb{P}(a_0, \dots, a_n) = \operatorname{Proj} A.$$

An alternative construction is the following: with respect to the k^*-action on the affine coordinates of $\mathbb{A}^{n+1}$ given by

$$\lambda(x_0, \dots, x_n) = (\lambda^{a_0} x_0, \dots, \lambda^{a_n} x_n)$$

$\mathbb{P}(a_o, \dots, a_n)$ is the quotient $(\mathbb{A}^{n+1} \setminus 0)/k^*$. The affine piece $x_i \neq 0$ is isomorphic to the quotient $\mathbb{A}^n / \mathbb{Z}_{a_i}$ where the group $\mathbb{Z}_{a_i}$ acts on the affine coordinates $\{y_0, \dots, \widehat{y_i}, \dots, y_n\}$ via

$$\varepsilon \colon y_j \longmapsto \varepsilon^{a_j} y_j, \quad \text{for all } j \neq i$$

where ε is a primitive a_ith root of unity. Here one can think of y_j as $x_j/(\sqrt[a_i]{x_i})^{a_j}$.

When no confusion can arise, we use either $\mathbb{P}$ or $\mathbb{P}(x_0, \dots, x_n)$ to denote $\mathbb{P}(a_0, \dots, a_n)$. Let q_i be the point $(0, \dots, 1, \dots, 0) \in \mathbb{P}$ where 1 is in the ith position. We call such points *vertices* of $\mathbb{P}$. The l-plane spanning $q_{i_1}, \dots, q_{i_l}$ will be denoted by $q_{i_1} \dots q_{i_l}$ and be called an l-stratum. The 1-dimensional strata will also be called *edges* and the 2-dimensional strata *faces*.

Definition 1.2 The graded ring R of a polarised K3 surface (X, D) is

$$R(X, D) = \bigoplus_{n \geq 0} H^0(X, \mathcal{O}_X(nD)) \quad \text{with} \quad X = \operatorname{Proj} R.$$

Since D is a $\mathbb{Q}$-ample divisor, R is finitely generated. One can go backward and forward between R and $X = \operatorname{Proj} R$.

Definition 1.3 Let X be a closed subvariety of $\mathbb{P}$ and $\pi \colon \mathbb{A}^{n+1} \setminus 0 \to \mathbb{P}$ the map which sends the point with affine coordinates $(x_0, \dots, x_n)$ to the point with weighted homogeneous coordinates $(x_0, \dots, x_n)$. The *punctured affine cone*, $\operatorname{Cone}^*(X)$, over X is defined to be $\pi^{-1}(X)$, and the *affine cone*, $\operatorname{Cone}(X)$, over X the closure of $\operatorname{Cone}^*(X)$ in $\mathbb{A}^{n+1}$.

The affine cone of a closed variety X is an algebraic variety in $\mathbb{A}^{n+1}$. The ideal of $\mathrm{Cone}(X)$ is the homogeneous ideal $I(X)$ of X in A. Note that one can think of $\mathrm{Cone}(X)$ as $\mathrm{Spec}\, R$ and X either as a quotient $X = \mathrm{Cone}^*(X)/k^*$ or $\mathrm{Proj}\, R$ where $R = A/I(X)$.

Definition 1.4 A closed subvariety X of dimension m in $\mathbb{P}$ is *quasismooth* if $\mathrm{Cone}(X)$ is smooth of dimension $m+1$ outside the vertex 0.

If X is quasismooth the singularities of X come from the k^*-action and are cyclic quotient singularities.

If the ideal $I(X)$ of X is generated by a regular sequence of homogeneous elements of A, then X is called a *weighted complete intersection of multidegree* $(d_1, \ldots, d_l)$. A sufficiently general element in the family of all weighted complete intersections of multidegrees $(d_1, \ldots, d_l)$ is denoted by $X(d_1, \ldots, d_l)$ in $\mathbb{P}(a_0, \ldots, a_n)$ where $l = n - \dim X$. We keep the same notation when X is not a complete intersection. When $I(X)$ is principal, we call X a *weighted hypersurface.*

Chapter 2

Numerical Data and the Corresponding Hilbert Function

This chapter is preparation for Chapters 3 and 5. Our aim is to study the graded rings of some numerical data which can give rise to polarised K3 surfaces (X, D), via the Riemann–Roch formula. In this chapter, we introduce the numerical data, namely the initial data and the resolution data associated to certain polarised varieties (X, D). We consider the Riemann–Roch formula P_n as a function of the numerical data, called the *Hilbert function*. We also study the resolution of cyclic quotient singularities to find an answer for the existence of possible candidates. This relation is explained in Chapter 5. As a consequence of the resolution, P_n on a polarised K3 surface (X, D) can be written in a more efficient way which leads to a simple formula for $h^0(X, nD)$ as in the case of a nonsingular K3 surface. This is an alternative and more geometric way of calculating graded ring via linear systems on a nonsingular K3 surface.

2.1 Numerical data

Let (X, D) be a polarised K3 surface and $R(X, D)$ the graded ring of (X, D) (see for definitions in Section 1.3 and 1.4).

Initial data

Definition 2.1 The *initial data* of a polarised *K3* surface (X, D) is D^2 and its basket of singularities. By *formal initial data* we mean a number $D^2 \in \mathbb{Q}$

together with a basket of singularities $\mathcal{B}$ (see Section 1.3.1 for the definition). The Hilbert function P_n corresponding to the formal initial data $\{D^2, \mathcal{B}\}$ is given by (1.2).

Remark 2.2 For our purposes we assume that the type b of singularity $\frac{1}{r}(a, -a)$ in $\mathcal{B}$ is $\leq \frac{r}{2}$. This is possible since b and $r - b$ have the same effect on P_n.

Remark 2.3 It is known (see Theorem III.9.20, [9]) that the number of A_{r-1} singularities on a K3 surface is finite and the bound for this is given by $\sum_r (r-1) \leq 19$.

Resolution data

Let (V, D) be a polarised surface with Du Val singularities. Corresponding to the Weil divisor D, we have a divisorial sheaf $\mathcal{O}_V(D)$ of rank 1 which corresponds to a whole divisor class of D. The Riemann–Roch formula deals with a divisor class, not with a divisor. Therefore we consider a resolution of singularities along a suitable divisor D' in the divisor class of D. By abuse of notation D' is called D.

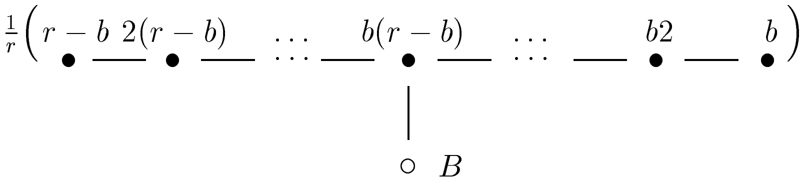
In the case of a surface with Du Val singularities there exists a minimal resolution σ of the singularities of (V, D) for some $D \in |D|$ such that $\sigma \colon (W, \sigma^*(D)) \to (V, D)$ with $\sigma^*(D) = B + \sum a_i C_i$ where B is the strict transform of D, the C_i are irreducible exceptional curves and $a_i \in \mathbb{Q}$. Since the C_i are exceptional and $K_W C_i = 0$, the C_i are (-2)-curves and form the well known A_n $(n \geq 1)$, D_n $(n \geq 4)$, E_6, E_7 or E_8 configurations (Chapter 3, [2]).

The *pull back* of a $\mathbb{Q}$-Cartier divisor D is defined to be

$$\sigma^*(D) := \frac{1}{r}\sigma^*(rD)$$

where r is the smallest integer such that rD is Cartier. Note that $\sigma^*(D)$ is also $\mathbb{Q}$-Cartier.

Proposition 2.4 *Let $P \in V$ be a germ of a cyclic quotient singularity $\frac{1}{r}(a, -a)$ of type b and D a $\mathbb{Q}$-ample Weil divisor on V. Then there exists a minimal resolution σ of the singularity along D, for some $D \in |D|$, and a divisor B meeting C_b transversely in a neighbourhood of the exceptional curves such that $\sigma^*(D) = B + \sum a_i C_i$. In other words, the resolution in W locally looks like an "almost star-shaped" graph with a central node B and a weighted chain of length $r - 1$ of (-2)-curves*



$(*)$

Notation For convenience we abbreviate $(*)$ by

$$\sigma^*(D) = B + \frac{1}{r}(r-b, 2(r-b), \dots, b(r-b), \dots, b2, b).$$

Proof. The singularity P is of the form $\frac{1}{r}(1,-1)$, and $\mathcal{O}_V(D) \cong \mathcal{L}_b$ which is generated by x^b, y^{r-b} as an $\mathcal{O}_V$-module. We denote $\operatorname{div} x^b$ and $\operatorname{div} y^{r-b}$ by L and M respectively, which are in the linear system $|D|$. From a toric resolution (see the Appendix of [1] and Section 8 of [6]), we have

$$\sigma^*(\operatorname{div} x^r) = rC_0 + (r-1, r-2, \dots, 1),$$
$$\sigma^*(\operatorname{div} y^r) = rC_r + (1, 2, \dots, r-1)$$

where C_0 and C_r correspond to the pull backs of $\pi(x=0)$ and $\pi(y=0)$ respectively under the toric resolution, and where π is the quotient map $\mathbb{A}^2 \to V$. Hence we can write

$$\sigma^* L = \frac{b}{r}\sigma^*(\operatorname{div} x^r),$$
$$\sigma^* M = \frac{r-b}{r}\sigma^*(\operatorname{div} y^r).$$

The minimum of $\sigma^* L - bC_0$ and $\sigma^* M - (r-b)C_r$ is

$$\frac{1}{r}(r-b, 2(r-b), \dots, b(r-b), b(r-b+1), \dots, b2, b).$$

Therefore

$$\sigma^* L = B_1 + \frac{1}{r}(r-b, \dots, b(r-b), \dots, b2, b),$$
$$\sigma^* M = B_2 + \frac{1}{r}(r-b, \dots, b(r-b), \dots, b2, b).$$

Since any principal effective divisor is locally zero against exceptional divisors, it is sufficient to consider either $\sigma^* L$ or $\sigma^* M$. $\square$

Corollary 2.5 *We have*

$$BE_i = 0 \text{ for } i \neq b, \text{ otherwise } 1,$$
$$D^2 = \sigma^*(D)\sigma^*(D) = \sigma^*(D)B,$$
$$D^2 = B^2 + \frac{1}{r}b(r-b).$$

Corollary-Definition 2.6 *If there is a basket of cyclic quotient singularities, then*

$$\sigma^*(D) = B + \sum_{(r,b)} \frac{1}{r}(r-b, 2(r-b), \ldots, b(r-b), \ldots, b2, b),$$
$$D^2 = B^2 + \sum_{(r,b)} \frac{1}{r}b(r-b), \quad \text{the resolution data,} \tag{2.1}$$

where $B^2 \in 2\mathbb{Z}$ and the fractional part of (2.1) corresponds to the contribution of a basket of cyclic quotient singularities to the Riemann–Roch formula and (r,b) represents a cyclic quotient singularity $\frac{1}{r}(a,-a)$ of type b in $\mathcal{B}$.

Proof. The idea from the proof of Proposition 2.4 can be linearly extended to a basket of singularities. □

By *formal resolution data* we mean a number D^2 written in the form

$$D^2 = B^2 + \sum_{(r,b)} \frac{1}{r}b(r-b), \quad -4 \leq B^2 \in 2\mathbb{Z} \tag{2.2}$$

where the sum is taken over a basket $\mathcal{B}$ and (r,b) represents a cyclic quotient singularity $\frac{1}{r}(a,-a)$ of type b with r and a coprime. This is a convenient way of encoding the formal initial data, that is a number D^2 and a basket of singularities.

By looking at the resolution data we can read off necessary information, namely B^2, the singularities and the types of the singularities. We give an example from Fletcher's list [9].

Example 2.7 Let x, y, z, t be the homogeneous coordinates on $\mathbb{P}(5,7,8,20)$ with weights $5, 7, 8, 20$ respectively. The general weighted quasismooth homogeneous polynomial $X\colon f(x,y,z,t) = 0$ of degree 40 in $\mathbb{P}(5,7,8,20)$ with $D = \mathcal{O}_X(1)$ has the resolution data (see Corollary-Definition 2.6)

$$D^2 = \frac{40}{5 \cdot 7 \cdot 8 \cdot 20} = -4 + \frac{3}{4} + 2 \times \frac{2 \cdot 3}{5} + \frac{6}{7}.$$

That is, $B^2 = -4$ and the singularities are of the following form

$$\begin{cases} \frac{1}{4}(5,7) = \frac{1}{4}(1,3) & \text{of type 1 along the } q_2q_3 \text{ stratum,} \\ 2 \times \frac{1}{5}(7,8) = 2 \times \frac{1}{5}(2,3) & \text{of type 2 along the } q_0q_3 \text{ stratum,} \\ \frac{1}{7}(8,20) = \frac{1}{7}(1,6) & \text{of type 1 at the vertex } q_1. \end{cases}$$

Indeed, since X is quasismooth the only singularities are due to the k^*-action. Therefore the singularities occur only on the edges and at the vertices of $\mathbb{P}$. Note that from the generality assumption of X the coefficients of f could be nonzero constant, 1 for brevity. Therefore the defining equation of X in a calculation is written as

$$f(x,y,z,t) = x^8 + y^5x + z^5 + t^2 + \cdots$$

where $\cdots$ refers to all other possible terms. Since x^8, z^5 and $t^2 \in f$ the vertices $q_0, q_2, q_3 \notin X$. The polynomial f does not contain any power of y so $q_1 \in X$. Locally at q_1,

$$f(x,1,z,t) = x^8 + x + z^5 + t^2 + \cdots \quad \text{in } \mathbb{A}^3/\langle\varepsilon\rangle$$

where ε is a primitive 7th root of unity and acts on the affine coordinates x, z, t of $\mathbb{A}^3$ by

$$\varepsilon\colon \begin{cases} x \longmapsto \varepsilon^5 x \\ z \longmapsto \varepsilon^8 z = \varepsilon z \\ t \longmapsto \varepsilon^{20} t = \varepsilon^6 t. \end{cases}$$

The derivative of f with respect to x is nonzero at this point. Hence by the inverse function theorem z, t are local coordinates around q_1. This gives rise to a singularity $\frac{1}{7}(1,6)$, which is A_6. Now consider a one dimensional singular stratum $q_{i_1}q_{i_2}$ of $\mathbb{P}$ such that $\mathrm{hcf}(a_{i_1}, a_{i_2}) \neq 1$. Along the stratum q_0q_3, which is the xt line, f is written as

$$f_{|(y=z=0)} = x^8 + t^2 + x^4t \subset \mathbb{P}(5,20)$$

There are exactly two intersection points, which correspond to two singularities $2 \times \frac{1}{5}(7,8)$ of type 2. Along the stratum q_2q_3, which is the zt line, f is written as

$$f_{|(x=y=0)} = z^5 + t^2 \subset \mathbb{P}(8,20).$$

There are two intersection points which coincide so there is only one singularity $\frac{1}{4}(5,7)$ of type 1. Therefore X has only the given cyclic quotient singularities.

Similarly, it can be seen that $X(8,9)$ in $\mathbb{P}(1,2,3,4,7)$ has the resolution data

$$D^2 = \frac{8\cdot 9}{1\cdot 2\cdot 3\cdot 4\cdot 7} = -2 + 2\times\frac{1}{2} + \frac{3}{7}.$$

That is, $B^2 = -2$ and the singularities are

$$\begin{cases} 2\times\frac{1}{2}(1,1) & \text{of type 1 along the stratum } q_1q_3,\\ \frac{1}{7}(3,4) & \text{of type 2 at the vertex } q_4. \end{cases}$$

2.2 The Riemann–Roch formula on a polarised K3 surface

On a polarised K3 surface (X,D) with Du Val singularities, formula (1.2) takes the form

$$P_n = \chi(X,\mathcal{O}_X(nD)) = 2 + \frac{1}{2}(nD)^2 - \sum_{(r,b)} \frac{\overline{nb}(r-\overline{nb})}{2r} \qquad (2.3)$$

for $n \geq 1$ and $P_0 = 1$, where $\bar{}$ denotes the smallest residue modulo r.

By using resolution data, we write P_n in a more efficient form. We divide the problem into two parts. First if (X,D) has no singularities, then D is nef and big ($D^2 > 0$) and these conditions, by the general theory of linear systems on a nonsingular K3 surface, imply that $h^0(nD) = 2 + \frac{1}{2}(nD)^2$ (see Saint-Donat [23]). Second, in the case of a singular K3 surface our aim is to give a simple formula for $h^0(X,nD)$ as in the first case.

First note that

$$H^0(S, n\sigma^*(D)) = H^0(X,nD), \quad \text{for every } n\in\mathbb{N}$$

so we can simplify the notation by replacing $\sigma^*(D)$ by D. Here σ is a minimal resolution $(S,\sigma^*(D)) \to (X,D)$. Notice that S is a nonsingular K3 surface. By the definition of a divisorial sheaf $\mathcal{O}_S(D)$ the fractional part of nD does not contribute to the calculation of $H^0(S,nD)$. Therefore

$$H^0(X,nD) = H^0(S,\llcorner nD \lrcorner), \quad \text{for every } n\in\mathbb{N}$$

where $\llcorner nD \lrcorner$ is the integral part of nD. The *integral part* of D is defined to be

$$\llcorner D \lrcorner = B + \sum \llcorner a_i \lrcorner C_i$$

where $\llcorner a_i \lrcorner$ is the maximal integer not greater than a_i.

We now define F_n as the sum of the $(r-1)$-dimensional array

$$F_n = \sum_{(r,b)} \frac{1}{r}(r-\overline{nb}, 2(r-\overline{nb}), \ldots, \overline{nb}(r-\overline{nb}), \overline{nb}(r-\overline{nb}-1), \ldots, 2\overline{nb}, \overline{nb}),$$

where $^{-}$ denotes the smallest residue modulo r, and write

$$D_n = nD - F_n$$

where

$$nD = nB + \sum_{(r,b)} \frac{n}{r}(r-b, 2(r-b), \ldots, b(r-b), \ldots, b2, b).$$

The following lemma is preparation for Theorem 2.9. It states that F_n is in the fixed part of $|nD|$.

Lemma 2.8 *Let (X, D) be as given above. Then*

(i) $F_nC_i = \begin{cases} 0 & \textit{if } i \neq \overline{nb}, \\ -1 & \textit{otherwise} \end{cases}$ *and* $D_nC_i = \begin{cases} 0 & \textit{if } i \neq \overline{nb}, \\ 1 & \textit{otherwise}; \end{cases}$

(ii) $F_n^2 = -\sum_{(r,b)} \dfrac{\overline{nb}(r-\overline{nb})}{r}$;

(iii) $\llcorner F_n \lrcorner$ *is in the fixed part of* $|\llcorner nD \lrcorner|$.

Proof. (i) is obvious since F_n is an arithmetic progression and $D_n = nD - F_n$. Clearly (i) implies (ii). To prove (iii), it is sufficient to prove that D_n is maximal nef against the exceptional divisors C_i, that is, $D_n + H$ is not nef if $H = \sum_{l=1}^{m} b_{i_l} C_{i_l}$, $b_{i_l} > 0$.

Observe that $|nD| = |\llcorner nD \lrcorner|$+fractional part and $nD = D_n + \llcorner F_n \lrcorner + (F_n - \llcorner F_n \lrcorner)$ imply that $\llcorner nD \lrcorner = D_n + \llcorner F_n \lrcorner$ and $F_n - \llcorner F_n \lrcorner =$ fractional part. We claim that $\llcorner F_n \lrcorner$ is in the fixed part of $|\llcorner nD \lrcorner|$. It will be proven by contradiction. We assume that there is only one singularity (the argument can easily be extended to the general case). Suppose that $D_n + H$ is nef against C_i for each $C_i \subset H$. If H contains only one exceptional curve, that is $H = b_jC_j$ for some $b_j > 0$, then

$$(D_n + H)C_j = \begin{cases} HC_j < 0 & \text{if } j \neq \overline{nb}, \\ 1 + HC_j < 0 & \text{otherwise,} \end{cases}$$

a contradiction.

Now assume that H contains at least two exceptional divisors. Without loss of generality H can be written

$$H = \sum_{i=1}^{m} b_i C_i$$

where $b_i > 0$ for $i = 1, m$, $b_i \geq 0$ otherwise, and $C_i C_{i+1} = 1$ for every $i = 1, \ldots, m$. If H does not contain $C_{\overline{nb}}$, that is $b_{\overline{nb}} = 0$, then by assumption $D_n C_i + HC_i \geq 0$, which gives $H^2 \geq 0$, a contradiction. So assume that H contains $C_{\overline{nb}}$. Then

$$(D_n + H)C_i = \begin{cases} HC_i \geq 0 & \text{if } i \neq \overline{nb}, \\ 1 + HC_i \geq 0 & \text{otherwise.} \end{cases}$$

Notice that $HC_{\overline{nb}} = -1$ otherwise $H^2 \geq 0$, a contradiction. When we calculate the intersections of H against the C_i, there are cancellations along the diagonals. Indeed,

$$\begin{aligned} HC_1 &= -2b_1 + b_2 \geq 0 \\ HC_2 &= b_1 - 2b_2 + b_3 \geq 0 \\ HC_3 &= b_2 - 2b_3 + b_4 \geq 0 \\ &\vdots \\ HC_{i-2} &= b_{i-3} - 2b_{i-2} + b_{i-1} \geq 0 \\ HC_{i-1} &= b_{i-2} - 2b_{i-1} + b_i \geq 0 \\ HC_i &= b_{i-1} - 2b_i + b_{i+1} = -1 \quad \text{for } i = \overline{nb} \\ HC_{i+1} &= b_i - 2b_{i+1} + b_{i+2} \geq 0 \\ HC_{i+2} &= b_{i+1} - 2b_{i+2} + b_{i+3} \geq 0 \\ &\vdots \\ HC_{m-1} &= b_{m-2} - 2b_{m-1} + b_m \geq 0 \\ HC_m &= b_{m-1} - 2b_m \geq 0. \end{aligned}$$

Add all HC_i except $i \neq \overline{nb}$ and conclude that

$$0 \leq \sum_{i \neq \overline{nb}} HC_i = -b_1 - (b_{\overline{nb}-1} - 2b_{\overline{nb}} + b_{\overline{nb}+1}) - b_m,$$

but since $b_1, b_m > 0$ and $HC_{\overline{nb}} = -1$, $\sum_{i \neq \overline{nb}} HC_i < 0$, which is a contradiction. Therefore D_n is maximal nef against the exceptional divisors. If we replace H by $\llcorner F_n \lrcorner$, then $\llcorner nD \lrcorner = D_n + \llcorner F_n \lrcorner$ is not nef which means that there exists an exceptional divisor C_i for some i in $\llcorner F_n \lrcorner$ such that $\llcorner nD \lrcorner C_i$ is negative. Hence C_i is in the fixed part of $|\llcorner nD \lrcorner|$. We can repeat the process by replacing $\llcorner F_n \lrcorner$ with $\llcorner F_n \lrcorner - C_i$ and continuing in this way we can prove that $\llcorner F_n \lrcorner$ is in the fixed part of $|\llcorner nD \lrcorner|$. □

Now we are ready to state the main theorem of this section.

Theorem 2.9 *Given (X,D) with cyclic quotient singularities, then for every integer $n \geq 0$*

(i) $h^0(X, nD) = 2 + \frac{1}{2}{D_n}^2$;

(ii) $h^0(X, nD) = h^0(S, D_n)$.

Proof. It can easily be seen that $(nD)^2 = D_n^2 - F_n^2$. Inserting this into (2.3), we get

$$P_n = 2 + \frac{1}{2}D_n^2 - \frac{1}{2}F_n^2 + \text{contribution}.$$

On the other hand, $\frac{1}{2}F_n^2$, by Lemma 2.8 (ii), is equal to the contribution. This proves (i). The proof of (ii) is divided into two parts corresponding to whether $h^0(X, nD) \geq 1$ or is equal to 0. First, we remark that if $nD = 0$, there is nothing to prove. Suppose that $h^0(S, nD) \geq 1$. Then we can assume that nD is effective. Any element $E \in |nD|$ is written as "good"+"bad" where "good" is a sum of exceptionals which by Lemma 2.8 (iii) contains F_n and "bad" is a sum of non-exceptionals. This gives a one-to-one correspondence between $|nD|$ and $|D_n|$ so (ii) holds. On the other hand $h^0(S, nD) = 0$ implies that $h^0(S, D_n) = 0$. To see this, notice that F_n being an effective divisor gives $h^0(S, D_n) \leq h^0(S, nD)$ by using the short exact sequence

$$0 \to \mathcal{O}(D_n) \to \mathcal{O}(nD) \to \mathcal{O}_{F_n}(nD) \to 0.$$

□

Corollary 2.10 *D_n^2 is even and ≥ -4. In particular, when $n = 1$, $D_1^2 = B^2 \geq -4$.*

Remark 2.11 The resolution data of a polarised K3 surface (X, D) should satisfy B^2 even and ≥ -4, and $\sum(r-1) \leq 19$ (see Corollary 2.10 and Remark 2.3).

The following question comes naturally:

$$D_n^2 \geq 0 \text{ if and only if } D_n \text{ is nef.}$$

It is clear that if D_n is nef then by the weak form of the Kleiman Criterion $D_n^2 \geq 0$ (see Reid [18], Corollary 2.3, p. 116). We point out that this question is not relevant to our study.

Example 2.12 Given the formal initial data $\{D^2 = 7/6, A_1 \text{ and } A_2\}$ it is known that there exists a K3 surface $X(7)\colon (f = 0) \subset \mathbb{P}(1,1,2,3)$ in Fletcher's thesis [9]. To see this, we will compute P_n by using Theorem 2.9. When we blow up the singularities along D the resolution is

$$D = B + \frac{1}{2}C_1 + \frac{2}{3}C_2 + \frac{1}{3}C_3$$

where $BC_i = 1$ for $i = 1, 2$ and $BC_3 = 0$. So $D^2 = DB = B^2 + 1/2 + 2/3$ implying that $B^2 = 0$. Immediately from here, calculate $\dim H^0(X, D) = \dim H^0(S, B) = 2$, which implies that there are only two generators x_1, x_2 of degree 1. Now consider

$$2D = 2B + C_1 + C_2 + \text{fractional part},$$

so $2B + C_1 + E_2$ is maximal nef against the exceptionals. Therefore, by Theorem 2.9,

$$h^0(X, 2D) = h^0(S, 2B + C_1 + C_2) = 2 + \frac{1}{2}(2B + C_1 + C_2)^2$$

which is four but there are already three generators x_1^2, x_2^2, x_1x_2. Therefore we need another generator y of degree 2. Similarly, $h^0(X, 3D) = 7$ gives one more generator z of degree 3 since there are six generators coming from degree 1 and 2, namely $S^3(x_1, x_2)$, the 3rd symmetric tensor power of (x_1, x_2), and $yS^1(x_1, x_2)$. If $n \geq 4$, it can easily be observed that there is no need for a new generator. When $n = 7$, $h^0(X, 7D) = 30$ and the number of the monomials of degree 7 in variables x_1, x_2, y, z is 31. Therefore there is a relation f of degree 7. If $n \geq 8$, observe that $\#\{\text{monomials of degree } n\} = \#\{\text{monomials of degree } n - 7\} + h^0(X, nD)$, which implies that in further calculations f is the only relation. Then $X(7) = \operatorname{Proj} k[x_1, x_2, y, z]/(f)$ is the weighted hypersurface in $\mathbb{P}(1,1,2,3)$, with the resolution data $\{D^2 = 0 + 1/2 + 2/3\}$.

For large n, it is very difficult to make this calculation. A more efficient way is to use the Hilbert function of a graded ring. We will explain this in a further section.

Chapter 3

The Hilbert Function Method

The question we are interested in is the following: given a numerical function P_n, does there exist a ring R with P_n as its Hilbert function? Usually, P_n is obtained from the numerical data for a polarised K3 surface (X, D) (or other classes of varieties) and we expect that $X = \operatorname{Proj} R$ together with $D = \mathcal{O}_X(1)$ and conversely $R = R(X, D)$. One way of approaching the question is to consider the Hilbert series of graded rings as a rational function in Theorem 3.1. This is a convenient way to cope with the difficulties of calculating P_n by using the generating function given by the Hilbert series. We explain how this approach leads to lists of possible K3 surfaces in Section 3.3. The question of existence will be addressed in Chapter 5.

All the results we get for polarised K3 surfaces can be extended to $\mathbb{Q}$-Fano 3 folds. This will be explained in Section 3.4. The question of existence will not be answered in this thesis.

3.1 The Hilbert series

Let the Hilbert series of P_n be $P(t) = \sum_{n=0}^{\infty} P_n t^n$ where P_n is given by an appropriate form of the Riemann–Roch formula.

Theorem 3.1 *Let (X, D) be a K3 surface with Du Val singularities and $R = R(X, D) = \bigoplus_{n \geq 0} H^0(X, nD)$ with $P_n = h^0(X, nD)$. Then the Hilbert series of P_n is as follows*

$$P(R,t) = \frac{(1+t)}{(1-t)} + \frac{1}{2}D^2 \frac{t(1+t)}{(1-t)^3} - \sum_{(r,b)} \frac{1}{2r} \frac{\sum_{n=1}^{r-1} \overline{bn}(r - \overline{bn})t^n}{1 - t^r} \tag{3.1}$$

where (r, b) represents the singularity $\frac{1}{r}(a, -a)$ in the basket $\mathcal{B}$ (see Section 1.3.1 for notation) such that $ab \equiv 1 \mod r$, and $^{-}$ denotes the smallest *residue modulo r.*

Proof. Take the sum of P_n over all $n \geq 0$, given by (2.3) (see Section 2.2), to obtain

$$P(R,t) = \sum_{n=0}^{\infty} P_n t^n = 1 + \sum_{n=1}^{\infty} \left(2 + \frac{1}{2}(nD)^2 - \sum_{(r,b)} \frac{\overline{nb}(r - \overline{nb})}{2r} \right) t^n$$

and rearrange into the following

$$P(R,t) = 1 + \sum_{n=1}^{\infty} 2t^n + \frac{1}{2}D^2 \sum_{n=1}^{\infty} n^2 t^n - \sum_{(r,b)} \frac{1}{2r} \sum_{n=1}^{\infty} \overline{nb}(r - \overline{nb}) t^n. \qquad (3.2)$$

In (3.2) the expressions $1 + 2t^2 + 2t^3 + \cdots$ and $(t + 2^2t^2 + 3^2t^3 + \cdots)D^2/2$ correspond to the first and second terms of (3.1), and for each element of the basket, the function $\overline{nb}(r - \overline{nb})$ repeats periodically over the intervals $[0, r]$, $[r, 2r], \ldots$ and is zero at the end points, giving the last term of (3.1). $\square$

Remark 3.2 Because D is $\mathbb{Q}$-ample, the graded ring $R(X, D)$ of (X, D) is a finitely generated ring, so a quotient ring of the Noetherian graded ring $A = k[x_0, \ldots, x_n]$ where $\mathrm{wt}(x_i) = a_i$ (see Reid [19], Section 2). Therefore, by Theorem 13.2, Chapter 5 in [16],

$$P(R,t) = \frac{f(t)}{\prod_i (1 - t^{a_i})} \qquad (3.3)$$

where $f(t)$ is a polynomial in $\mathbb{Z}[t]$. It is known that in the case of a weighted complete intersection $X(d_1, \ldots, d_l) \subset \mathbb{P}(a_0, \ldots, a_n)$, where $l = n - \dim X$,

$$f(t) = \prod_{i=1}^{l} (1 - t^{d_i})$$

(see Dolgachev [7]).

Example 3.3 The Hilbert series $P(R,t)$ of $X(8,9)$ in $\mathbb{P}(1,2,3,4,7)$ with $D^2 = -2 + 2 \times 1/2 + 2 \cdot 5/7$ is

$$\begin{aligned} &\frac{(1-t^8)(1-t^9)}{(1-t)(1-t^2)(1-t^3)(1-t^4)(1-t^7)} \\ &\qquad = \frac{(1+t)}{(1-t)} + \frac{3}{2 \cdot 7} \frac{t(1+t)}{(1-t)^3} - 2 \times \frac{1}{2 \cdot 2} \frac{t}{1-t^2} \\ &\qquad\qquad - \frac{1}{2 \cdot 7} \frac{10t + 12t^2 + 6t^3 + 6t^4 + 12t^5 + 10t^6}{1 - t^7}. \end{aligned}$$

Here the left hand side gives information about the degrees of the defining equations of X and the ambient space in which X is embedded and the right hand side corresponds to the numerical data (see Theorem 3.1).

3.2 Searching strategy

To produce a list of candidates for polarised K3 surfaces (X, D) of given codimension, we consider all possible baskets of singularities on a K3 surface given by a choice of some numerical data (see Chapter 2), the possibilities being a few thousand (see Remark 2.11). We fix a priori $\operatorname{codim} X$, so we are only in R with $\leq \operatorname{codim} X + 3$ generators. Here by generators of R we mean generators of the graded ring A where $R = A/I(X)$ (see Remark 3.2). For each basket, we have $0 \leq P_1 \leq 3 + \operatorname{codim} X$ and a bound for P_2 related to P_1.

The search is based on the values of P_1, where $0 \leq P_1 \leq 7$ (and usually ≤ 6, see Section 4.3), and P_2. We fix P_1 and P_2 and get a bound on $\sum_{(r,b)} b$ (see Lemma 3.7), hence a lower bound on the r. For each list $(b_1, \ldots, b_k)$ of b we consider all possible lists $(r_1, \ldots, r_k)$ of r in a lexicographic order. There are several mechanisms which can kill the cases $(r_1, \ldots, r_k)$ and $(r'_1, \ldots, r'_k)$ where $r'_i > r_i$ for some i and $r'_j \geq r_j$ for $j \neq i$. Each $(r_1, \ldots, r_k)$ imposes new generators on the ring. The reason is that by Lemma 3.4, each $\frac{1}{r}(a, -a)$ usually implies that $R(X, D)$ has some extra generators in certain degrees. This can be used together with $3 + \operatorname{codim} X$ to exclude many cases; especially effective when $2 \leq P_1$, they cut the list of possible candidates to quite a small number. Similarly when we increase r to the next possible value r', Proposition 3.6 tell us that there will be at least one new generator, but an old generator(s) may be lost. In all of our cases we have found that after a certain value of r all of the old generators are retained. This means that there will be no K3 surfaces of small codimension for further iterations. See Section 4.3 where we carry out the search for $2 \leq P_1 \leq 6$. Then we have to work case by case on each possible item, to find if (X, D) exists or, if not, to exclude it.

Lemma 3.4 *Let (V, D) be a polarised surface having a cyclic quotient singularity $\frac{1}{r}(a, -a)$ with some multiplicity c where a and r are coprime. Then there exist two generators x, y of $R(V, D)$ where $\operatorname{wt}(x)$ and $\operatorname{wt}(y)$ are congruent to $a, -a$ modulo r. Moreover, depending on whether $c = 1$ or > 1, either there is at least a generator z of degree r or there are at least two different generators z_1, z_2 where $\operatorname{wt}(z_i) \equiv 0 \mod r$.*

Proof. The singularity of V arises from the $\mathbb{C}^*$-action and hence the ring R

has two generators x, y with weights a_1 and a_2 which satisfy $a_i \equiv \pm a \pmod r$ for $i = 1, 2$ and $a_1 + a_2 = lr$ for some l. Moreover, since rD is a Cartier divisor which is locally isomorphic to $\mathcal{O}_V$, there exists a further generator z such that $\mathrm{wt}(z) \equiv 0 \pmod r$. If $c > 1$ then the singularities occur at least on a one dimensional stratum. This implies that there are at least two generators whose weights are divisible by r. $\square$

Remark 3.5 Lemma 3.4 is also a consequence of Theorem 3.1.

As we explained above Proposition 3.6 plays a crucial role in terminating the search.

Proposition 3.6 *Let $\{D^2, \mathcal{B}\}$ and $\{D'^2, \mathcal{B}'\}$ be formal initial data for K3 surfaces, where $\mathcal{B} = \{(r_i, b_i)\}_{i \in I}$ and $\mathcal{B}' = \{(r_i', b_i)\}_{i \in I}$ such that $r_i \leq r_i'$. Assume that the corresponding numerical functions P_n and P_n', given by the Riemann–Roch formula, satisfy $P_1 = P_1'$. Then*

$$P_n \leq P_n' \quad \textit{for every } n \quad (< \textit{ for at least one } n).$$

Proof. We assume that $\mathcal{B}$ and $\mathcal{B}'$ differ at only term $r < r'$ where $(r, b) \in \mathcal{B}$ and $(r', b) \in \mathcal{B}'$. This can easily be extended to general baskets.

The Riemann–Roch formula gives us

$$P_n = 2 + \frac{n^2}{2}(D)^2 - \frac{\overline{nb}(r - \overline{nb})}{2r} + x_n$$

and

$$P_n' = 2 + \frac{n^2}{2}(D')^2 - \frac{\widetilde{nb}(r' - \widetilde{nb})}{2r'} + x_n$$

where $\overline{\ }$, $\widetilde{\ }$ denote the smallest residues modulo r, r' respectively and x_n is the sum of the contributions of the common singularities in $\mathcal{B}$ and $\mathcal{B}'$. By using the assumption $P_1 = P_1'$ we can get rid of D^2 and D'^2 and some common terms in P_n and P_n'. This implies that it is sufficient to prove that

$$\Delta_n(r, b) \leq \Delta_n(r', b) \quad \text{for every } n$$

where

$$\Delta_n(r, b) = \frac{n^2 b - \overline{nb}}{2} - \frac{(nb - \overline{nb})(nb + \overline{nb})}{2r} \tag{3.4}$$

and $\Delta_n(r', b)$ is given by replacing r by r' and $\overline{\ }$ by $\widetilde{\ }$ in (3.4). (This notation was first used by Fletcher for different purposes; see Fletcher [9], Definition

3.5.) One can easily check that $\Delta_n(r_i, b_i) \in \mathbb{Z}$, for each singularity (r_i, b_i) and each n, by using the Riemann–Roch formula or directly by a numerical calculation. We can write

$$nb = lr + \overline{nb} \quad \text{and} \quad nb = l'r' + \widetilde{nb}.$$

for some positive integers l, l'. Since $r < r'$, $0 \le l' \le l$. Now the proof is divided into the following two cases:

(i) $l = l'$,

(ii) $l > l'$.

First notice that

$$\Delta_n(r, b) \le \Delta_n(r', b) \Longleftrightarrow ll'(r' - r) + \widetilde{nb}(l - l' - 1) + \overline{nb}(l - l' + 1) \ge 0. \tag{3.5}$$

Suppose that (i) holds. Since $nb = lr + \overline{nb} = lr' + \widetilde{nb}$ and $r < r'$, $\overline{nb} \ge \widetilde{nb}$. It follows that by replacing l' by l on the right hand side of (3.5) we have

$$\Delta_n(r, b) \le \Delta_n(r', b).$$

Suppose that (ii) holds. Since $r' > r$ and $l > l'$, the inequality on the right hand side of (3.5) holds. Thus $\Delta_n(r, b) \le \Delta_n(r', b)$. For large n $r' < nb$ and so $l' > 0$. The expression on the right hand side of (3.5) is always greater than zero. Hence $\Delta_n(r, b) < \Delta_n(r', b)$ for some n, as required. $\square$

Lemma 3.7 *Given a basket of cyclic quotient singularities (r, b) on a polarised K3 surface (X, D) then*

(i) $D^2 = 2P_1 - 4 + \sum_{(r,b)} c \times \dfrac{b(r-b)}{r} > 0$ *where c is the number of times (r, b) appears. Furthermore, $2P_1 - 4 = B^2$ where B comes from the resolution data in Chapter 2, Corollary 2.6.*

(ii) $P_2 = 4P_1 - 6 + \sum_{(r,b)} cb$, *where $b \le \frac{r}{2}$.*

Proof. (i) follows from the definition of P_1 and a $\mathbb{Q}$-ample Weil divisor D. For (ii) we can assume that $b \le \frac{r}{2}$. This is possible since b and $r - b$ have the same effect on the Riemann–Roch formula. We can get rid of D^2 in P_2 by using (i) to obtain

$$P_2 = 4P_1 - 6 + \sum_{(r,b)} c\Delta_2(r, b)$$

Since $2b \le r$, $2b \equiv 0$ or $2b \mod r$. If $2b \equiv 0 \mod r$ then $r = 2$, otherwise $r > 2$. In both cases, that is $r \ge 2$,

$$\Delta_2(r, b) = b.$$

□

In Lemma 3.7 P_2 gives most of the information we need, namely the generators of the ring R and the maximum number of singularities which can occur, especially when $2 \le P_1 \le 3 + \operatorname{codim} X$. We explain this point in Section 4.3. One can also get a general formula P_i as in the case of P_2 as long as $ib \le r$ for each (r, b) in $\mathcal{B}$.

3.3 Outline of the method

Having fixed singularities and a number D^2 (or equivalently, a value of P_1, see Lemma 3.7 (i)), an outline of the method is as follows:

Step 1. Consider the Hilbert series $P(t)$ of P_n as a rational function $P(R, t)$ in Theorem 3.1.

Step 2. Choose suitable module generators y_1, y_2, y_3 of $\mathrm{wt}(y_i) = s_i$ so that

$$P(R/\langle y_1, y_2, y_3\rangle, t) = (1 - t^{s_1})(1 - t^{s_2})(1 - t^{s_3})P(R, t) \qquad (3.6)$$

is a finite polynomial with positive coefficients and of reasonably small degree in $A = k[x_0, \dots, x_n]$ of $\mathrm{wt}(x_i) = a_i$. We call (s_1, s_2, s_3) a "good choice".

Step 3. Guess the generators by analysing $P(R/\langle y_1, y_2, y_3\rangle, t)$ in Step 2 together with Lemma 3.4 and using this as a trial-and-error strategy on the following step:

Step 4. The generators satisfy the condition

$$\prod_{i=0}^{n}(1 - t^{a_i})P(R, t) = f(t) \qquad (3.7)$$

where $f(t)$ is a polynomial with integer coefficients which comes from the resolution structure of the graded ring R and is calculated in Chapter 4 (see also (3.3)).

We now describe the first three steps in more detail.

Step 1 gives information how to choose module generators of R in Step 2 to make a finite polynomial but it does not guarantee a polynomial with positive coefficients.

Remark 3.8 R is a finite $k[y_1, y_2, y_3]$-module and a Cohen–Macaulay ring of dimension 3 which implies that $R/\langle y_1, y_2, y_3\rangle$ is Cohen–Macaulay of dimension 0, and a vector space over k where $\{y_1, y_2, y_3\}$ is a regular sequence. Geometrically, it means that $C(X)$ is cut out transversally by three hyperplanes. Hence the P_n of the new ring $R/\langle y_1, y_2, y_3\rangle$ are zero for all but a finite number of n.

The remark states only the existence of module generators in Step 2, on the other hand it does not give explicitly how to choose them. We can overcome these difficulties by choosing weights s_i sufficiently large, such that some of them are divisible by the indices of the singularities, to kill the periodicity of P_n, or alternatively, the denominators of $P(R, t)$. In this way we can always ensure that (3.6) is a finite polynomial with positive coefficients. But the aim is, at the same time, to make the degree of the equation "reasonably small", which means that we hope that (3.6) is the Hilbert series of an Artinian ring of small codimension.

Step 3 is difficult to automate but by Lemma 3.4 and using a trial-and-error strategy on the steps we can guess what the generators can be as long as the ring is "small", especially, in the case of codimension ≤ 3.

If we succeed in Step 4 we say that there is a candidate $X \subset \mathbb{P}(a_0, \dots, a_n)$ for a K3 surface. The existence of candidates will be proven in Chapter 5.

3.4 $\mathbb{Q}$-Fano 3 folds

We are interested in finding possible candidates for general $\mathbb{Q}$-Fano 3 folds V of codimensions 3 and 4 embedded in weighted projective spaces, especially when $P_1 = 0$ (see $P_n = h^0(X, -nK_V)$ in (1.3)). The formal initial data is a number $(-K_V)^3 \in \mathbb{Q}$ and a basket $\mathcal{B}$ of terminal cyclic quotient singularities $\frac{1}{r}(1, -1, b)$ of type b where r is coprime to b. Without loss of generality we can assume that $b \leq \frac{r}{2}$ because b and $r - b$ have same effect on P_n. This is important for computing P_2.

By the result of Kawamata [12] on a $\mathbb{Q}$-Fano 3 fold, we have

$$24 = (-K_V)c_2(V) + \sum (r - \frac{1}{r}), \quad (-K_V)c_2(V) > 0.$$

Therefore the number of singularities $\frac{1}{r}(1, -1, b)$ on a $\mathbb{Q}$-Fano 3 fold is bounded, and $\sum_r (r - 1) \leq 23$. This implies that the computer search will be finite.

Some results on a $\mathbb{Q}$-Fano 3 folds

We state without proof some results on a $\mathbb{Q}$-Fano 3 fold. The proofs can be done in a similar way as in the case of a polarised K3 surface.

Let V be a $\mathbb{Q}$-Fano 3 fold and $R(V, -K_V) = \bigoplus_{n\geq 0} H^0(V, -nK_V)$ its ring with $P_n = h^0(X, -nK_V)$, $P_0 = 1$. Then

(i) the Hilbert series of P_n is of the form

$$P(R,t) = \frac{(-K_V)^3}{2}\frac{(t+t^2)}{(1-t)^4} + \frac{(1+t)}{(1-t)^2} - \sum_{(r,b)} \frac{1}{2r}\frac{\sum_{n=1}^{r-1} \overline{bn}(r-\overline{bn})t^n}{(1-t)(1-t^r)}$$

where (r,b) represents the singularity $\frac{1}{r}(a,-a,1)$ of index r and type b in the basket $\mathcal{B}$ such that $ab \equiv 1 \mod r$, and $^-$ denotes the smallest residue modulo r. Observe that, by replacing $(-K_V)^3$ by D^2, the product of $P(R,t)$ and $(1-t)$ has the same form as the Hilbert series of a polarised K3 surface (X, D) (compare (3.1)).

(ii) Suppose that V has a terminal quotient singularity $\frac{1}{r}(a,-a,1)$ with multiplicity c, where a and r are coprime. Then there exist four generators x, y, z, t of $R(V, -K)$ where at least three of $\mathrm{wt}(t), \mathrm{wt}(z), \mathrm{wt}(y), \mathrm{wt}(x)$ are congruent to $0, a, -a$ modulo r. This is a consequence of (i) (compare Lemma 3.4). If $c > 1$ there exist at least two generators whose weights are divisible by r.

(iii) The given formal initial data $\{(-K)^3 \in \mathbb{Q}, \text{ basket } \mathcal{B}\}$, if we increase one of the singularities (r,b) (that is, increase the index r of the singularity under consideration), keeping the type b fixed, then this increases P_n (compare Proposition 3.6).

(iv) $(-K_V)^3$ can be expressed in terms of P_1 and the contributions coming from singularities. Namely,

$$(-K_V)^3 = 2P_1 - 6 + \sum_{(r,b)} c \times \frac{b(r-b)}{r} > 0$$

where c is the number of singularities (r,b) repeated. In particular, if $P_1 = 0$ then

$$(-K_V)^3 = -6 + \sum_{(r,b)} c \times \frac{b(r-b)}{r} > 0.$$

(v) In general,

$$P_2 = 5P_1 - 10 + \sum_{(r,b)} cb$$

since $b \leq \frac{r}{2}$. When $P_1 = 0$ we have

$$P_2 = -10 + \sum_{(r,b)} cb,$$

together with

$$10 \leq \sum cb \leq 12.$$

If $\sum_{(r,b)} cb = 12$ then the first singularity is of the form $\frac{1}{2}(1, -1, 1)$ with some multiplicity.

Note that at most six different types of singularities (repetitions allowed) can occur.

In the case of a $\mathbb{Q}$-Fano 3 fold V of codim 3, the shape of the resolution of R over $A = k[x_0, \dots, x_6]$ with $\mathrm{wt}(x_i) = a_i$ looks like

$$\prod_{i=0}^{6}(1 - t^{a_i})P(R,t) = 1 - \sum_{i=1}^{2m+1} t^{d_i} + \sum_{i=1}^{2m+1} t^{e_i} - t^s$$

where the d_i are the degree of the relations and the e_i are the degree of the syzygies, and $K_V = \mathcal{O}_V(-1)$ which is $K_V = \mathcal{O}_V(s - \sum a_i)$.

We follow the same strategy for $\mathbb{Q}$-Fano 3 folds as for polarised K3 surfaces checking that the method holds Steps 1–4 (see Section 3.3) modified as appropriate. The Maple program written for a polarised K3 surface embedded in weighted projective space (see the Appendix of this chapter) can easily be extended to a $\mathbb{Q}$-Fano 3 fold. In my search I proved that there are no Pfaffian $\mathbb{Q}$-Fano 3 folds in codim 3 with $P_1 = 0$. When $P_1 = 0$, rings are usually "very big". On the other hand, we do have the following three candidates for $\mathbb{Q}$-Fano 3 folds in codim 4.

Table 3.1: Candidates for $\mathbb{Q}$-Fano 3 folds in codimension 4

Nb.	**Candidate for $\mathbb{Q}$-Fano 3 fold**	**Formal Initial Data**
1	$V \subset \mathbb{P}(2,2,3,3,4,4,5,5)$	$-6 + 11 \times \frac{1}{2} + \frac{2}{3}$
2	$V \subset \mathbb{P}(2,3,4,5,6,7,8,9)$	$-6 + 7 \times \frac{1}{2} + \frac{2}{3} + \frac{3 \cdot 5}{8}$
3	$V \subset \mathbb{P}(2,5,6,7,8,9,10,11)$	$-6 + 7 \times \frac{1}{2} + \frac{4}{5} + \frac{3 \cdot 4}{7}$

Let $f(t)$ be $\prod_{i=0}^{7}(1-t^{a_i})P(R,t)$ where $P(R,t)$ is the Hilbert series of R over $A=[x_0,\dots,x_7]$. The polynomial $f(t)$ for each candidate in Table 3.1 is the following:

1. $f(t)=t^{27}-3t^{19}-3t^{18}-3t^{17}+2t^{15}+6t^{14}+6t^{13}+2t^{12}-3t^{10}-3t^{9}$
   $-3t^{8}+1$

2. $f(t)=t^{43}-2t^{31}-t^{30}-2t^{29}-2t^{28}-t^{27}+2t^{24}+2t^{23}+3t^{22}+3t^{21}$
   $+2t^{20}+2t^{19}-t^{16}-2t^{15}-2t^{14}-t^{13}-2t^{12}+1$

3. $f(t)=t^{57}-t^{41}-t^{40}-2t^{39}-t^{38}-2t^{37}-t^{36}-t^{35}+t^{32}+2t^{31}+2t^{30}$
   $+3t^{29}+3t^{28}+2t^{27}+2t^{26}+t^{25}-t^{22}-t^{21}-2t^{20}-t^{19}-2t^{18}$
   $-t^{17}-t^{16}+1$

From these, we can read off information about degrees of relations and syzygies. For example, in the case of candidate #1 in Table 3.1 we expect that if it exists it should have 9 relations of degrees $8^3, 9^3, 10^3$ and 16 syzygies of degrees $12^2, 13^6, 14^6, 15^2$. Candidate #2 looks related to Fletcher's example $X(12,14)\subset\mathbb{P}(2,3,4,5,6,7)$ with the singularities $\frac{1}{5}(1,-1,2), 7\times\frac{1}{2}(1,-1,1), 2\times\frac{1}{3}(1,-1,1)$. The existence could possibly be proven by projecting from the line spanned by q_6 and q_7 to Fletcher's example as we will see for K3 surfaces in Chapter 5. We do not study this further in this thesis but we hope to return to it later on.

Chapter 4

Practical Calculations

In this chapter we work through the practical calculation of finding the possible candidates (X, D) where $\operatorname{codim} X \leq 3$, particularly $2 \leq P_1 \leq 6$. It is advantageous to have access to a computer that runs Maple (see Appendix 4.3) to check whether the method holds Steps 2–4 in Chapter 3, Section 3.3 for the given choice of (s_1, s_2, s_3).

4.1 Weighted complete intersections of codimension ≤ 2

Let X be a weighted complete intersection. Then it is given by $X = \operatorname{Proj} R$ where $R = A/I(X)$ and $A = k[x_0, \ldots, x_n]$ with $\operatorname{wt}(x_i) = a_i$. Since X is c. i., then R is Cohen–Macaulay. Therefore $\operatorname{pd}_A R = \operatorname{codim} X$ where the projective dimension of R, $\operatorname{pd}_A R$, denotes the minimum of the lengths of projective resolutions of R over A.

If $\operatorname{codim} X = 1$, then $I(X) = \langle f \rangle$ and the resolution of R is as follows

$$0 \to A \xrightarrow{f} A \to R \to 0.$$

If $\operatorname{codim} X = 2$, then $I(X) = \langle f, g \rangle$ and by the Hilbert–Burch theorem (see Introduction) the resolution of R has the following form

$$0 \to A \xrightarrow{(-g,f)} A \oplus A \xrightarrow{\binom{f}{g}} A \to R \to 0$$

(see Eisenbud [8] for more details).

From the resolution of R, we can calculate the Hilbert series and the canonical divisor of X by taking the Hilbert function and $\mathcal{H}om(\quad, \omega_{\mathbb{P}})$ of the resolution respectively. In general the canonical divisor of a weighted complete intersection $X(d_1, \ldots, d_l)$ in $\mathbb{P}(a_0, \ldots, a_n)$ is $K_X = \mathcal{O}_X(d_1 + \cdots + d_l - a_0 - \cdots - a_n)$ and the Hilbert series is of the form

$$P(R,t) = \frac{f(t)}{\prod_{i=0}^{n}(1-t^{a_i})}$$

where

$$f(t) = \prod_{i=1}^{l}(1-t^{d_i})$$

(see [7]).

Now we give some examples.

Example 4.1 Let the initial data be

$$D^2 = -4 + 5 \times 1/2 + 3/4 + 4/5.$$

Then by Theorem 3.1 $P(R,t)$ is written as the rational function

$$\begin{aligned} P(R,t) = \frac{1+t}{1-t} + \frac{1}{2\cdot 20}\frac{t(1+t)}{(1-t)^3} - 5 \times \frac{1}{2\cdot 2}\frac{t}{1-t^2} \\ - \frac{1}{2\cdot 4}\frac{3t+4t^2+3t^3}{1-t^4} - \frac{1}{2\cdot 5}\frac{4t+6t^2+6t^3+4t^4}{1-t^5}. \end{aligned} \tag{4.1}$$

We can always spot the first nonzero generator(s) and typically a few more in higher degrees by choosing (s_1, s_2, s_3) such that

$$(1-t^{s_1})(1-t^{s_2})(1-t^{s_3})P(R,t)$$

is a finite polynomial with positive coefficients . This can be achieved by choosing (s_1, s_2, s_3) sufficiently large and such that the index of every singularity divides one of the s_i to kill the denominators of $P(R,t)$ in (4.1). For example, choose $(10, 20, 30)$ then

$$(1-t^{10})(1-t^{20})(1-t^{30})\sum P_n t^n = 1 + t^2 + 2t^4 + t^5 + 2t^6 + \cdots + t^{60},$$

a finite polynomial with positive coefficients. From the coefficients of this polynomial, we can observe that $P_1 = 0$, $P_2 = 1$, $P_3 = 0$, $P_4 = 2$ and $P_5 = 1$. It follows that there exist generators x, y, z of degrees $2, 4, 5$ respectively. A good choice for (s_1, s_2, s_3) can be $(2, 4, 5)$. The reason is that certainly we need something which is divisible by 4 and 5 to kill the denominators of $P(R,t)$ and also the s_i are relatively small integers having this property. Now repeat the same calculation for $(2,4,5)$ as we did for the previous choice, we hope to get a polynomial with positive coefficients. In fact,

$$(1-t^2)(1-t^4)(1-t^5)P(R,t) = 1+t^{11}.$$

This looks like the Hilbert series of an Artinian ring with a further generator t of degree 11. Hence a plausible candidate is $X \subset \mathbb{P}(2,4,5,11)$. To see the structure of its resolution, we multiply the previous equation by $(1-t^{11})$ which gives

$$(1-t^2)(1-t^4)(1-t^5)(1-t^{11})P(R,t) = 1-t^{22}.$$

Therefore the shape of the resolution of $R(X,D)$ over $A = k[x,y,z,t]$ is

$$0 \to A(-22) \xrightarrow{f} A \to R \to 0$$

where f is the defining equation of X in degree 22. The canonical divisor is $K_X = \mathcal{O}_X(22-2-4-5-11)$, which is trivial. It can be seen that $\mathrm{Cone}(X)$ is smooth by using Bertini's theorem in the case of a complete intersection as in Example 2.7 (see Fletcher [9] for more details). Therefore this is a K3 surface $X(22)$ in $\mathbb{P}(2,4,5,11)$ of codim 1 with given initial data in Fletcher's list.

Example 4.2 Consider the case $D^2 = -4 + 1/2 + 2/3 + 3/4 + 2\cdot 3/5 + 8/9$. Then it follows from Theorem 3.1 that

$$\begin{aligned} P(R,t) = \frac{1+t}{1-t} &+ \frac{1}{2\cdot 180}\frac{t(1+t)}{(1-t)^3} - \frac{1}{2\cdot 2}\frac{t}{1-t^2} - \frac{1}{2\cdot 3}\frac{2t+2t^2}{1-t^3} \\ &- \frac{1}{2\cdot 4}\frac{3t+4t^2+3t^3}{1-t^4} - \frac{1}{2\cdot 5}\frac{6t+4t^2+4t^3+6t^4}{1-t^5} \\ &- \frac{1}{2\cdot 9}\frac{8t+14t^2+18t^3+20t^4+20t^5+18t^6+14t^7+8t^8}{1-t^9}. \end{aligned}$$

As in the previous example, to detect a few generators we choose (s_1,s_2,s_3) suitably such that

$$Q(t) = (1-t^{s_1})(1-t^{s_2})(1-t^{s_3})P(R,t) \tag{4.2}$$

is a finite polynomial with positive coefficients. For example, choose sufficiently large $(20,30,72)$ to kill the denominators of $P(R,t)$ and to guarantee that $Q(t)$ is a polynomial with positive coefficients. It follows that

$$(1-t^{20})(1-t^{30})(1-t^{72})P(R,t) = 1+t^8+t^9+t^{10}+t^{12}+t^{15}+\cdots+t^{122}$$

is a finite polynomial with positive coefficients. Hence it can easily be seen that $P_i = 0$ for $i = 1, \dots, 7$ and $P_8 = P_9 = P_{10} = P_{12} = P_{15} = 1$. In other words, there exist generators x, y, z, t, u of degrees $8, 9, 10, 12, 15$ respectively. Since each of the indices of the singularities divide at least one of $8, 9, 10$, the degrees of x, y, z, a good choice for (s_1, s_2, s_3) can be $(8, 9, 10)$. Now multiplying $P(R, t)$ by $(1-t^8)(1-t^9)(1-t^{10})$ gives

$$(1-t^8)(1-t^9)(1-t^{10})P(R,t) = 1 + t^{12} + t^{15} + t^{27}.$$

Hence a plausible candidate is $X \subset \mathbb{P}(8, 9, 10, 12, 15)$. To see the structure of its resolution, multiply again by $(1-t^{12})(1-t^{15})$ to obtain

$$(1-t^8)(1-t^9)(1-t^{10})(1-t^{12})(1-t^{15})P(R,t) = 1 - t^{24} - t^{30} + t^{54}.$$

This implies that the shape of the resolution of $R(X, D)$ over $A = k[x, y, z, t, u]$ is

$$0 \to A(-54) \xrightarrow{(-g,f)} A(-24) \oplus A(-30) \xrightarrow{\binom{f}{g}} A \to R \to 0$$

where f and g are the defining equations of X in degrees $30, 24$ respectively. The canonical divisor is $K_X = \mathcal{O}_X(24 + 30 - 8 - 9 - 10 - 12 - 15)$, which is trivial. Again it is not hard to check that it is quasismooth. This is a K3 surface $X(24, 30) \subset \mathbb{P}(8, 9, 10, 12, 15)$ of codim 2, which appears in the list of Fletcher.

4.2 Pfaffians in codimension 3

The generic form for Gorenstein rings of codim 3 is given by Buchsbaum and Eisenbud [4] namely, it is shown that an ideal I of codim 3 in a regular local ring A is Gorenstein if and only if it is the ideal of $2m$th order Pfaffians or the submaximal Pfaffians of some $(2m+1) \times (2m+1)$ skew-symmetric matrix, and also if $R = A/I$ is Gorenstein, then the free resolution (over A) of R is self-dual, which gives us that the resolution of R is written as follows

$$0 \to A \xrightarrow{P^t} F^* \xrightarrow{M} F \xrightarrow{P} A$$

where F is a free A-module of rank $2m+1$, M a $(2m+1) \times (2m+1)$ skew-symmetric matrix, P a $(2m+1) \times 1$ matrix and P^t denotes the transpose of P.

The generators Pf_i of I, called the *relations* of R, are the $2m$th order Pfaffians of M which are given as the Pfaffians[1] of the $2m \times 2m$ submatrices

[1] The *Pfaffian* of a skew-symmetric matrix is the square root of its determinant. Formulae in the case $m = 2$ are given in Remark 4.5.

obtained by omitting the ith row and the ith column of M, and the degree of the relations are the degree of the entries of P. A *syzygy* is a relation among the relations.

The resolution of a graded ring

In view of the above preamble, we can rewrite the resolution of R in the following form

$$0 \to A(-s) \xrightarrow{P^t} \bigoplus_{i=1}^{2m+1} A(-e_i) \xrightarrow{M} \bigoplus_{i=1}^{2m+1} A(-d_i) \xrightarrow{P} A \to R \to 0 \tag{4.3}$$

where M is a $(2m+1)\times(2m+1)$ skew-symmetric matrix and P is a $(2m+1)\times 1$ vector such that the degrees of the entries are given by the degrees d_i of the relations, and the degrees of the syzygies are given by the e_i. It can be easily observed that $s - d_i = e_i$.

The following Lemma 4.3 is important to find out the degrees of the generators and syzygies of the possible candidates if there are any cancellations in $f(t)$.

Lemma 4.3 *$d_j = e_i$ can hold for at most one i and j. In this situation, $j = i$. Moreover, $d_j = e_j = s/2$.*

Proof. Suppose that $d_j = e_i$ holds for $j \neq i$. Since $M = (m_{ij})$ is skew-symmetric and the degree of the entries are given by $\deg m_{ij} = e_i - d_j$, $m_{ij} = -m_{ji}$ is a unit. Therefore $d_i = e_j$. Without loss of generality we can assume that $d_1 \leq \cdots \leq d_{2m+1}$ and $e_1 \leq \cdots \leq e_{2m+1}$. It follows that

$$d_i = e_j = d_j = e_i.$$

This reduces M to a lower dimensional matrix. Therefore $i = j$. Since $s = d_j + e_j$, $d_j = e_j = s/2$.

The Hilbert series

By taking the Hilbert function over the resolution, we can easily deduce that

$$P(R,t) = \frac{f(t)}{\prod_{i=0}^{n}(1 - t^{a_i})}$$

where

$$f(t) = 1 - \sum_{i=1}^{2m+1} t^{d_i} + \sum_{i=1}^{2m+1} t^{e_i} - t^s.$$

For $m = 2$, by Lemma 4.3 either four relations and four syzygies or five relations and five syzygies can appear in $f(t)$. If the first case happens, that is $d_i = e_i$ for a unique i, we can find the degree of the fifth equation by simply dividing s by 2. There are cases where this occurs, such as $\#4, \#6, \#7, \ldots$ in Table 5.1.

The canonical sheaf

We know that $\omega_{\mathbb{P}} = \mathcal{O}_{\mathbb{P}}(-\sum a_i)$ and $\omega_V = \mathcal{E}xt^{N-n}(\mathcal{O}_V, \omega_{\mathbb{P}})$ where $\mathbb{P} = \mathbb{P}(a_0, \ldots, a_N)$ and $\dim V = n$. Take $\mathcal{H}om(\ \ , \omega_{\mathbb{P}})$ of the resolution (4.3). Deduce that $\omega_V = \mathcal{O}_V(s - \sum a_i)$. In case of a K3 surface X, the following are equivalent:

(i) $\omega_X \cong \mathcal{O}_X$.

(ii) $s = \sum a_i$.

Remark 4.4

(i) Consider the resolution of R for $m = 2$

$$0 \to A(-s) \xrightarrow{P^t} \bigoplus_{i=1}^{5} A(-e_i) \xrightarrow{M} \bigoplus_{i=1}^{5} A(-d_i) \xrightarrow{P} A \to R \to 0$$

where M is a 5×5 skew-symmetric matrix and P is a vector such that the degrees of the entries are given by the degrees d_i of the relations. The rank of M can be $0, 2, 4$. If $\operatorname{rk} M|_{p_i} = 0, 4$, then either $\operatorname{Cone}(X)$ is singular at p_i or $p_i \notin \operatorname{Cone}(X)$. Therefore we consider only $\operatorname{rk} M|_{p_i} = 2$.

(ii) $\operatorname{Cone}(X)$ is nonsingular at p_i if and only if either $p_i \notin \operatorname{Cone}(X)$ or both $p_i \in \operatorname{Cone}(X)$ and there are three linearly independent Pfaffians locally at p_i.

(iii) The singularities of X come from the $\mathbb{C}^*$-action if X is quasismooth.

Remark 4.5 Let $M = (a_{ij})$ be a 5×5 skew-symmetric matrix. Then the submaximal Pfaffians, or the 4th Pfaffians, Pf_i of M are given by taking the square root of the determinants of the submatrices obtained by deleting the ith row and the ith column of M. These are given by the formulas:

$$\mathrm{Pf}_1 = a_{23}a_{45} - a_{24}a_{35} + a_{25}a_{34}, \quad \mathrm{Pf}_2 = a_{13}a_{45} - a_{14}a_{35} + a_{15}a_{34},$$
$$\mathrm{Pf}_3 = a_{12}a_{45} - a_{14}a_{25} + a_{15}a_{24}, \quad \mathrm{Pf}_4 = a_{12}a_{35} - a_{13}a_{25} + a_{15}a_{23},$$
$$\mathrm{Pf}_5 = a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23}.$$

Example 4.6 Let the formal initial data be

$$D^2 = -4 + \frac{4}{5} + \frac{5}{6} + 2 \times \frac{2 \cdot 3}{5}.$$

Then by Theorem 3.1

$$\begin{aligned} P(R,t) = {} & \frac{1+t}{1-t} + \frac{1}{2 \cdot 30} \frac{t(1+t)}{(1-t)^3} - \frac{1}{2 \cdot 5} \frac{4t + 6t^2 + 6t^3 + 4t^4}{1-t^5} \\ & - \frac{1}{2 \cdot 6} \frac{5t + 8t^2 + 9t^3 + 8t^4 + 5t^5}{1-t^6} - 2 \times \frac{1}{2 \cdot 5} \frac{6t + 4t^2 + 4t^3 + 6t^4}{1-t^5}. \end{aligned}$$

We are looking for a good choice for (s_1, s_2, s_3) such that

$$(1-t^{s_1})(1-t^{s_2})(1-t^{s_3}) \sum P_n t^n$$

is a finite polynomial with positive coefficients. Indeed, some of the s_i must be divisible by 6 and 5 to kill the denominators of $P(R,t)$. We also expect two generators whose weights are divisible by 5 since the singularities of type 2, $2 \times \frac{1}{5}(3,-3)$, occur at least on a one dimensional stratum. So (s_1, s_2, s_3) can be $(5,5,6)$. After multiplying $P(R,t)$ by $(1-t^5)(1-t^5)(1-t^6)$ we obtain

$$(1-t^5)(1-t^5)(1-t^6)P(R,t) = 1 + t^7 + t^8 + t^9 + t^{16}. \tag{4.4}$$

This looks like the Hilbert series of an Artinian ring with further generators of degrees $7, 8, 9$ so a plausible candidate is $X \subset \mathbb{P}(5,5,6,7,8,9)$. Let x_1, x_2, y, z, t, w be homogeneous coordinates on $\mathbb{P}(5,5,6,7,8,9)$ with weights $5,5,6,7,8,9$ respectively. To see the structure of its resolution of R, we multiply $(1-t^7)(1-t^8)(1-t^9)$ by (4.4) giving

$$\begin{aligned} (1-t^5)(1-t^5)(1-t^6)(1-t^7)(1-t^8)(1-t^9)P(R,t) = 1 - t^{14} - t^{15} \\ - t^{16} - t^{17} - t^{18} + t^{22} + t^{23} + t^{24} + t^{25} + t^{26} - t^{40}. \end{aligned}$$

This means that the shape of the resolution of R over $A = k[x_1, x_2, y, z, t, w]$ takes the form

$$\begin{aligned} 0 \to A(-40) \xrightarrow{P^t} A(-22) \oplus A(-23) \oplus A(-24) \oplus A(-25) \oplus A(-26) \\ \xrightarrow{M} A(-18) \oplus A(-17) \oplus A(-16) \oplus A(-15) \oplus A(-14) \xrightarrow{P} A \to R \to 0. \end{aligned}$$

That is, it is expected that there are five relations in degrees $14, 15, 16, 17, 18$ and five syzygies in degrees $22, 23, 24, 25, 26$. It is clear that K_X is trivial since $s = \sum a_i = 40$.

The last thing is to check whether X is quasismooth or not. It is enough to examine whether $\mathrm{Cone}(X)$ is smooth or not on each stratum of $\mathbb{P}$. Now consider a 5×5 skew-symmetric matrix M which comes from the resolution of R so that the degrees of the entries of M are $e_i - d_j$. Now we fill in the entries of M with general elements of A as follows:

$$M = \begin{pmatrix} 0 & l_1(x_1,x_2) & y & z & t \\ & 0 & z & t & w \\ & & 0 & w & q(x_1,x_2) \\ & -\mathrm{sym} & & 0 & l_2(x_1,x_2)y \\ & & & & 0 \end{pmatrix}$$

where the monomials l_i, q are linear and quadratic respectively. After a change of coordinates we can assume that both l_1 and l_2 contain x_1 and x_2 and that $q = x_1x_2$. Now consider all the possible strata of $\mathbb{P}$.
q_3, q_4, q_5: Since some powers of z, t, w appear in Pf_i for some i $q_3, q_4, q_5 \notin X$.
q_2: The Pfaffians contain no monomial of the form y^n for some n so $q_2 \in X$. Consider the affine piece $y = 1$. Since $\mathrm{rk}\, M|_{q_2} = 2$ there are locally three Pfaffians in $\mathbb{A}^5/\langle\varepsilon\rangle$

$$\begin{aligned} \mathrm{Pf}_2 &= -x_1 + wt + zq + x_2, \\ \mathrm{Pf}_4 &= -w + ql_1 + tz, \\ \mathrm{Pf}_5 &= -t + z^2 + wl_1 \end{aligned}$$

which are linearly independent where ε is a primitive 6th root of unity and acts on the affine coordinates x_1, x_2, z, t, w of $\mathbb{A}^5$ via

$$\varepsilon\colon (x_1, x_2, z, t, w) \longmapsto (\varepsilon^5 x_1, \varepsilon^5 x_2, \varepsilon z, \varepsilon^2 t, \varepsilon^3 w)$$

By the inverse function theorem x_2, z are local coordinates around q_2, which gives rise to a singularity $\frac{1}{6}(5,7)$, that is A_5 of type 1.
q_0q_1: Since $y = z = t = w = 0$

$$M|_{q_0q_1} = \begin{pmatrix} 0 & l_1(x_1,x_2)|_{q_0q_1} & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 \\ & & 0 & 0 & q(x_1,x_2)|_{q_0q_1} \\ & -\mathrm{sym} & & 0 & 0 \\ & & & & 0 \end{pmatrix}$$

which gives us three intersection points on $X \cap q_0q_1$ which are the solutions of $l_1(x_1,x_2)q(x_1,x_2)|_{q_0q_1} = 0$. These are the points $q_0 = (1,0,0,0,0,0)$, $q_1 = (0,1,0,0,0,0)$ and $(a,b,0,0,0,0)$ where a and b are nonzero. On q_0 $\operatorname{rk} M|_{q_0} = 2$ and there are three linearly independent equations locally around q_0

$$\begin{aligned}
\mathrm{Pf}_3 &= -y + zw - t^2 + \cdots, \\
\mathrm{Pf}_4 &= -x_2 + yw - zt + \cdots, \\
\mathrm{Pf}_5 &= -w + yt - z^2 + x_2w.
\end{aligned}$$

This means that $\mathrm{Cone}(X)$ is smooth at q_0. By the inverse function theorem z, t are local coordinates around q_0. Under the $\mathbb{C}^*$-action it gives rise to a cyclic quotient singularity $\frac{1}{5}(7,8)$, that is A_4 of type 2. On q_1 $\operatorname{rk} M|_{q_1} = 2$. Again write all possible Pfaffians, observe that three of them are linearly independent locally at q_1

$$\begin{aligned}
\mathrm{Pf}_3 &= -y + zw - t^2 + \cdots, \\
\mathrm{Pf}_4 &= -x_1 + yw - tz + \cdots, \\
\mathrm{Pf}_5 &= -w + yt - z^2 + \cdots.
\end{aligned}$$

By the inverse function theorem z, t are local coordinates around q_1. This gives a singularity $\frac{1}{5}(7,8)$ which is A_4 of type 2. On $(a,b,0,0,0,0)$ we can again use the change of coordinates $x_1 = l_1(x_1,x_2)$ and $x_2 = x_2$. Then our point becomes $(0,1,0,0,0,0)$ and M

$$\begin{pmatrix}
0 & x_1 & y & z & t \\
 & 0 & z & t & w \\
 & & 0 & w & l_1(x_1,x_2)x_2 \\
 & -\mathrm{sym} & & 0 & l_2'(x_1,x_2)y \\
 & & & & 0
\end{pmatrix}$$

with $\operatorname{rk} M|_{(0,1,0,0,0,0)} = 2$ where $l_2{}'$ is a linear function in x_1, x_2. Notice that there are three linearly independent Pfaffians locally around q_1

$$\begin{aligned}
\mathrm{Pf}_1 &= -t + w^2 + zyx_2 + \cdots, \\
\mathrm{Pf}_2 &= -z + wt + x_1y^2 + \cdots, \\
\mathrm{Pf}_4 &= -x_1 + yw - zt + \cdots.
\end{aligned}$$

By the inverse function theorem, w and y are local coordinates at (0,1,0,0,0,0) which give a cyclic quotient singularity $\frac{1}{5}(9,6)$ of type 1. When we check the quasismoothness candition on the higher dimensional strata the calculations become unmanageable. For this reason we carry out the calculation in the next chapter by using Macaulay (see Example 5.9). This example is #61 in Table 5.1.

Example 4.7 Consider the case:

$$D^2 = 2 \times \frac{1}{2} + \frac{3}{4}.$$

It is easy to see that $P_1 = 2$ and $P_2 = 5$. This gives us at least four generators, say x_1, x_2, y_1, y_2, of R in degrees $1,1,2,2$. By Lemma 3.4, we must have at least one more generator whose weight is divisible by 4. Therefore a good choice of (s_1, s_2, s_3) can be $(1,1,4)$. An alternative way is to consider

$$P(R,t) = \frac{1+t}{1-t} + \frac{7}{2\cdot 4}\frac{t(1+t)}{(1-t)^3} - 2\times\frac{1}{2\cdot 2}\frac{t}{1-t^2} - \frac{1}{2\cdot 4}\frac{3t+4t^2+3t^3}{1-t^4}.$$

To kill its denominators, one of the s_i must be divisible by 4, say $s_3 = 4$. Also, it is already known that there are four generators of weights $1,1,2,2$ hence it is enough to choose the rest of the s_i from the set of weights $1,1,2,2$, for example, $s_1 = s_2 = 1$. After multiplying $P(R,t)$ by $(1-t)(1-t)(1-t^4)$ we obtain

$$P(R/\langle x_1, x_2, z\rangle, t) = 1 + 2t^2 + t^3 + 2t^4 + t^6. \tag{4.5}$$

This looks like the Hilbert series of an Artinian ring with further generators z, t in degrees $3, 4$ and therefore a plausible candidate is X in $\mathbb{P}(1,1,2,2,3,4)$ that we find below. To find out the structure of the ring R, multiply (4.5) by $(1-t^2)(1-t^2)(1-t^3)$ to obtain

$$\begin{aligned}(1-t)^2(1-t^2)^2(1-t^3)(1-t^4)P(R,t) &= 1 - t^4 - 2t^5 \\ &\quad - 2t^6 + 2t^7 + 2t^8 + t^9 - t^{13}. \quad (4.6)\end{aligned}$$

From here we can read off the shape of the resolution of $R(X,D)$ over $A = k[x_1, x_2, y_1, y_2, z, t]$, namely:

$$\begin{aligned}0 \to A(-13) &\xrightarrow{P^t} A(-7)\oplus A(-7)\oplus A(-8)\oplus A(-8)\oplus A(-9) \\ &\xrightarrow{M} A(-6)\oplus A(-6)\oplus A(-5)\oplus A(-5)\oplus A(-4) \xrightarrow{P} A \to R \to 0.\end{aligned}$$

That is, it is expected that there are five relations in degrees $4,5,5,6,6$ and five syzygies in degrees $7,7,8,8,9$. The canonical divisor is trivial since $s = \sum a_i = 13$.

We will show that X is quasismooth. In other words, X has only cyclic quotient singularities. It is sufficient to check that $\mathrm{Cone}(X)$ is smooth on each strata of $\mathbb{P}$. The equations of X come from 4th order Pfaffians of a skew symmetric M with degrees

$$\begin{pmatrix} 0 & 1 & 2 & 2 & 3 \\ & 0 & 2 & 2 & 3 \\ & & 0 & 3 & 4 \\ & -\mathrm{sym} & & 0 & 4 \\ & & & & 0 \end{pmatrix}.$$

After a change of coordinates we assume that

$$M = \begin{pmatrix} 0 & x_1 & y_2 & y_1 & * \\ 0 & 0 & y_1 + f_2(x_1, x_2) & * & * \\ & & 0 & z & f_4(x_1, x_2, y_1, y_2, z) \\ -\mathrm{sym} & & & 0 & t \\ & & & & 0 \end{pmatrix}$$

where f_2 is a function of x_1, x_2 in degree 2, f_4 is a function of x_1, x_2, y_1, y_2, z in degree 4 and * denotes any possible combinations in the variables x_1, x_2, y_1, y_2, z, t in the appropriate degree.
q_0, q_1, q_2, q_3, q_4: Some powers of x_1, x_2, y_1, y_2, z appear in Pf_i for some i, therefore $q_0, q_1, q_2, q_3, q_4 \notin X$.
q_5: This implies that $x_1 = x_2 = y_1 = y_2 = z = 0$. Then

$$M|_{q_5} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 \\ & & 0 & 0 & 0 \\ & -\mathrm{sym} & & 0 & t \\ & & & -t & 0 \end{pmatrix}$$

with $\operatorname{rk} M|_{q_5} = 2$ and so $q_5 \in X$. Consider the affine piece $t = 1$. Locally around q_5, we have three linearly independent Pfaffians in $\mathbb{A}^5/\langle\varepsilon\rangle$

$$\begin{aligned}
\mathrm{Pf}_1 &= -x_1 + f_5\\
\mathrm{Pf}_2 &= -y_1 + h_6,\\
\mathrm{Pf}_3 &= -y_2 + g_6
\end{aligned}$$

where ε is a primitive 4th root of unity, acting on the affine coordinates x_1, x_2, y_1, y_2, z of $\mathbb{A}^5$ via

$$\varepsilon\colon (x_1, x_2, y_1, y_2, z) \longmapsto (\varepsilon x_1, \varepsilon x_2, \varepsilon^2 y_1, \varepsilon^2 y_2, \varepsilon^3 z),$$

and f_5, h_6, g_6 are the function of x_1, x_2, y_1, y_2, z in degrees $5, 6, 6$ respectively. In other words $\mathrm{Cone}(X)$ is nonsingular along the t-axis. By the inverse function theorem x_2, z are local coordinates and under the $\mathbb{C}^*$-action it gives a cyclic quotient singularity $\frac{1}{4}(1,3)$ which is A_3.
$q_2q_3q_5$: This means that $x_1 = x_2 = z = 0$. Then

$$M|_{q_2q_3q_5} = \begin{pmatrix} 0 & 0 & y_2 & y_1 & 0\\ 0 & 0 & y_1 & y_2 + y_1 & 0\\ -y_2 & -y_1 & 0 & 0 & y_1^2 + y_1y_2 + y_2^2\\ -y_1 & -y_2 - y_1 & 0 & 0 & t\\ 0 & 0 & -(y_1^2 + y_2^2 + y_1y_2) & -t & 0 \end{pmatrix}$$

and

$$\operatorname{rk} M|q_2q_3q_5 \leq 2 \Longleftrightarrow \operatorname{rk} \begin{pmatrix} -y_2 & -y_1 & 0 & 0 & y_1^2 + y_2^2 + y_1y_2\\ -y_1 & -y_2 - y_1 & 0 & 0 & t \end{pmatrix} \leq 1. \quad (4.7)$$

The right hand side of (4.7) gives that there are three Pfaffians on $q_2q_3q_5$

$$\begin{aligned}
\mathrm{Pf}_1 &= -y_1t + (y_1 + y_2)(y_1^2 + y_2^2 + y_1y_2),\\
\mathrm{Pf}_2 &= -y_2t + y_1(y_1^2 + y_2^2 + y_1y_2),\\
\mathrm{Pf}_5 &= y_2(y_1 + y_2) - y_1^2.
\end{aligned}$$

It follows that $X \cap q_2q_3q_5$ consists of 2 points and that the rank of M at each of these points is two. Since y_1, y_2 and t are non zero at these points, it can be observed that locally

$$\begin{aligned}
\mathrm{Pf}_2 &= -y_2t + y_1(y_1^2 + y_2^2 + y_1y_2) + \cdots,\\
\mathrm{Pf}_3 &= -x_1t + g_4(x_1, x_2, y_1, y_2, z),\\
\mathrm{Pf}_5 &= -x_1z + y_2(y_1 + y_2) - y_1^2 + \cdots,
\end{aligned}$$

are linearly independent. Since these points are in the affine piece $(y_1 = 1)$, by the inverse function theorem, x_1, z are local coordinates and hence these give rise to $2A_1$. It is difficult to check the quasismoothness condition on all of the other higher dimensional strata. This difficulty can be overcome by using unprojection, which will be defined in the next chapter. This example is #13 in Table 5.1.

4.3 Calculation of plausible K3 surfaces of codimension ≤ 3 for $2 \leq P_1 \leq 6$

Under the assumption of Lemma 3.7, we have

$$P_2 = 4P_1 - 6 + \sum_{(r_i,b_i)} c_i b_i$$

where b_i corresponds to the type of the singularity r_i and c_i is the number of times singularity (r_i, b_i) appears.

Recall that $X(d_1, \ldots, d_l) \subset \mathbb{P}(a_0, \ldots, a_n)$ denotes a sufficiently general element in the family of all weighted varieties given by the zero locus of $f_1, \ldots, f_l$ where $\deg f_i = d_i$.

Let $P_1 = 6$. Then there exists a complete intersection K3 surface $X(2,2,2)$ in $\mathbb{P}(1,1,1,1,1,1)$ by Bertini's theorem.

Suppose that $P_1 = 5$. So there is at most one singularity. If there are no singularities then there is a c. i. K3 surface $X(2,3)$ of codim 2 in $\mathbb{P}(1,1,1,1,1)$. If there is a singularity then by Lemma 3.4 it is of the form $\frac{1}{2}(1,-1)$. Clearly $X(2,3,3,3,3)$ in $\mathbb{P}(1,1,1,1,1,2)$ with A_1 is a candidate for a K3 surface (see the Application in the appendix of this chapter).

Suppose that $P_1 = 4$. So $10 \leq P_2 \leq 12$, that is $0 \leq \sum c_i b_i \leq 2$. Hence there are at most two singularities. Suppose that there are no singularities, that is $P_2 = 10$, which gives us a K3 surface $X(4)$ of codim 1 in $\mathbb{P}(1,1,1,1)$. Suppose that there is only one singularity, that is $P_2 = 11$. Then we have:

$r = 2 \implies$ there is a c. i. K3 surface $X(3,3)$ in $\mathbb{P}(1,1,1,1,2)$ with A_1,

$r = 3 \implies$ $X(3,3,4,4,4)$ in $\mathbb{P}(1,1,1,1,2,3)$ with A_2 is a candidate for a K3 surface.

When $r \geq 4$ we have six generators coming from the case $r = 3$. In addition we have at least one more generator by either Proposition 3.6 or Lemma 3.4. It follows that there are no K3 surfaces of codim 3 for $r \geq 4$. Suppose that there are two singularities. Then $P_2 = 12$. It follows that $r_1 = r_2 = 2$. Hence $X(3,3,3,3,4)$ in $\mathbb{P}(1,1,1,1,2,2)$ with $2A_1$ is a candidate for a K3 surface.

Let $P_1 = 3$. So,

$$6 \leq P_2 = 6 + \sum_i c_i b_i \leq 9 \quad \Longleftrightarrow \quad 0 \leq \sum_i c_i b_i \leq 3$$

which implies that there are at most two different kinds of singularities that can appear in the following forms:

$$c \times \frac{1}{r}(1,-1) \quad \text{or} \quad \left\{\frac{1}{r_1}(1,-1),\ \frac{1}{r_2}(1,-1)\right\}$$

where $c \leq 3$ and $r_1 \neq r_2$. If $P_2 = 9$, then $X(3,3,4,4,4)$ in $\mathbb{P}(1,1,1,2,2,2)$ with $3A_1$ is a plausible candidate for a K3 surface. Let $P_2 = 8$ and consider, first, only the first possibility $2 \times \frac{1}{r}(1,-1)$. This gives us

$r = 2 \quad \Longrightarrow \quad$ there is a K3 surface $X(3,4)$ in $\mathbb{P}(1,1,1,2,2)$ with $2A_1$.

When $r \geq 3$, by Lemma 3.4 there are at least two more generators whose weights are divisible by r. Hence the minimum number of generators is greater than six and so the corresponding case will not be considered here. Next, assume the second possibility, that is, there are two singularities of different indices. By Lemma 3.4 $r_1 = 2$ and $r_2 = 3$ which implies that $X(3,4,4,4,5)$ in $\mathbb{P}(1,1,1,2,2,3)$ with A_1, A_2 is a candidate for a K3 surface. Let $P_2 = 7$. Then we have only one singularity which is of the form $\frac{1}{r}(1,-1)$. Now we proceed with the algorithm:

$r = 2 \quad \Longrightarrow \quad$ there is a K3 surface $X(5)$ in $\mathbb{P}(1,1,1,2)$ with A_1,
$r = 3 \quad \Longrightarrow \quad$ there is a K3 surface $X(4,4)$ in $\mathbb{P}(1,1,1,2,3)$ with A_2,
$r = 4 \quad \Longrightarrow \quad$ $X(4,4,5,5,6)$ in $\mathbb{P}(1,1,1,2,3,4)$ with A_3 is a candidate for a K3 surface.

When $r \geq 5$ we have six generators coming from the case $r = 4$ plus at least one new generator by Proposition 3.6. Therefore there are no K3 surfaces of codim 3 for $r \geq 5$. We can obtain the same result by using Lemma 3.4. Let $P_2 = 6$, then there are no singularities. However there is a nonsingular K3 surface $X(6)$ of codim 1 in $\mathbb{P}(1,1,1,3)$.

Assume that $P_1 = 2$. It is easily seen that $3 \leq P_2 \leq 7$. Let $P_2 = 7$, then $X(4,4,4,4,4)$ in $\mathbb{P}(1,1,2,2,2,2)$ with $5A_1$ is a candidate for a K3 surface. Let $P_2 = 6$, then the generators of R with weights $\{1,1,2,2,2\}$ imply that at most two kinds of singularities can occur. If $cb = 4$, then by Lemma 3.4 $b = 1$. Now we start the algorithm:

$r = 2 \quad \Longrightarrow \quad$ there is a K3 surface $X(4,4)$ in $\mathbb{P}(1,1,2,2,2)$ with $4A_1$

and when $r \geq 3$ by Lemma 3.4 there are at least two more generators divisible by r so that $\mathrm{codim}(X) \geq 4$ which is not considered. If $c_1b_1 + c_2b_2 = 4$, then there are two kinds of singularities of different indices r_1, r_2. These should be $3A_1$ and A_2 by Lemma 3.4 and hence $X(4,4,4,5,5)$ in $\mathbb{P}(1,1,2,2,2,3)$ is a candidate for a K3 surface with these singularities. Let $P_2 = 5$. If $cb = 3$, then the following possibilities can occur:

(a) $r \geq 2$ and $b = 1$,

(b) $r \geq 7$ and $b = 3$.

Suppose that (a) holds. Then we have

$r = 2 \implies$ there is a K3 surface $X(6)$ in $\mathbb{P}(1,1,2,2)$ with $3A_1$

and when $r \geq 3$ we have $P_3 = 11$, not a case of codim 3 as before. Assume that (b) holds. Then $P_3 \geq 9$ and so by Lemma 3.4 r should be 5 which contradicts $r \geq 7$. If $c_1b_1 + c_2b_2 = 3$, then we have the following cases:

(i) $c_1 = 2$ and $b_1, c_2, b_2 = 1$,

(ii) $c_2 = 2$ and $c_1, b_1, b_2 = 1$,

(iii) $b_1 = 2$ and $c_1, c_2, b_2 = 1$,

(iv) $b_2 = 2$ and $c_1, b_1, c_2 = 1$.

Suppose that (i) holds. If $r_1 \neq 2$, then $P_3 = 11$ which gives more than seven generators. So consider $r_1 = 2$ and increase r_2 inductively, then

$r_2 = 3 \implies$ there is a K3 surface $X(4,5)$ in $\mathbb{P}(1,1,2,2,3)$ with $2A_1, A_2$,
$r_2 = 4 \implies$ $X(4,5,5,6,6)$ in $\mathbb{P}(1,1,2,2,3,4)$ with $2A_1, A_3$ is a candidate for a K3 surface

and when $r_2 \geq 5$ there are no K3 surfaces X of codim 3 by arguing as before. Suppose that (ii) holds. This means that a singularity of index r_2 occurs on a one dimensional stratum. Therefore there are at least two new generators whose weights are divisible by r_2. By Lemma 3.4 $r_2 = 3$ which implies $r_1 = 2$ and so $X(4,4,5,5,6)$ in $\mathbb{P}(1,1,2,2,3,3)$ with $A_1, 2A_2$ is a candidate for a K3 surface. Assume that (iii) holds. So $r_1 \geq 5$. By Lemma 3.4 we need one more generator whose weight is congruent to 3, but $P_3 = 10$. This implies that there are two more generators of degree 3. Hence $\mathrm{codim}\, X \geq 5$. Suppose that (iv) holds. Then $r_2 \geq 5$ and so $P_3 \geq 9$, but the only possibilities of r_1 and r_2 are 2 and 5 respectively, otherwise $P_3 \geq 10$ when $r_2 \geq 7$.

Therefore $X(4,5,6,6,7)$ in $\mathbb{P}(1,1,2,2,3,5)$ with A_1, A_4 is a candidate for a K3 surface. If $c_1b_1 + c_2b_2 + c_3b_3 = 3$, then $P_3 \geq 10$ which gives at least two more generators of weight 3 but we need one more generator whose weight is r_3. Therefore $\operatorname{codim}(X) \geq 4$. Let $P_2 = 4$. This implies that at most two kinds of singularities can occur. Suppose that $cb = 2$. Then

(a) $c = 2$ and we have:

$$\begin{aligned} r = 2 &\implies \text{there is a K3 surface } X(8) \text{ in } \mathbb{P}(1,1,2,4) \text{ with } 2A_1, \\ r = 3 &\implies \text{there is a K3 surface } X(4,6) \text{ in } \mathbb{P}(1,1,2,3,3) \text{ with } 2A_2. \end{aligned}$$

When $r \geq 4$ there are no K3 surfaces X of codim 3 by arguing as before, using either Proposition 3.6 or Lemma 3.4.

(b) $b = 2$ which implies $r \geq 5$, therefore

$$r = 5 \implies \text{there is a K3 surface } X(6,6) \text{ in } \mathbb{P}(1,1,2,3,5) \text{ with } A_4.$$

When $r \geq 7$, $P_3 = 8$. This means that we have at least five generators of weights ≤ 3. By Lemma 3.4 we need two more new generators. Hence there are no K3 surfaces X of codim 3 for $r \geq 7$.

If $c_1b_1 + c_2b_2 = 2$, then $r_1 \neq r_2$. We are going to consider the following two cases:

(i) $r_1 \neq 2$,

(ii) $r_1 = 2$.

Suppose that (i) holds. Then $P_3 = 8$ which implies that there are two new generators of degree 3 and hence by Lemma 3.4 $r_1 = 3$ and $r_2 = 4$. This gives a candidate $X(4,5,6,6,7)$ in $\mathbb{P}(1,1,2,3,3,4)$ with A_2, A_3 for a K3 surface. If either r_1 or r_2 is ≥ 4, then $\operatorname{codim}(X) \geq 4$ by arguing as before. Now suppose that (ii) holds. Then

$$\begin{aligned} r_2 = 3 &\implies \text{there is a K3 surface } X(7) \text{ in } \mathbb{P}(1,1,2,3) \text{ with } A_1, A_2, \\ r_2 = 4 &\implies \text{there is a K3 surface } X(5,6) \text{ in } \mathbb{P}(1,1,2,3,4) \text{ with } A_1, A_3, \\ r_2 = 5 &\implies X(6,6,7,7,8) \text{ in } \mathbb{P}(1,1,2,3,4,5) \text{ with } A_1, A_4 \text{ is a candidate} \\ &\qquad \text{for a K3 surface.} \end{aligned}$$

If $r_2 \geq 6$, then there are no K3 surfaces X of codim 3 by arguing as before. Let $P_2 = 3$. This implies that there is only one singularity of type 1 such that

$$\begin{array}{lll}
r=2 & \Longrightarrow & \text{there is a K3 surface } X(12) \text{ in } \mathbb{P}(1,1,4,6) \text{ with } A_1, \\
r=3 & \Longrightarrow & \text{there is a K3 surface } X(10) \text{ in } \mathbb{P}(1,1,3,5) \text{ with } A_2, \\
r=4 & \Longrightarrow & \text{there is a K3 surface } X(9) \text{ in } \mathbb{P}(1,1,3,4) \text{ with } A_3, \\
r=5 & \Longrightarrow & \text{there is a K3 surface } X(6,8) \text{ in } \mathbb{P}(1,1,3,4,5) \text{ with } A_4, \\
r=6 & \Longrightarrow & X(6,7,8,9,10) \text{ in } \mathbb{P}(1,1,3,4,5,6) \text{ with } A_5 \text{ is a candidate} \\
 & & \text{for a K3 surface.}
\end{array}$$

If $r \geq 7$, then there are no K3 surfaces X of codim 3.

So far when $2 \leq P_1 \leq 6$, we have fourteen plausible K3 surfaces of codim 3 with the given singularities plus a c. i. nonsingular K3 surface of codim 3.

Appendix

Maple program

The following program, written in the Maple V Release 4 language, is essentially a prearranged group of commands. The reference [17] for Maple V explains how to use Maple V as a programming language.

```
#Final() should be considered as
#Final(k,D^2,r_1,..,r_k,b_1,..,b_k,a_0,...,a_l)
#=\prod_{i=0}^{l}(1-t^{a_i})P(t) where k is the number of singularities,
#the r_j are the indices of the singularities, the b_j are the types of
#the singularities, the a_i are the weights of either the ring generators
#or the module generators and P(t)=\sum P_nt^n is the Hilbert series
#of P_n for the given initial data, considered as a rational function
#using Theorem 3.1.
Final := proc()
      local sum1, sum2, sum3, sum4, j;
      sum1 := Mult(args[j] $j=(3+2*args[1])..nargs);
      sum2 := simplify(expand(Cpoly(args[j] $j=2..(2+2*args[1]))*sum1));
      sum3 := simplify(simplify(expand(sum1*(1+t))/(1-t))
              +simplify(expand(1/2*args[2]*sum1*t*(1+t))/((1-t)^3)));
      sum4 := sort(simplify(expand(sum2+sum3)));
      sum1*P(t) = sum4;
end:
#Mult(): A procedure for calculating the product of (1-t^{a_i}) over i.
Mult := proc()
```

```
     local s, i;
     s := 1;
     for i from 1 by 1 to nargs do
        if args[i]>0 then
        s := (1-t^args[i])*s;
        fi;
     od;
     s;
end:
#Cpoly(): A procedure for rewriting the power series of the sum of the
#contributions of the singularities as a finite sum of some polynomials.
Cpoly := proc()
      local k, l, n, h, j, sum1, sum2, sum3;
      k := (nargs-1)/2;
      for h from 1 by 1 to k do
          sum1[h] := 0 ;
      od;
      sum2 := 0;
      h := 1;
      for l from 2 by 1 to k+1 do
          if args[l]-1>0 then
             for n from 1 by 1 to args[l]-1 do
                 sum1[h] := sum1[h]+Contr(args[l],n*args[k+h+1])*t^n;
             od;
          fi;
          sum2 := sum2 - simplify(1/(2*args[l])*sum1[h]/(1-t^args[l]));
          h := h+1;
      od;
      sum2;
end:
#The following procedure computes twice the contribution C_n(Q) of
#a singularity Q of index r and type b for each n.
Contr := proc(r,nb)
      local result, j;
      j := nb mod r;
      result := j*(r-j);
      result;
end:
#Save: Record the state of all variables and procedures
#in filename.m.
save filename.m
```

Application

We start Maple sessions with the command "read filename.m". This is followed by running the procedure Final.

```
> maple -x
> read filename.m
```

The following calculation shows how the structure of the resolution of a candidate for a codimension 3 K3 surface X with the singularity A_1 of type 1 and $D^2 = 13/2$ in $\mathbb{P}(1,1,1,1,1,2)$ looks like via the procedure Final:

```
> Final(1,13/2,2,1,1,1,2);
           2         2             4     3     2
  (1 - t )  (1 - t ) P(t) = t + 3t + 5t + 3t + 1
> Final(1,13/2,2,1,1,1,2,1,1,1);
          5      2              7     5      4      3   2
(1 - t ) (1 - t ) P(t) = - t  + t  + 4t  - 4t - t + 1
```

This implies that a candidate for a K3 surface with given data is defined by the common zero locus of five equations of degrees $2,3,3,3,3$. It has also five syzygies of degrees $4,4,4,4,5$.

The following example computes the structure of the resolution of a codimension 3 possible K3 surface with singularities $3A_1, A_6, A_6$ of types $1,1,1,1,3$ and $D^2 = 1/14$ in $\mathbb{P}(2,5,6,7,7,8)$:

```
> Final(5,1/14,2,2,2,7,7,1,1,1,1,3,2,7,7);
           7 2         2            16     11     10     8     6     5
    (1 - t )   (1 - t ) P(t) = t   + t   + t   + t  + t  + t  + 1
> Final(5,1/14,2,2,2,7,7,1,1,1,1,3,2,7,7,5,6,8);
       8         6         5         7 2        2              35    23
(1 - t ) (1 - t ) (1 - t ) (1 - t )  (1 - t ) P(t) = -t   + t
          22     21     20     19     16     15     14     13     12
     + t   + t   + t   + t   - t   - t   - t   - t   - t   + 1
```

In other words, there are five relations in degrees $12,13,14,15,16$ and five syzygies in degrees $19,20,21,22,23$.

Chapter 5

Unprojection-Projection

In this chapter, we construct examples of K3 surfaces of codimensions 3 and 4 from smaller codimension K3 surfaces whose rings are much simpler. This can be done in two ways: one way is to use birational geometry and the other way is to use commutative algebra. In all cases except Example 5.14 in codim 4 we explain how to use birational geometry to get quasismooth K3 surfaces. In Example 5.14 we construct the surface (X, D) through an $(X, 2D)$-model.

5.1 Introduction

Unprojection is an inverse operation of projection. What we mean here by projection is the following: we have a K3 surface X with A_n type singularities such that the resolution of these singularities forms a configuration of curves on a nonsingular K3 surface S (as explained in Chapter 2) then contracting some suitable subconfiguration on S with some assumptions gives another K3 surface Y of smaller codimension with A_n type singularities, containing some curves C_i. The inverse operation is to construct a K3 surface Y with A_n singularities, containing the curves C_i and use this to get a K3 surface X of higher codimension. Going forward and backward between Y and X is a birational operation, called *unprojection-projection*.

Now we demonstrate the simplest example of unprojection-projection between K3 surfaces of codimensions 2 and 1. In the first part of the example we consider projection, in the second part we consider unprojection.

Example 5.1 (A) We start by assuming that a K3 surface $X = X(3,3) \subset \mathbb{P}(1,1,1,1,2)$ with A_1 and a $\mathbb{Q}$-ample Weil divisor $D = \mathcal{O}_X(1)$, where $D^2 = 9/2$, exists then we want to construct a nonsingular K3 surface $Y = X(4) \subset \mathbb{P}^3$ with an ample divisor D', containing the line $C\colon x_1 = x_2 = 0$. This can be done in two ways: the first way is to blow up the singularity along D and contract down a suitable subconfiguration as explained in the above preamble. The second way is the following: pick the point $(0,0,0,0,1)$, which is an A_1 singularity, and do projection away from this point to get $Y = X(4)$ as desired.

We now give details of the two constructions. Let $x_1, \dots, x_4, y$ be homogeneous coordinates on $\mathbb{P}(1,1,1,1,2)$ with weights $1,1,1,1,2$.

(1) Using the same notation as in Chapter 2 the resolution of the singular point along D is $\sigma\colon (S, \sigma^*(D)) \to (X, D)$ such that $\sigma^*(D) = B + \frac{1}{2}C_1$ where $B^2 = 4$ and $BC_1 = 1$ (see Proposition 2.4). Now choose the subconfiguration B and consider the corresponding map $\phi_{|B|}$ from S to $\mathbb{P}^3$. If the linear system $|B|$ is not hyperelliptic then the image $Y = \phi_{|B|}(X) \subset \mathbb{P}^3$ is a K3 surface of degree 4 (see Saint-Donat [23]) with the divisor $D' = (\phi_{|B|})_*(B)$, and contains a line $C = \phi_{|B|}(C_1)$ since $BC_1 = 1$. In this case it is easy to choose coordinates such that the equations of C are chosen among the coordinates. However we use a construction which is applicable in more general situations.

Consider the following exact sequence:

$$0 \to \mathcal{O}_Y(-C) \to \mathcal{O}_Y \to \mathcal{O}_C \to 0. \tag{5.1}$$

Twisting by D' (in general we may need to twist (5.1) by nD' where $n > 1$) and then taking cohomology, we have

$$0 \to H^0(Y, \mathcal{O}_Y(D' - C)) \to H^0(Y, \mathcal{O}_Y(D')) \to H^0(C, \mathcal{O}_C(D')) \to \dots .$$

Since $H^0(Y, \mathcal{O}_Y(D')) = H^0(X, \mathcal{O}_X(B))$ the middle vector space has dimension 4, and the right hand vector space has dimension 2 by Riemann–Roch on C_1. So we conclude that

$$\dim H^0(Y, \mathcal{O}_Y(D' - C)) \geq 2.$$

Without loss of generality we can choose elements x_1, x_2 of that space whose zero set will be a line containing C in $\mathbb{P}^3$. Since C is a line it is given by

$$C\colon x_1 = x_2 = 0.$$

(2) Since X has a singularity $\frac{1}{2}(1,-1)$ locally at $y = 1$ it has two local coordinates, say x_3, x_4, so that the defining equations of X can be written as follows

$$x_1 y = -a_2 \quad \text{and} \quad x_2 y = a_1,$$

or equivalently,

$$y = -\frac{a_2}{x_1} = \frac{a_1}{x_2}$$

where a_1 and a_2 are general homogeneous polynomials of degree 3 in $k[x_1, \dots, x_4]$. That means that y is a rational function on Y which has a pole along C. Projecting X away from the point $(0, 0, 0, 0, 1)$ gives

$$Y \colon a_1x_1 + a_2x_2 = 0 \subset \mathbb{P}^3.$$

(B) We begin by constructing a nonsingular K3 surface $Y = X(4)$ of codim 1 in $\mathbb{P}^3$, containing the line $C \colon x_1 = x_2 = 0$ and then obtain a quasismooth K3 surface $X = X(3, 3)$ of codim 2 with A_1 in $\mathbb{P}(1, 1, 1, 1, 2)$ from Y by using birational geometry.

Let $x_1, \dots, x_4$ be homogeneous coordinates on $\mathbb{P}^3$. Define the linear system $\mathcal{L}$ of all homogeneous polynomials of degree 4 on $\mathbb{P}^3$ containing the line C and take a sufficiently general element $f = a_1x_1 + a_2x_2 \in \mathcal{L}$ where a_1 and a_2 are sufficiently general homogeneous polynomials of degree 3 in $k[x_1, x_2, x_3, x_4]$. Denote by Y the variety defined by the equation $f = 0$ in $\mathbb{P}^3$. The base locus of the linear system $\mathcal{L}$ is the line C. By Bertini's theorem (see [11], Remark III.10.9.2) Y is nonsingular away from C. Therefore it is sufficient to check that Y is nonsingular along C. The singular locus of Y along C is $(a_1(0, 0, x_3, x_4) = a_2(0, 0, x_3, x_4) = 0) \subset \mathbb{P}(x_3, x_4)$. But a_1 and a_2 are general polynomials, therefore the singular locus is empty. Hence Y is a nonsingular K3 surface in $\mathbb{P}^3$, containing the line C. Now set a new generator of degree 2

$$y = \frac{a_1}{x_2} = -\frac{a_2}{x_1}$$

and define a rational map $\varphi \colon Y \subset \mathbb{P}^3 \dashrightarrow X \subset \mathbb{P}(1, 1, 1, 1, 2)$, where X is the complete intersection $x_1y = -a_2$ and $x_2y = a_1$, by

$$(x_1, x_2, x_3, x_4) \longmapsto (x_1, x_2, x_3, x_4, y).$$

It can easily be observed that φ is a birational map, and in fact an isomorphism $Y \setminus C \xrightarrow{\sim} X \setminus q_4$ where $q_4 = (0, 0, 0, 0, 1)$. Indeed, the inverse map is projection from q_4. Thus it suffices to see that $\mathrm{Cone}(X)$ is nonsingular at q_4. The rank of the Jacobian matrix of $\mathrm{Cone}(X)$ is two at this point. Therefore $\mathrm{Cone}(X)$ is nonsingular. The singularities of X arise due to the singularities of $\mathbb{P}$ and occur only at the vertex q_4 which is a singularity of type A_1 in X.

5.2 Existence of K3 surfaces in codimension 3

We state the main theorem of this section.

Theorem 5.2 *The list of 70 candidates is complete and they all exist.*

Completeness of this list comes from an exhaustive computer search. For existence we work on model examples from the list. The strategy is as follows:

1. Reduce the codimension of candidates by projecting and then lift back by using birational geometry.

2. From first principles prove nonsingularity.

The seventy candidates will be dealt with as follows (see Table 5.1). The first example in the list is a complete intersection $X(2,2,2) \subset \mathbb{P}^5$ and can be done by Bertini's theorem as explained in Example 5.3. Sixty four candidates can be done by the first type of unprojection and two candidates by the second type of unprojection (see below for definitions). We explain how to construct explicitly quasismooth K3 surfaces of codim 3 from either K3 surfaces of codim 1 or codim 2 by using birational geometry or unprojection-projection. In the remaining three candidates unprojection-projection is either too complicated or can not be used. We overcome these difficulties by directly proving nonsingularity of a special example using Macaulay.

Example 5.3 Any k general hypersurfaces of $\mathbb{P}^n$ define an $(n-k)$-dimensional complete intersection by Bertini's theorem. For $n = 5$, to define a surface in $\mathbb{P}^5$, we must choose $k = 3$, and therefore the only c. i. K3 surface in $\mathbb{P}^5$ is $X(2,2,2)$.

Type I unprojection

Let Y be a projectively Gorenstein variety of codimension c embedded in weighted projective space $\mathbb{P}(a_0, \dots, a_n)$, containing a projectively Gorenstein variety C of codimension $c+1$. Let $I(C)$ be the ideal of C generated by homogeneous polynomials h_i for $i = 1, \dots, l$, and $I(Y)$ the ideal of Y generated by f_i for $i = 1, \dots, k$. Then there exist a rational function v on Y which has a pole along C, that is,

$$v = \frac{m_1}{h_1} = \cdots = \frac{m_l}{h_l}$$

for some homogeneous polynomials m_i such that X defined by

$$f_1 = \cdots = f_k = vh_1 - m_1 = \cdots = vh_l - m_l = 0$$

is a projectively Gorenstein variety of codimension $c+1$ in $\mathbb{P}(a_0, \ldots, a_n, a_{n+1})$ where $\mathrm{wt}(v) = a_{n+1}$. This construction is called unprojection. (See Introduction or Kustin–Miller [15], Theorem 1.5 and Reid [21]).

Note that a projectively Gorenstein variety means that the projective coordinate ring of the variety is Gorenstein.

Example 5.4 To prove the existence of a quasismooth K3 surface $X = X(2,3,3,3,3)$ in $\mathbb{P}(1,1,1,1,1,2)$ with A_1, we construct a nonsingular K3 surface $Y = X(2,3)$ of codim 2 in $\mathbb{P}^4$ containing the line $C\colon x_1 = x_2 = x_3 = 0$.

The first question to ask is how we know that Y and C are the right choices. In the first part of Example 5.1 we show that there are two ways of getting Y and C. We start by assuming that X is given and then we either blow up the singularity and then contract down a suitable configuration or project X away from the singular point $q_5 = (0,0,0,0,0,1)$ to get Y and C. We briefly describe the latter construction. Let $x_1, \ldots, x_5, y$ be homogeneous coordinates on $\mathbb{P}(1,1,1,1,1,2)$ with weights $1,1,1,1,1,2$. Since X has a singularity $\frac{1}{2}(1,-1)$ it has local coordinates, say x_4, x_5, at q_5. In other words, there are three linearly independent equations of X at q_5, which are $x_1 y = g_1, x_2 y = g_2, x_3 y = g_3$ for some homogeneous polynomials $g_i \in k[x_1, \ldots, x_5]$. When we write down a 5×5 skew-symmetric matrix whose submaximal Pfaffians are the defining equations of X we see that the other two equations of degrees $2,3$ are in $k[x_1, \ldots, x_5]$ and contain the line $C\colon x_1 = x_2 = x_3 = 0$, and we denote Y by the zero locus of these two equations. We project X away from q_5 to get Y. From now on in all of our examples we write down Y immediately without any explanation. One can work out the details as explained above.

Now we construct X from Y. Let $x_1, \ldots, x_5$ be homogeneous coordinates on $\mathbb{P}^4$. Define linear systems $\mathcal{L}_1$ and $\mathcal{L}_2$ of all homogeneous polynomials of degrees 2 and 3 respectively on $\mathbb{P}^4$, containing the line C. Let $f_i = \sum_{j=1}^{3} a_{ij} x_j \in \mathcal{L}_i$ be sufficiently general polynomials for $i = 1,2$ where a_{ij}'s are sufficiently general in $k[x_1, \ldots, x_5]$. Define

$$Y\colon (f_1 = f_2 = 0) \subset \mathbb{P}^4.$$

Both of the linear systems have base locus C so that by Bertini's theorem, the general members of these linear systems are smooth outside C. To prove that Y is smooth on C, it is sufficient to show that for sufficiently general a_{ij}, the Jacobian matrix of f_1, f_2, with respect to x_1, x_2, x_3, has rank 2 at every point of the line C, that is,

$$\mathrm{rk} \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}\Big|_C = 2.$$

For example, take the following matrix on C

$$\begin{pmatrix} 0 & x_5 & x_4 \\ x_5^2 & x_4^2 & 0 \end{pmatrix}$$

which has rank 2 at every point of $C = \mathbb{P}(x_4, x_5)$. Hence Y is nonsingular along C.

We want to find a rational function on Y which has a pole along C. Notice that Y can be given by a system of linear equations

$$M\mathbf{x}^t = 0 \tag{5.2}$$

where $M = (a_{ij})$ is a 2×3 matrix and $\mathbf{x} = (x_1, x_2, x_3)$. Now by Cramer's rule, we can solve (5.2) to get

$$y\mathbf{x} = \bigwedge^2 M$$

where $\bigwedge^2 M$ is the vector of the 2×2 minors of M. Define X by the following five equations

$$y\mathbf{x} = \bigwedge^2 M \quad \text{and} \quad M\mathbf{x}^t = 0$$

which are also the submaximal Pfaffians of the following 5×5 skew-symmetric matrix

$$\begin{pmatrix} 0 & x_1 & -x_2 & a_{13} & a_{23} \\ & 0 & x_3 & a_{12} & a_{22} \\ & & 0 & a_{11} & a_{21} \\ & -\text{sym} & & 0 & -y \\ & & & & 0 \end{pmatrix}.$$

Let φ be the rational map $\varphi\colon Y \subset \mathbb{P}^4 \to X \subset \mathbb{P}(1,1,1,1,1,2)$ given by

$$(x_1, x_2, x_3, x_4, x_5) \longmapsto (x_1, x_2, x_3, x_4, x_5, y).$$

We claim that $Y \backslash C \longrightarrow X \backslash q_5$, where $q_5 = (0,0,0,0,0,1)$, is an isomorphism. Restricting φ to $Y \setminus C$ gives a birational morphism $Y \setminus C \to X \setminus q_5$. The inverse of φ is projection from q_5. This proves the claim. To show that X is quasismooth, it is sufficient to show that the cone is smooth at q_5. It can easily be observed that $y\mathbf{x} = \bigwedge^2 M$ gives three linearly independent equations locally at q_5. This implies that X is quasismooth with a singularity A_1.

Example 5.5 We show the existence of a quasismooth K3 surface $X = X(6,6,6,7,7) \subset \mathbb{P}(1,2,3,3,3,4)$ with $3A_2$, A_3 by constructing a quasismooth K3 surface $Y = X(6,6) \subset \mathbb{P}(1,2,3,3,3)$ with $4A_2$ containing the line $C\colon y = z_1 = z_2 = 0$.

Let x, y, z_1, z_2, z_3 be homogeneous coordinates on $\mathbb{P}(1,2,3,3,3)$ of weights $1,2,3,3,3$ respectively. Let $\mathcal{L}$ be the linear system of all homogeneous polynomials of degree 6 containing C with respect to weights $1,2,3,3,3$. Let f_1, $f_2 \in \mathcal{L}$ be sufficiently general elements, that is,

$$\begin{aligned} f_1 &= a_{11}y + a_{12}z_1 + a_{13}z_2, \\ f_2 &= a_{21}y + a_{22}z_1 + a_{23}z_2 \end{aligned}$$

where a_{ij} are sufficiently general. A general K3 surface Y of codim 2 containing C in $\mathbb{P}(1,2,3,3,3)$ is given by

$$M\mathbf{x}^t = 0 \tag{5.3}$$

where M is the 2×3 matrix (a_{ij}) and $\mathbf{x} = (y, z_1, z_2)$.

By Bertini's theorem, the singularities of general elements in the linear systems lie on the base locus $\mathrm{Cone}(C)$. To show that Y is quasismooth it is sufficient to prove that the cone is smooth along $\mathrm{Cone}(C)$ except at the origin. For sufficiently general a_{ij}, we can ensure that the Jacobian matrix of f_i, with respect to y, z_1, z_2,

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \quad \text{along } \mathrm{Cone}^*(C)$$

has rank 2. For example, choose

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}\Bigg|_C = \begin{pmatrix} x^4 & 0 & z_3 \\ 0 & z_3 & x^3 \end{pmatrix},$$

so that the matrix has rank 2 at every point of $\mathrm{Cone}^*(C)$. We claim that Y has $4A_2$ singularities. Indeed, the singularities of Y come from the singularities of $\mathbb{P}$ and occur only on the vertices, edges and faces of $\mathbb{P}$. It is not difficult to see that the point q_4 is the only vertex which gives rise to a singularity A_2. Now we consider the face $q_2q_3q_4$ of $\mathbb{P}$. The homogeneous polynomials f_1, f_2 on $q_2q_3q_4$ can be written in the homogeneous coordinates z_1, z_2, z_3 of $\mathbb{P}(3,3,3)$. Since $\mathbb{P}(3,3,3)$ is isomorphic to $\mathbb{P}^2$ and f_1, f_2 are of degree 2 in $\mathbb{P}^2$, by Bézout's theorem, $(f_1 = f_2 = 0)$ in $\mathbb{P}^2$ consists of exactly four points counted with multiplicity, including the point q_4. By the inverse function theorem, x, y are local coordinates and so each point is of type $\frac{1}{3}(1,2)$. Hence the claim follows.

Now we are ready to construct X with $3A_2$, A_3 in $\mathbb{P}(1,2,3,3,3,4)$ from Y with $4A_2$ in $\mathbb{P}(1,2,3,3,3)$. By Cramer's rule, we can solve (5.3) to get a new generator t of degree 4 such that $ty = m_{23}$, $tz_1 = m_{13}$ and $tz_2 = m_{12}$ where the m_{ij} are the minors of the 2×3 matrix M. This gives three new equations together with f_1, f_2. Define X by these equations which are just the submaximal Pfaffians of the following 5×5 skew-symmetric matrix

$$\begin{pmatrix} 0 & t & a_{11} & a_{12} & a_{13} \\ & 0 & a_{21} & a_{22} & a_{23} \\ & & 0 & z_2 & -z_1 \\ & -\mathrm{sym} & & 0 & y \\ & & & & 0 \end{pmatrix}.$$

The last thing is to show that X has only the four singularities $3A_2, A_3$. Consider the rational map $\varphi\colon Y \subset \mathbb{P}(1,2,3,3,3) \to X \subset \mathbb{P}(1,2,3,3,3,4)$ given by

$$(x, y, z_1, z_2, z_3) \longmapsto (x, y, z_1, z_2, z_3, t).$$

Restricting φ to $Y \setminus C$ gives an isomorphism $Y \setminus C \xrightarrow{\sim} X \setminus q_5$. The inverse map is projection from the point q_5. Three singularities $3A_2$ of Y excluding the point $(0,0,0,0,1)$ correspond to three points p_i of X under the isomorphism for $i = 1,2,3$. Since $Y \setminus C \cup 3A_2 \simeq X \setminus p_1 \cup p_2 \cup p_3 \cup q_5$ it is sufficient to check that X is quasismooth at these points p_i and q_5. We have three Pfaffians, namely $ty = m_{23}$, $tz_1 = m_{13}$ and $tz_2 = m_{12}$, which are linearly independent locally at q_5. This gives rise to a singularity A_3. At the p_i the Jacobian matrix of the cone has rank 3 because f_1, f_2 and $tz_1 = m_{13}$ form a linearly independent set locally at these points. This gives three more singularities $3A_2$. Hence X is a quasismooth K3 surface with $3A_2, A_3$ in $\mathbb{P}(1,2,3,3,3,4)$.

So far we have showed how to get a K3 surface of codim 3 from a K3 surface of codim 2. Now we give an interesting construction of a K3 surface of codim 3 from a K3 surface of codim 1. The main point of this is to get a codim 4 example by a third unprojection.

Example 5.6 We prove the existence of a quasismooth K3 surface $X = X(3,3,3,3,4)$ in $\mathbb{P}(1,1,1,1,2,2)$ with $2A_1$, and containing the line $C\colon x_1 = x_2 = x_3 = y_2 = 0$. For this, we construct $Z = X(4)$ in $\mathbb{P}^3$, containing the two disjoint lines $C_1 \colon x_1 = x_2 = 0$ and $C_2 \colon x_3 = x_4 = 0$ and having an ordinary double point at $q_3 = (0,0,0,1) \in C_1$. As we can see, Z is not quasismooth.

Let x_1, x_2, x_3, x_4 be homogeneous coordinates on $\mathbb{P}^3$ and define the linear system $\mathcal{L}$ of all homogeneous polynomials of degree 4 on $\mathbb{P}^3$ containing C_1 and C_2. Let $f \in \mathcal{L}$ be sufficiently general and Z the variety given by the zero locus of f. The base locus of the linear system is the union of C_1 and C_2, and by Bertini's theorem the singularities of X lie in that set. The defining equation of Z can be written

$$f = ax_1 + bx_2 = cx_3 + dx_4$$

where $a = a_1x_3 + a_2x_4$, $b = b_1x_3 + b_2x_4$, $c = a_1x_1 + b_1x_2$ and $d = a_2x_1 + b_2x_2$. By an appropriate choice of a and b, we can arrange for Z to have $q_3 \in C_1$ as an ordinary double point (that is, a double point with nondegenerate tangent cone), and no other singularities. This imposes conditions on a and b: it implies that a and b do not contain x_4^3, but contain a monomial $x_4^2x_i$ for some $i \in \{1,2,3\}$.

Now set new generators y_1, y_2 of degree 2 as follows:

$$y_1 = \frac{b_1x_3 + b_2x_4}{x_1} = -\frac{a_1x_3 + a_2x_4}{x_2},$$
$$y_2 = -\frac{a_2x_1 + b_2x_2}{x_3} = \frac{a_1x_1 + b_1x_2}{x_4}.$$

This gives five relations

$$y_1x_1 = b_1x_3 + b_2x_4, \quad y_1x_2 = -(a_1x_3 + a_2x_4),$$
$$y_2x_3 = -(a_2x_1 + b_2x_2), \quad y_2x_4 = a_1x_1 + b_1x_2,$$
$$y_1y_2 = a_1b_2 - b_1a_2$$

which are the submaximal Pfaffians of the following matrix:

$$\begin{pmatrix} 0 & -x_1 & x_2 & -x_3 & x_4 \\ & 0 & y_2 & b_2 & b_1 \\ & & 0 & a_2 & a_1 \\ & -\mathrm{sym} & & 0 & y_1 \\ & & & & 0 \end{pmatrix}.$$

The variety defined by these five equations is denoted by X in $\mathbb{P}(1,1,1,1,2,2)$. Since a_2 and b_2 do not contain x_4^2, it contains the line C. Define a rational map $\varphi \colon Z \subset \mathbb{P}^3 \to X \subset \mathbb{P}(1,1,1,1,2,2)$ via

$$(x_1, x_2, x_3, x_4) \longmapsto (x_1, x_2, x_3, x_4, y_1, y_2).$$

This is a birational map and an isomorphism $Z \setminus C_1 \cup C_2 \tilde{\to} X \setminus C \cup q_4 \cup q_5$ where $q_4 = (0,0,0,0,1,0)$ and $q_5 = (0,0,0,0,0,1)$. Indeed, the inverse map is projection from the line generated by q_4 and q_5. It is enough to check that X is quasismooth along the line C and at the points q_4 and q_5. At q_4 there are locally three linearly independent equations $x_1 = b_1x_3 + b_2x_4$, $x_2 = -(a_1x_3 + a_2x_4)$ and $y_2 = a_1b_2 - b_1a_2$. This gives a singularity A_1. Similarly, the equations $x_3 = -(a_2x_1 + b_2x_2)$, $x_4 = a_1x_1 + b_1x_2$ and $y_1 = a_1b_2 - b_1a_2$ are locally linearly independent at q_5. So q_5 is a singular point of type A_1. We claim that X is quasismooth along C. To prove the claim, we write the defining equations explicitly

$$
\begin{aligned}
y_1x_1 &= x_4^2(\alpha_1x_1 + \beta_1x_2 + \gamma_1x_3) + x_4g_1(x_1,x_2,x_3) + g_2(x_1,x_2,x_3),\\
y_1x_2 &= -(x_4^2(\alpha_2x_1 + \beta_2x_2 + \gamma_2x_3) + x_4g_3(x_1,x_2,x_3) + g_4(x_1,x_2,x_3)),\\
y_2x_3 &= -(a_2x_1 + b_2x_2),\\
y_2x_4 &= a_1x_1 + b_1x_2,\\
y_1y_2 &= a_2b_1 - b_2b_1
\end{aligned}
$$

where the g_i are homogeneous polynomials with the appropriate degrees in x_1, x_2, x_3, and $\alpha_i, \beta_i, \gamma_i \in k$ for every i. Without loss of generality we can assume that a_1 and b_1 do not contain any terms involving x_4 as a multiple since we can include these terms in a_2 and b_2. If $x_4 = 0$ on C then we have only one point q_4 at which X is quasismooth. So assume $x_4 = 1$. The Jacobian matrix J of X along C with respect to the homogeneous coordinates is

$$
\begin{pmatrix}
\alpha_1 - y_1 & \beta_1 & \gamma_1 & 0 & 0 & 0\\
\alpha_2 & \beta_2 + y_1 & \gamma_2 & 0 & 0 & 0\\
a_2 & b_2 & y_2 & 0 & 0 & 0\\
a_1 & b_1 & 0 & 0 & 0 & x_4\\
0 & 0 & 0 & 0 & 0 & y_1
\end{pmatrix}
\quad \text{along } C.
$$

Now

$$
\operatorname{rk} J = 3 \iff \operatorname{rk} \begin{pmatrix} \alpha_1 - y_1 & \beta_1 & \gamma_1\\ \alpha_2 & \beta_2 + y_1 & \gamma_2 \end{pmatrix} = 2
$$

which is equivalent to the 2×2 minors of the matrix, denoted by M, on the right hand side having an empty solution. Locally at $x_4 = 1$

$$
f = (\alpha_2x_1 + \beta_2x_2 + \gamma_2x_3)x_1 + (\alpha_1x_1 + \beta_1x_2 + \gamma_1x_3)x_2 + \cdots
$$

where $\cdots$ refers to higher degree terms. Denote by $f^{(2)}$ the homogeneous polynomial of degree 2 in f, that is, the tangent cone of f. This can be written in the form

$$(x_1, x_2, x_3)N(x_1, x_2, x_3)^t$$

where

$$N = \begin{pmatrix} \alpha_2 & \frac{1}{2}(\alpha_1+\beta_2) & \frac{1}{2}\gamma_2 \\ \frac{1}{2}(\alpha_1+\beta_2) & \beta_1 & \frac{1}{2}\gamma_1 \\ \frac{1}{2}\gamma_2 & \frac{1}{2}\gamma_1 & 0 \end{pmatrix}.$$

Now $\det N = \frac{1}{4}(-\gamma_1(\alpha_2\gamma_1 - \beta_2\gamma_2) + \gamma_2(\gamma_1\alpha_1 - \gamma_2\beta_1)) \neq 0$ since the tangent cone is nondegenerate. This implies that γ_1 and γ_2 cannot be zero at the same time. Assume that $\gamma_1 = 0$. Then β_1 is nonzero since $\det N \neq 0$, which implies that $\operatorname{rk} M = 2$. Similarly, if $\gamma_2 = 0$ then α_2 is nonzero and therefore $\operatorname{rk} M = 2$. Now assume that both are nonzero. The 2×2 minors of M are

$$\begin{gathered} (\alpha_1 - y_1)(\beta_2 + y_1) - \beta_1\alpha_2 = 0, \\ (\alpha_1 - y_1)\gamma_2 - \alpha_2\gamma_1 = 0, \\ \beta_1\gamma_2 - \gamma_1(\beta_2 + y_1) = 0 \end{gathered}$$

and we note that the first minor is generated by other two minors, namely

$$\begin{aligned} &(\alpha_1 - y_1)(\beta_2 + y_1) - \beta_1\alpha_2 = \\ &\qquad \frac{(y_1+\beta_2)}{\gamma_2}((\alpha_1 - y_1)\gamma_2 - \alpha_2\gamma_1) - \frac{\alpha_2}{\gamma_2}(\beta_1\gamma_2 - \gamma_1(\beta_2 + y_1)). \end{aligned}$$

Now assume that the last two minors have a common solution. Then since γ_1 and γ_2 are both nonzero we can multiply the first of these by γ_1, the second by $-\gamma_2$ and add together to get $4\det N = 0$ which contradicts the assumption. Hence X is nonsingular along C. Therefore it has only two singularities $2A_1$.

Type II unprojection

The construction we give for the Type II unprojection is based on notes of Reid. We start by constructing a hypersurface $Y \subset \mathbb{A}^4$ containing a nonnormal curve to obtain a codim 3 Gorenstein $X \subset \mathbb{A}^6$.

Let $\Gamma \subset \mathbb{A}^4$ be the nonnormal variety parametrised by

$$x = r^2, \quad y = r^3, \quad z = s \quad \text{and} \quad t = rs.$$

Its four defining equations can be given by the nongeneric determinantal form

$$\operatorname{rk}\begin{pmatrix} x & z & y & t \\ y & t & x^2 & xz \end{pmatrix} \le 1. \tag{5.4}$$

A general hypersurface Y containing Γ is defined by the zero locus of

$$f = A(xt - yz) + B(y^2 - x^3) + C(x^2z - yt) + D(t^2 - xz^2),$$

where A, B, C, D are general polynomials in $k[x, y, z, t]$. By analogy with the Type I unprojection we are looking for rational sections of $\mathcal{O}_Y(n)$ with a pole along Γ. Consider the following matrix N associated to f:

$$N = \begin{pmatrix} x & y & xC - zD \\ z & t & xB \\ y & x^2 & xA + tD \\ t & xz & zA - yB + tC \end{pmatrix}.$$

We are not concerned here with where this matrix originally came from. Notice that the 3×3 minors of N equal f times x, z, y, t respectively, and that

$$(t, -y, z, -x)N = (0, 0, 0, 0) \quad \text{and} \quad (xz, -x^2, t, -y)N = (0, 0, f).$$

Since the 3×3 minors of N are divisible by f and $\operatorname{rk} N = 2$ on Y the equations

$$N \begin{pmatrix} u \\ v \\ 1 \end{pmatrix} = 0 \quad \text{on } Y \tag{5.5}$$

have a unique solution $u, v \in K(Y)$. Clearly, u, v are rational functions which become regular on multiplying by any 2×2 minors of the matrix in (5.4).

This construction leads to a codim 3 Gorenstein subvariety X whose defining equations are the submaximal Pfaffians of a 5×5 skew-symmetric matrix M where

$$M = \begin{pmatrix} 0 & x & z & y & t \\ & 0 & v & D & -u \\ & & 0 & u + C & B \\ & -\text{sym} & & 0 & A + xv \\ & & & & 0 \end{pmatrix}$$

in $k[x, y, z, t, u, v]$. In other words, we add u, v to the ring $R(Y)$ of Y in order to get the Gorenstein ring $R(Y)[u, v]$ of codimension 3, the ring of X.

To be more precise, the four submaximal Pfaffians Pf_i for $i = 2, \ldots, 4$ are the four equations in (5.5), and Pf_1 can be written as a combination of the others since $M(\mathrm{Pf}_1, \ldots, \mathrm{Pf}_5) = 0$. Now consider the map $Y \subset \mathbb{A}^4 \to X \subset \mathbb{A}^6$ via

$$(x, y, z, t) \longmapsto (x, y, z, t, u, v).$$

This is a birational map and an isomorphism $Y \setminus \Gamma \to X \setminus q_5$. Hence $\operatorname{codim} X = 3$. Since X is given by the maximal Pfaffians of M, it is Gorenstein. Because of the isomorphism it is sufficient to check that X is smooth at q_5. Since we have the 3 Pfaffians, namely

$$vy = \ldots, \quad vt = \ldots, \quad \text{and} \quad vA = \ldots,$$

X is smooth at q_5 if and only if A contains a nonzero linear term in x, z.

Now we apply this construction to K3 surfaces.

Example 5.7 We want to construct $X = X(7, 8, 8, 9, 10) \subset \mathbb{P}(2, 3, 3, 4, 4, 5)$ having singularities $3A_1, 3A_2, A_3$ via constructing a codim 1 K3 surface $Y = X(12) \subset \mathbb{P}(2, 3, 3, 4)$ with singularities $3A_1, 4A_2$ containing a projectively nonnormal curve Γ.

Let x, y_1, y_2, z_1 be homogeneous coordinates on $\mathbb{P}(2, 3, 3, 4)$ with weights $2, 3, 3, 4$ respectively. The curve Γ is given in parametric form by

$$x = r^2, \quad y_1 = r^3, \quad y_2 = s, \quad z_1 = rs.$$

The defining equations of Γ can be put into determinantal form

$$\operatorname{rk} \begin{pmatrix} x & y_2 & y_1 & z_1 \\ y_1 & z_1 & x^2 & xy_2 \end{pmatrix} \leq 1.$$

Notice that this is not a generic determinantal form because two of the minors are combinations of others. Now define the linear system $\mathcal{L}$ of all homogeneous polynomials of degree 12 with respect to weights $2, 3, 3, 4$ containing Γ:

$$\mathcal{L} = \big\{ A(xz_1 - y_2y_1) + B(x^3 - y_1^2) + C(x^2y_2 - y_1z_1) + D(xy_2^2 - z_1^2) \big| A, B, C, D \text{ are homogeneous polynomials in degrees } 6, 6, 5, 4 \text{ resp. in } k[x, y_1, y_2, z_1] \big\}.$$

Let $f \in \mathcal{L}$ be general element, that is,

$$f = A(xz_1 - y_2y_1) + B(x^3 - y_1^2) + C(x^2y_2 - y_1z_1) + D(xy_2^2 - z_1^2)$$

where A, B, C, D are general homogeneous polynomials. Define a general hypersurface $Y \subset \mathbb{P}(2,3,3,4)$ by the zero locus of f. By Bertini's theorem $\mathrm{Cone}(Y)$ is nonsingular away from $\mathrm{Cone}(\Gamma)$. Also, $\mathrm{Cone}(Y)$ is smooth along $\mathrm{Cone}(\Gamma)$ except at the origin. Under the $\mathbb{C}^*$-action Y has singularities $3A_1, 4A_2$.

Now we write down the 4×3 matrix N associated to f:

$$N = \begin{pmatrix} x & y_1 & xC + y_2 D \\ y_2 & z_1 & -xB \\ y_1 & x^2 & xA - z_1 D \\ z_1 & xy_2 & y_2 A + y_1 B + z_1 C \end{pmatrix}$$

whose 3×3 minors are f times x, y_1, y_2, z_1 respectively. The equations

$$N \begin{pmatrix} t \\ z_2 \\ 1 \end{pmatrix} = 0 \quad \text{on } Y$$

have a unique solution $t, z_2 \in K(Y)$ of degrees $5, 4$. This gives four equations which are four submaximal Pfaffians of the following matrix:

$$M = \begin{pmatrix} 0 & x & y_2 & y_1 & z_1 \\ & 0 & z_2 & -D & -t \\ & & 0 & t + C & -B \\ & -\mathrm{sym} & & 0 & A + xz_2 \\ & & & & 0 \end{pmatrix}.$$

The fifth equation can be seen from the matrix M or alternatively can be written as a combination of the others since these are not linearly independent. Define X by these five Pfaffians associated to M. The original hypersurface Y can be obtained by eliminating t and z_2 from the defining equations of X.

To see that X is a quasismooth K3 surface with singularities $3A_1, 3A_2, A_3$ in $\mathbb{P}(2,3,3,4,4,5)$ consider a rational map $\varphi \colon Y \to X$ given by

$$(x, y_1, y_2, z_1) \longmapsto (x, y_1, y_2, z_1, z_2, t).$$

It is a morphism away from Γ and moreover an isomorphism $Y \setminus \Gamma \xrightarrow{\sim} X \setminus q_4$. For, we begin with a map

$$\pi_L : \mathbb{P}(2,3,3,4,4,5) \setminus L \longmapsto \mathbb{P}(2,3,3,4)$$

which is projection from the line L: $x = y_1 = y_2 = z_1 = 0$. Restricting π_L to X gives a morphism from $X \setminus q_4$ to $Y \setminus \Gamma$ which is the inverse of φ. Since $Y \setminus \Gamma \simeq X \setminus q_4$ it suffices to check that $\mathrm{Cone}(X)$ is smooth at the point q_4. Locally at q_4 Pfaffians $\mathrm{Pf}_1, \mathrm{Pf}_4, \mathrm{Pf}_5$ form a linearly independent set. By the inverse function theorem y_2, t are local coordinates. This gives a singularity A_3. Similarly, it can be proven that X has the further singularities $3A_1, 3A_2$.

The next example can be done in a similar way. Here we just state the result and give the machinery.

Example 5.8 $X(10, 11, 12, 13, 14) \subset \mathbb{P}(3, 4, 5, 5, 6, 7)$ with $2A_2, A_3, A_4, A_4$ is a quasismooth K3 surface of codim 3. This follows from first constructing a general hypersurface $X(18) \subset \mathbb{P}(3, 4, 5, 6)$ with $3A_2, A_3, A_1, A_4$, containing the nonnormal curve Γ given by nongeneric determinantal form

$$\mathrm{rk}\begin{pmatrix} z & y & u & t \\ u & t & z^2 & yz \end{pmatrix} \le 1$$

where y, z, t, u are homogeneous coordinates on $\mathbb{P}(3, 4, 5, 6)$ with weights $3, 4, 5, 6$ respectively.

Nonsingularity of the remaining three candidates via Macaulay

The remaining three candidates (X, D) in the list are numbers 61, 65 and 70. We will consider just the first of these; the other two can be done in a similar way.

Recall that X is defined by the 5 Pfaffians Pf_i in the weighted projective space $\mathbb{P}(x_0, \dots, x_5)$ with weights $\mathrm{wt}(x_i) = a_i$. Let $J(X)$ be the Jacobian matrix of X given by a 5×6 matrix

$$J(X) = (\frac{\partial \,\mathrm{Pf}_i}{\partial x_i}).$$

Denote by $\mathrm{Sing}\, X$ the singular locus of X, which is defined by the 3×3 minors of $J(X)$ together with the ideal $I(X)$ of X. To check the nonsingularity of X it suffices to prove that $\mathrm{Sing}\, X = \emptyset$. We will show this by computing the ideal $I(\mathrm{Sing}\, X)$ via Macaulay.

Example 5.9 We consider candidate 61. Let a, b, c, d, e, f be homogeneous coordinates on $\mathbb{P}(5, 5, 6, 7, 8, 9)$ with weights $5, 5, 6, 7, 8, 9$. The defining equations of X can be represented by a skew symmetric matrix M which, after a suitable change of coordinates, is of the form

$$M = \begin{pmatrix} 0 & a & c & d & e \\ -a & 0 & d & e & f \\ -c & -d & 0 & f & a^2+ab+b^2 \\ -d & -e & -f & 0 & bc \\ -e & -f & -a^2-ab-b^2 & -bc & 0 \end{pmatrix}.$$

Here we assume that all of the coefficients of the entries of M are 1. We think of 1 as a generic number. We start a Macaulay session by defining a ring r.

```
% ring r
characteristic : 0
number of variables : 6
6 variables : a b c d e f
variables weights : 5 5 6 7 8 9
monomial order : reverse lexicographic order
% mat M
number of rows: 5
number of columns: 5
input of entries:
(1,1)=0
(2,1)=-a
      .
      .
```

The notations $I(X)$, $J(X)$, $I(\text{sing } X)$ are not suitable for Macaulay therefore we denote these by IX, JX, ISingX.

```
% pfaff M IX
% jacob IX JX
% wedge JX 3 N
```

where N is a 10×20 matrix whose entries give the ideal of Sing X.

The command "std" calculates the standard basis of $I(\text{Sing } X)$. The command "putstd" allows to view standard basis as a matrix.

```
% flatten N ISingX
% std ISingX ISingX_std
% putstd ISingX_std ISingX_pstd
```

We can now observe the standard basis by simply typing:

```
% type ISingX_pstd
```

The standard basis contains

$$
\begin{aligned}
&f^3, \quad b^6, \quad a^5 + 55/6e^2f - 7/2df^2, \\
&d^3 - 3/2cde - ae^2 + 1/2c^2f - 1/2adf, \\
&e^3 + 4a^3f + 7/2a^2bf + 4ab^2f + 10def - 13/2cf^2, \\
&c^4 + 29/12a^3f + 55/12a^2bf + 35/12ab^2f + 41/6def - 31/4cf^2, \dots .
\end{aligned}
$$

The zero locus of these equations is empty therefore Sing X is empty, as required. In Example 4.6, we show that (X, D) has the given singularities. This completes the existence of this surface with the given features.

5.2.1 List of K3 surfaces in codimension 3

Table 5.1: Weighted K3 surfaces in codimension 3

Nb.	**Weighted K3 surface**	**Resolution Data**
1	$X(2,2,2) \subset \mathbb{P}(1,1,1,1,1,1)$	8
2	$X(2,3,3,3,3) \subset \mathbb{P}(1,1,1,1,1,2)$	$6 + \frac{1}{2}$
3	$X(3,3,4,4,4) \subset \mathbb{P}(1,1,1,1,2,3)$	$4 + \frac{2}{3}$
4	$X(3,3,3,3,4) \subset \mathbb{P}(1,1,1,1,2,2)$	$4 + 2 \times \frac{1}{2}$
5	$X(3,3,4,4,4) \subset \mathbb{P}(1,1,1,2,2,2)$	$2 + 3 \times \frac{1}{2}$
6	$X(4,4,5,5,6) \subset \mathbb{P}(1,1,1,2,3,4)$	$2 + \frac{3}{4}$
7	$X(3,4,4,4,5) \subset \mathbb{P}(1,1,1,2,2,3)$	$2 + \frac{1}{2} + \frac{2}{3}$
8	$X(4,4,4,4,4) \subset \mathbb{P}(1,1,2,2,2,2)$	$0 + 5 \times \frac{1}{2}$

continued on next page

Table 5.1: Weighted K3 surfaces in codimension 3

Nb.	Weighted K3 surface	Resolution Data
9	$X(6,7,8,9,10) \subset \mathbb{P}(1,1,3,4,5,6)$	$0 + \frac{5}{6}$
10	$X(4,4,4,5,5) \subset \mathbb{P}(1,1,2,2,2,3)$	$0 + 3\times\frac{1}{2}+\frac{2}{3}$
11	$X(4,4,5,5,6) \subset \mathbb{P}(1,1,2,2,3,3)$	$0 + \frac{1}{2}+2\times\frac{2}{3}$
12	$X(4,5,6,6,7) \subset \mathbb{P}(1,1,2,2,3,5)$	$0 + \frac{1}{2}+\frac{2\cdot 3}{5}$
13	$X(4,5,5,6,6) \subset \mathbb{P}(1,1,2,2,3,4)$	$0 + 2\times\frac{1}{2}+\frac{3}{4}$
14	$X(5,6,6,7,8) \subset \mathbb{P}(1,1,2,3,4,5)$	$0 + \frac{1}{2}+\frac{4}{5}$
15	$X(4,5,6,6,7) \subset \mathbb{P}(1,1,2,3,3,4)$	$0 + \frac{2}{3}+\frac{3}{4}$
16	$X(4,5,5,6,6) \subset \mathbb{P}(1,2,2,2,3,3)$	$-2 + 5\times\frac{1}{2}+\frac{2}{3}$
17	$X(6,6,7,7,8) \subset \mathbb{P}(1,2,2,3,4,5)$	$-2 + 4\times\frac{1}{2}+\frac{4}{5}$
18	$X(5,5,6,6,6) \subset \mathbb{P}(1,2,2,3,3,3)$	$-2 + 2\times\frac{1}{2}+3\times\frac{2}{3}$
19	$X(6,7,8,9,10) \subset \mathbb{P}(1,2,2,3,5,7)$	$-2 + 2\times\frac{1}{2}+\frac{3\cdot 4}{7}$
20	$X(8,9,10,11,12) \subset \mathbb{P}(1,2,4,5,6,7)$	$-2 + 3\times\frac{1}{2}+\frac{6}{7}$
21	$X(6,6,6,7,7) \subset \mathbb{P}(1,2,3,3,3,4)$	$-2 + 3\times\frac{2}{3}+\frac{3}{4}$
22	$X(6,8,9,10,11) \subset \mathbb{P}(1,2,3,3,5,8)$	$-2 + \frac{2}{3}+\frac{3\cdot 5}{8}$
23	$X(6,8,9,10,11) \subset \mathbb{P}(1,2,3,4,5,7)$	$-2 + \frac{3}{4}+\frac{3\cdot 4}{7}$
24	$X(10,11,12,13,14) \subset \mathbb{P}(1,3,5,6,7,8)$	$-2 + 2\times\frac{2}{3}+\frac{7}{8}$
25	$X(8,9,9,10,10) \subset \mathbb{P}(1,3,4,4,5,6)$	$-2 + 2\times\frac{3}{4}+\frac{5}{6}$
26	$X(8,8,9,9,10) \subset \mathbb{P}(1,3,4,4,5,5)$	$-2 + \frac{3}{4}+2\times\frac{4}{5}$
27	$X(14,15,16,17,18) \subset \mathbb{P}(1,5,7,8,9,10)$	$-2 + \frac{2\cdot 3}{5}+\frac{9}{10}$
28	$X(8,9,10,12,13) \subset \mathbb{P}(1,3,4,4,5,9)$	$-2 + \frac{3}{4}+\frac{2\cdot 7}{9}$
29	$X(8,9,10,12,13) \subset \mathbb{P}(1,3,4,5,6,7)$	$-2 + \frac{5}{6}+\frac{2\cdot 5}{7}$
30	$X(16,17,18,19,20) \subset \mathbb{P}(1,6,8,9,10,11)$	$-2 + \frac{1}{2}+\frac{2}{3}+\frac{10}{11}$
31	$X(12,13,14,15,16) \subset \mathbb{P}(1,4,6,7,8,9)$	$-2 + \frac{1}{2}+\frac{3}{4}+\frac{8}{9}$

continued on next page

Table 5.1: Weighted K3 surfaces in codimension 3

Nb.	Weighted K3 surface	Resolution Data
32	$X(11,12,12,13,14) \subset \mathbb{P}(1,4,5,6,7,8)$	$-2 + \frac{1}{2}+\frac{4}{5}+\frac{7}{8}$
33	$X(10,11,12,12,13) \subset \mathbb{P}(1,4,5,6,6,7)$	$-2 + \frac{1}{2}+\frac{5}{6}+\frac{6}{7}$
34	$X(9,10,10,11,12) \subset \mathbb{P}(1,3,4,5,6,7)$	$-2 + \frac{2}{3}+\frac{3}{4}+\frac{6}{7}$
35	$X(8,9,10,10,11) \subset \mathbb{P}(1,3,4,5,5,6)$	$-2 + \frac{2}{3}+\frac{4}{5}+\frac{5}{6}$
36	$X(6,6,7,7,8) \subset \mathbb{P}(1,2,3,3,4,4)$	$-2 + \frac{1}{2}+\frac{2}{3}+2\times\frac{3}{4}$
37	$X(6,7,7,8,8) \subset \mathbb{P}(1,2,3,3,4,5)$	$-2 + \frac{1}{2}+2\times\frac{2}{3}+\frac{4}{5}$
38	$X(7,8,8,9,10) \subset \mathbb{P}(1,2,3,4,5,6)$	$-2 + 2\times\frac{1}{2}+\frac{2}{3}+\frac{5}{6}$
39	$X(6,7,8,8,9) \subset \mathbb{P}(1,2,3,4,4,5)$	$-2 + 2\times\frac{1}{2}+\frac{3}{4}+\frac{4}{5}$
40	$X(6,7,8,9,10) \subset \mathbb{P}(1,2,3,4,5,5)$	$-2 + \frac{1}{2}+\frac{4}{5}+\frac{2\cdot 3}{5}$
41	$X(6,6,7,8,9) \subset \mathbb{P}(1,2,3,3,4,5)$	$-2 + \frac{2}{3}+\frac{3}{4}+\frac{2\cdot 3}{5}$
42	$X(6,7,8,9,10) \subset \mathbb{P}(1,2,3,3,4,7)$	$-2 + \frac{1}{2}+\frac{2}{3}+\frac{2\cdot 5}{7}$
43	$X(5,6,6,6,7) \subset \mathbb{P}(1,2,2,3,3,4)$	$-2 + 3\times\frac{1}{2}+\frac{2}{3}+\frac{3}{4}$
44	$X(5,6,6,7,8) \subset \mathbb{P}(1,2,2,3,3,5)$	$-2 + 2\times\frac{1}{2}+\frac{2}{3}+\frac{2\cdot 3}{5}$
45	$X(10,12,13,14,15) \subset \mathbb{P}(3,3,4,5,7,10)$	$-4 + 3\times\frac{2}{3}+\frac{3\cdot 7}{10}$
46	$X(12,14,15,16,19) \subset \mathbb{P}(3,4,5,7,8,11)$	$-4 + \frac{3\cdot 5}{8}+\frac{3\cdot 8}{11}$
47	$X(10,12,14,15,17) \subset \mathbb{P}(2,3,5,7,8,9)$	$-4 + \frac{3\cdot 5}{8}+\frac{4\cdot 5}{9}$
48	$X(10,12,14,15,17) \subset \mathbb{P}(2,3,5,5,7,12)$	$-4 + \frac{2\cdot 3}{5}+\frac{5\cdot 7}{12}$
49	$X(9,10,11,12,14) \subset \mathbb{P}(2,2,3,5,7,9)$	$-4 + 4\times\frac{1}{2}+\frac{4\cdot 5}{9}$
50	$X(6,7,8,8,9) \subset \mathbb{P}(2,2,3,3,4,5)$	$-4 + 5\times\frac{1}{2}+\frac{2}{3}+\frac{2\cdot 3}{5}$
51	$X(14,15,16,17,18) \subset \mathbb{P}(2,5,6,7,9,11)$	$-4 + \frac{1}{2}+\frac{5}{6}+\frac{5\cdot 6}{11}$
52	$X(9,10,11,12,14) \subset \mathbb{P}(2,3,4,5,7,7)$	$-4 + 2\times\frac{1}{2}+\frac{3\cdot 4}{7}+\frac{2\cdot 5}{7}$
53	$X(12,13,14,15,16) \subset \mathbb{P}(2,5,6,7,7,8)$	$-4 + 3\times\frac{1}{2}+\frac{6}{7}+\frac{3\cdot 4}{7}$
54	$X(8,9,10,11,12) \subset \mathbb{P}(2,3,3,4,5,8)$	$-4 + 2\times\frac{1}{2}+2\times\frac{2}{3}+\frac{3\cdot 5}{8}$

continued on next page

Table 5.1: Weighted K3 surfaces in codimension 3

Nb.	Weighted K3 surface			Resolution Data		
55	$X(8,9,10,10,11)$	$\subset$	$\mathbb{P}(2,3,3,4,5,7)$	-4	$+$	$\frac{1}{2}+3\times\frac{2}{3}+\frac{3\cdot4}{7}$
56	$X(8,9,9,10,10)$	$\subset$	$\mathbb{P}(2,3,4,4,5,5)$	-4	$+$	$3\times\frac{1}{2}+2\times\frac{3}{4}+\frac{2\cdot3}{5}$
57	$X(8,8,9,9,10)$	$\subset$	$\mathbb{P}(2,3,3,4,5,5)$	-4	$+$	$\frac{1}{2}+2\times\frac{2}{3}+2\times\frac{2\cdot3}{5}$
58	$X(8,9,10,11,12)$	$\subset$	$\mathbb{P}(2,3,4,4,5,7)$	-4	$+$	$4\times\frac{1}{2}+\frac{3}{4}+\frac{2\cdot5}{7}$
59	$X(7,8,8,9,10)$	$\subset$	$\mathbb{P}(2,3,3,4,4,5)$	-4	$+$	$3\times\frac{1}{2}+3\times\frac{2}{3}+\frac{3}{4}$
60	$X(14,15,16,17,18)$	$\subset$	$\mathbb{P}(3,5,6,7,8,11)$	-4	$+$	$\frac{2}{3}+\frac{5}{6}+\frac{4\cdot7}{11}$
61	$X(14,15,16,17,18)$	$\subset$	$\mathbb{P}(5,5,6,7,8,9)$	-4	$+$	$\frac{4}{5}+2\times\frac{2\cdot3}{5}+\frac{5}{6}$
62	$X(12,13,14,15,16)$	$\subset$	$\mathbb{P}(4,4,5,6,7,9)$	-4	$+$	$2\times\frac{1}{2}+2\times\frac{3}{4}+\frac{2\cdot7}{9}$
63	$X(12,13,14,15,16)$	$\subset$	$\mathbb{P}(3,4,5,6,7,10)$	-4	$+$	$\frac{1}{2}+\frac{2}{3}+\frac{4}{5}+\frac{3\cdot7}{10}$
64	$X(10,11,12,13,14)$	$\subset$	$\mathbb{P}(3,4,5,5,6,7)$	-4	$+$	$2\times\frac{2}{3}+\frac{3}{4}+\frac{4}{5}+\frac{2\cdot3}{5}$
65	$X(12,13,14,15,16)$	$\subset$	$\mathbb{P}(4,5,5,6,7,8)$	-4	$+$	$\frac{1}{2}+\frac{3}{4}+2\times\frac{4}{5}+\frac{2\cdot3}{5}$
66	$X(10,11,12,12,13)$	$\subset$	$\mathbb{P}(3,4,4,5,6,7)$	-4	$+$	$\frac{1}{2}+\frac{2}{3}+2\times\frac{3}{4}+\frac{2\cdot5}{7}$
67	$X(10,11,12,13,14)$	$\subset$	$\mathbb{P}(2,3,4,5,7,9)$	-4	$+$	$\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\frac{4\cdot5}{9}$
68	$X(9,10,10,11,12)$	$\subset$	$\mathbb{P}(2,3,4,5,5,7)$	-4	$+$	$\frac{1}{2}+\frac{3}{4}+\frac{2\cdot3}{5}+\frac{3\cdot4}{7}$
69	$X(8,9,10,11,12)$	$\subset$	$\mathbb{P}(2,3,4,5,5,6)$	-4	$+$	$3\times\frac{1}{2}+\frac{2}{3}+\frac{4}{5}+\frac{2\cdot3}{5}$
70	$X(16,17,18,19,20)$	$\subset$	$\mathbb{P}(5,6,7,8,9,10)$	-4	$+$	$\frac{1}{2}+\frac{2}{3}+\frac{4}{5}+\frac{2\cdot3}{5}+\frac{6}{7}$

5.3 Some examples in codimension 4

In this section we give two types of constructions of K3 surfaces (X, D) in codim 4. Namely, the first construction is through unprojection and the second through an $(X, 2D)$-model. The latter construction is explained in Example 5.14.

We state an important remark, pointed out by Reid.

Remark 5.10 Let X be a projectively Gorenstein subscheme of codim 3, containing a projectively Gorenstein subscheme C of codim 4, that is the canonical sheaves of X and C are $\omega_X = \mathcal{O}_X(k_X)$ and $\omega_C = \mathcal{O}_C(k_C)$ for some

$k_X, k_C \in \mathbb{Z}$ respectively. Then there exists a homomorphism $s\colon \mathcal{I}_C \to \mathcal{O}(-l)$ which is a basis of $\mathcal{H}om(I_C, \mathcal{O}_X(-l))$ at every point of C if $l = k_C - k_X < 0$.

Note that this remark can also be stated for any codimension, in particular, to get codim ≥ 5 examples. Moreover, it leads to the existence of codim ≥ 4 rings but it does not explicitly give the defining equations of the surfaces we construct. In the case of codim 4 we work on particular examples under some mild conditions, which we can give explicitly by defining equations, and furthermore by using birational geometry we show that the surfaces we construct are quasismooth.

The next example leads to the general construction of codim 4 K3 surfaces with explicit equations from successive constructions in codimensions 2 and 3. 112 examples in Table 5.2 can also be obtained in this way.

Example 5.11 We want to construct a codim 4 K3 surface X having A_6 in $\mathbb{P}(1,1,3,4,5,6,7)$ by constructing successively codim 2 and codim 3 K3 surfaces.

Let x_1, x_2, z, t, u be homogeneous coordinates on $\mathbb{P}(1,1,3,4,5)$ with weights $1,1,3,4,5$ respectively. Now define two linear systems $\mathcal{L}_1$, $\mathcal{L}_2$ of all homogeneous polynomials of degrees $6,8$ respectively in $k[x_1,x_2,z,t,u]$ with respect to weights $1,1,3,4,5$, containing the line

$$C_1\colon x_1 = z = t = 0.$$

Let $f_i = x_1 a_{i1} + z a_{i2} + t a_{i3} \in \mathcal{L}_i$ be sufficiently general for $i = 1,2$ where the a_{ij} are sufficiently general homogeneous polynomial in $k[x_1,x_2,z,t,u]$. Define a general surface

$$Z = X(6,8)\colon (f_1 = f_2 = 0) \subset \mathbb{P}(1,1,3,4,5). \tag{5.6}$$

By Bertini's theorem the singularities of $\mathrm{Cone}(Z)$ lie on the base locus $\mathrm{Cone}(C_1)$. We want Z to have an ordinary double point $q_1 = (0,1,0,0,0)$ and A_4 along C_1 where A_4 corresponds to the point $q_4 = (0,0,0,0,1)$. Obviously, this puts some restrictions on a_{ij}'s. Clearly, locally at q_4 we have

$$f_1 = x_1 + \cdots \quad \text{and} \quad f_2 = z + \cdots \subset \mathbb{A}^4/\langle\varepsilon\rangle$$

where ε is a primitive 5th root of unity and $\cdots$ refers to other terms. By the inverse function theorem x_2, t are local coordinates. This implies that q_4 is a singularity of type A_4. The surface Z having a singularity q_1 is equivalent to $\mathrm{rk}\, J(Z)_{|q_1} \leq 1$ where the Jacobian matrix $J(Z)$ of Z along C_1 is

$$\frac{\partial(f_1,f_2)}{\partial(x_1,z,t)}\Big|_{C_1} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}\Big|_{C_1}.$$

Since a_{ij}'s are general $\operatorname{rk} J(Z)_{|q_1} = 1$. By replacing f_2 by f_2 with a suitable multiple of $x_2^2 f_1$ subtracted, we can assume that the a_{2j} do not contain any pure powers of x_2. From here it can be observed that $\operatorname{Cone}(Z)$ is nonsingular on $\operatorname{Cone}(C_1)$ except at the origin and q_1.

Now by Cramer's rule we can solve (5.6) to get $v\mathbf{x} = \bigwedge^2 M$ where $M = (a_{ij})$ is a 2×3 matrix and $\mathbf{x} = (x_1, z, t)$. Define $Y = X(6,7,8,9,10) \subset \mathbb{P}(1,1,3,4,5,6)$ by the five equations

$$v\mathbf{x} = \bigwedge^2 M \quad \text{and} \quad M\mathbf{x}^t = 0$$

which are also the submaximal Pfaffians of the following 5×5 skew-symmetric matrix

$$N = \begin{pmatrix} 0 & -t & z & a_{21} & -a_{11} \\ & 0 & -x_1 & a_{22} & -a_{12} \\ & & 0 & a_{23} & -a_{13} \\ & -\mathrm{sym} & & 0 & -v \\ & & & & 0 \end{pmatrix}$$

where the entries of N are taken in the ring $k[x_1, x_2, z, t, u, v]$. We want to show that Y is a quasismooth K3 surface with A_5, containing the line

$$C_2 \colon x_1 = z = t = u = 0$$

in $\mathbb{P}(1,1,3,4,5,6)$. Since $x_2^\alpha \notin a_{2j}$ for any nonzero α, Y contains the line C_2. Define a rational map $\varphi \colon Z \dashrightarrow Y$ by

$$(x_1, x_2, z, t, u) \longmapsto (x_1, x_2, z, t, u, v).$$

The map φ restricted to $Z \setminus C_1$ gives an isomorphism $Z \setminus C_1 \xrightarrow{\sim} Y \setminus C_2$. Indeed, consider the map from $Y \setminus q_5$ to Z, which is projection from the point q_5, and so the inverse map is the restriction of this map to $Y \setminus C_2$. To see that $\operatorname{Cone}(Y)$ is smooth it is enough to consider $\operatorname{Cone}(Y)$ on $\operatorname{Cone}(C_2)$ excluding the origin since $\operatorname{Cone}(Z)$ is nonsingular outside $\operatorname{Cone}(C_1)$. Now observe that $(f_1 = 0)$ is nonsingular at q_1 and $(f_2 = 0)$ is singular at q_1. Consider $(f_2 = 0)$ on $(f_1 = 0)$ and calculate the tangent cone of $(f_2 = 0)$ on $(f_1 = 0)$ at q_1. It suffices to show that $\operatorname{Cone}(Y)$ being smooth corresponds to the tangent cone of $(f_2 = 0)$ on $(f_1 = 0)$ at q_1 being nondegenerate. One can prove this by using the same argument as that used in Example 5.6. Locally at q_5 there are three linearly independent equations of Y, which give rise to a singularity A_5. Remark 5.10 says there exists a generator w of degree 7 which has a pole along C_2. This means that

$$w = \frac{A}{x_1} = \frac{B}{z} = \frac{C}{t} = \frac{D}{u}$$

is a rational function of weight 7 on Y, having a pole along C_2 for some homogeneous polynomials $A, B, C, D \in k[x_1, x_2, z, t, u, v]$. This gives four relations, namely:

$$x_1 w = A, \quad zw = B, \quad tw = C \quad \text{and} \quad uw = D.$$

Now denote by X the variety given by the nine equations:

$$v\mathbf{x} = \bigwedge^2 M, \quad M\mathbf{x}^t = 0 \quad \text{and} \quad \mathbf{y}w = (A, B, C, D)$$

where $\mathbf{x} = (x_1, z, t)$ and $\mathbf{y} = (x_1, z, t, u)$. Notice that the first five equations are the defining equations of Y. Since a_{2j} does not contain a pure power of x_2 for $j = 1, 2, 3$ we can write

$$a_{2j} = x_1\lambda_1^j + z\lambda_2^j + t\lambda_3^j + u\lambda_4^j$$

where $\lambda_i^j \in k[x_1, x_2, z, t, u]$ for $i = 1, 2, 3, 4$. To find A, B, C, D explicitly we rewrite the defining equations Pf_k of Y in terms of λ_i^j by substituting for the a_{2j}:

$$(\mathrm{Pf}_1, \mathrm{Pf}_2, \mathrm{Pf}_3, \mathrm{Pf}_4, \mathrm{Pf}_5) = \mathbf{y}H$$

where

$$H = \begin{pmatrix} v + a_{13}\lambda_1^2 - a_{12}\lambda_1^3 & a_{13}\lambda_1^1 - a_{11}\lambda_1^3 & a_{12}\lambda_1^1 - a_{11}\lambda_1^2 & a_{11} & a_{21} \\ a_{13}\lambda_2^2 - a_{12}\lambda_2^3 & a_{13}\lambda_2^1 - a_{11}\lambda_2^3 - v & a_{12}\lambda_2^1 - a_{11}\lambda_2^2 & a_{12} & a_{22} \\ a_{13}\lambda_3^2 - a_{12}\lambda_3^3 & a_{13}\lambda_3^1 - a_{11}\lambda_3^3 & a_{12}\lambda_3^1 - a_{11}\lambda_3^2 + v & a_{13} & a_{23} \\ a_{13}\lambda_4^2 & a_{13}\lambda_4^1 & a_{12}\lambda_4^1 - a_{11}\lambda_4^2 & 0 & 0 \end{pmatrix}.$$

First we begin by finding A and B. Since $zA - x_1 B \in I(Y)$, some combination $\beta_1 \mathrm{Pf}_1 + \beta_2 \mathrm{Pf}_1$ of Pf_1 and Pf_2 can be made to contain a term which is a multiple of $a_{13}t$. Then we can add a multiple of Pf_4 to $\beta_1 \mathrm{Pf}_1 + \beta_2 \mathrm{Pf}_1$ to make it zero along $x_1 = z = 0$. This is possible since Pf_4 is zero along C_1. Explicitly:

$$\begin{aligned}-(a_{13}\lambda_4^1)\,\mathrm{Pf}_1 + (a_{13}\lambda_4^2)\,\mathrm{Pf}_2 - (-\lambda_4^1(a_{13}\lambda_3^2 - a_{12}\lambda_3^3)\\ + \lambda_4^2(a_{13}\lambda_3^1 - a_{11}\lambda_3^3))\,\mathrm{Pf}_4 = x_1A + zB\end{aligned}$$

where

$$\begin{aligned}A =&a_{13}\lambda_4^1(a_{13}\lambda_2^2 - a_{12}\lambda_2^3 - a_{12}\lambda_3^2) - a_{13}\lambda_4^2(a_{13}\lambda_2^1 - a_{11}\lambda_2^3 - v - a_{12}\lambda_3^1)\\ &+ a_{12}\lambda_3^3(a_{12}\lambda_4^1 - a_{11}\lambda_4^2),\\ B =& - a_{13}\lambda_4^1(v + a_{13}\lambda_1^2 - a_{12}\lambda_1^3 - a_{11}\lambda_3^2) + a_{13}\lambda_4^2(a_{13}\lambda_1^1 - a_{11}\lambda_1^3 - a_{11}\lambda_3^1)\\ &+ a_{11}\lambda_3^3(-a_{12}\lambda_4^1 + a_{11}\lambda_4^2).\end{aligned}$$

Similarly,

$$\begin{aligned}(-a_{11}\lambda_4^2 + a_{12}\lambda_4^1)\,\mathrm{Pf}_1 - (a_{13}\lambda_4^2)\,\mathrm{Pf}_3 + (\lambda_4^1(-a_{13}\lambda_2^2 + a_{12}\lambda_2^3)\\ + \lambda_4^2(-a_{11}\lambda_2^3 + a_{13}\lambda_2^1))\,\mathrm{Pf}_4 = x_1C - tA\end{aligned}$$

and

$$\begin{aligned}(-a_{13}\lambda_2^1 + a_{11}\lambda_2^3 + v + a_{12}\lambda_3^1)\,\mathrm{Pf}_1 + (a_{13}\lambda_2^2 - a_{12}\lambda_2^3 - a_{12}\lambda_3^2)\,\mathrm{Pf}_2\\ + (a_{12}\lambda_3^3)\,\mathrm{Pf}_3 + (E)\,\mathrm{Pf}_4 = -x_1D + uA\end{aligned}$$

where

$$\begin{aligned}C =&(-a_{11}\lambda_4^2 + a_{12}\lambda_4^1)(v + a_{13}\lambda_1^2 - a_{12}\lambda_1^3 + a_{11}\lambda_2^3)\\ &- a_{13}\lambda_4^2(a_{12}\lambda_1^1 - a_{11}\lambda_1^2 - a_{11}\lambda_2^1) - a_{13}\lambda_4^1(\lambda_2^2a_{11}),\\ D =& - (-a_{13}\lambda_2^1 + a_{11}\lambda_2^3 + v + a_{12}\lambda_3^1)(v + a_{13}\lambda_1^2 - a_{12}\lambda_1^3)\\ &- (a_{13}\lambda_2^2 - a_{12}\lambda_2^3 - a_{12}\lambda_3^2)(a_{13}\lambda_1^1 - a_{11}\lambda_1^3)\\ &- (a_{12}\lambda_3^3)(a_{12}\lambda_1^1 - a_{11}\lambda_1^2) - a_{11}(-\lambda_3^1(-a_{12}\lambda_2^3 + a_{13}\lambda_2^2)\\ &+ \lambda_3^2(-a_{11}\lambda_2^3 + a_{13}\lambda_2^1 - v) - \lambda_3^3(-a_{11}\lambda_2^2 + a_{12}\lambda_2^1)),\\ E =& - \lambda_3^1(-a_{12}\lambda_2^3 + a_{13}\lambda_2^2) + \lambda_3^2(-a_{13}\lambda_2^3 + a_{13}\lambda_2^1 - v) - \lambda_3^3(-a_{11}\lambda_2^2 + a_{12}\lambda_2^1).\end{aligned}$$

We want to see that X is a quasismooth K3 surface with A_6. Clearly, X has a singularity A_6 since locally at this point q_6 there are four linearly independent equations

$$x_1 = A, \quad z = B, \quad t = C \quad \text{and} \quad u = D.$$

Consider the rational map $\varphi\colon Y \to X$ given by

$$(x_1, x_2, z, t, u, v) \longmapsto (x_1, x_2, z, t, u, v, w).$$

This is a birational map and an isomorphism $Y \setminus C_2 \tilde{\to} X \setminus q_6$. In fact, the inverse map is projection from the point q_6.

Example 5.12 We want to construct a quasismooth K3 surface Z of codim 4 with singularities A_1, A_2 in $\mathbb{P}(1,1,1,1,2,2,3)$ by constructing two subrings of the graded ring $R(X,D)$, corresponding to two K3 surfaces X, Y of codim 3.

Recall that in Example 5.6 a K3 surface $X = X(3,3,3,3,4)$ with $2A_1$, containing the line

$$C\colon x_1 = x_2 = x_3 = y_2 = 0,$$

in $\mathbb{P}(1,1,1,1,2,2)$ is constructed via $X(4)$ in $\mathbb{P}^3$ containing two distinct lines

$$C_1\colon x_1 = x_2 = 0, \quad C_2\colon x_3 = x_4 = 0$$

and the ordinary double point $(0,0,0,1)$. Since X and $C \cong \mathbb{P}(1,2)$ are projectively Gorenstein and their canonical divisors $\omega_X = \mathcal{O}_X$ and $\omega_C = \mathcal{O}_C(-3)$, by Remark 5.10 there exists a generator $z\colon \mathcal{I}_C \to \mathcal{O}(3)$ in degree 3. This implies that z has a pole along C which gives four relations $f_6, \dots, f_9$. Denote by Z $V(f_1, \dots, f_9)$ where the first five relations $f_1, \dots, f_5$ are the defining equations of X. In this case, to check that Z is quasismooth we have to know defining equations of Z explicitly. To see this, we consider the same $X(4)$ as in Example 5.6 to construct a different K3 surface $Y = X(3,3,4,4,4)$ with A_2 in $\mathbb{P}(1,1,1,1,2,3)$ via a K3 surface of codim 2. The generator

$$y_1 = \frac{a}{x_2} = -\frac{b}{x_1}$$

which gives a K3 surface $X(3,3)$ with A_1 in $\mathbb{P}(1,1,1,1,2)$, containing the line

$$C_3\colon x_1 = x_2 = x_3 = 0$$

since a and b do not contain x_4^3. So rewrite the defining equations f_1, f_2 of $X(3,3)$

$$\begin{aligned} y_1x_2 &= n_1x_1 + n_2x_2 + n_3x_3, \\ y_1x_1 &= m_1x_1 + m_2x_2 + m_3x_3 \quad \text{respectively} \end{aligned} \tag{5.7}$$

where $a = n_1x_1 + n_2x_2 + n_3x_3$, $-b = m_1x_1 + m_2x_2 + m_3x_3$, and n_i, m_i are homogeneous polynomials of degree 2 in $k[x_1, x_2, x_3, x_4]$. Now by Cramer's rule we can solve (5.7) to get

$$z\mathbf{x} = \bigwedge^2 M$$

where $\mathbf{x} = (x_1, x_2, x_3)$ and M is a 2×3 matrix such that $M\mathbf{x}^t = (f_1, f_2)$. These three equations $z\mathbf{x} = \bigwedge^2 M$ together with the equations f_1, f_2 give a K3 surface $Y = X(3,3,4,4,4)$ with A_2 in $\mathbb{P}(1,1,1,1,2,3)$.

So far we have got eight equations explicitly and now consider all these together with choosing n_1, n_2, m_1, m_2 divisible by x_4, and $n_3 = a_1, m_3 = b_1$ such that $n_1x_1 + n_2x_2 + n_3x_3 = a_1x_3 + a_2x_4$ and $m_1x_1 + m_2x_2 + m_3x_3 = b_1x_3 + b_2x_4$. Multiply $y_2x_4 = a_1x_1 + b_1x_2$ by z and substitute for zx_1 and zx_2 as appropriate to deduce that the last equation f_9 is $zy_2 = ((a_1m_2 - b_1n_2)a_1 + (a_1m_1 - b_1n_1)b_1)/x_4$.

Define a rational map $\varphi\colon X \subset \mathbb{P}(1,1,1,1,2,2) \to Z \subset \mathbb{P}(1,1,1,1,2,2,3)$ via

$$(x_1, x_2, x_3, x_4, y_1, y_2) \longmapsto (x_1, x_2, x_3, x_4, y_1, y_2, z).$$

This is a birational map and an isomorphism $X \setminus q_4 \cup q_5 \cup C \xrightarrow{\sim} Z \setminus q_5 \cup q_6$ where q_4, q_5 are vertices in $\mathbb{P}(1,1,1,1,2,2)$ and q_5, q_6 are vertices in $\mathbb{P}(1,1,1,1,2,2,3)$. Since $X \setminus q_4 \cup q_5 \cup C$ is nonsingular it is sufficient to prove that Z is quasismooth at q_5, q_6. Locally at q_5 f_3, f_4, f_5, f_9 are linearly independent which gives a singularity A_1. Locally at q_6 $f_6, \dots, f_9$ are linearly independent. This gives rise to a singularity A_2.

The following example is a construction of K3 surface of codim 4 from a K3 surface of codim 2. This type of unprojection allows us to use Bertini's theorem on codimension 2 K3 surfaces to prove quasismoothness of codimension 4 K3 surfaces.

Example 5.13 Let $x_1, \dots, x_5$ be homogeneous coordinates on $\mathbb{P}^4$. We begin by constructing a nonsingular K3 surface $X(2,3)$ containing two distinct lines

$$C_1\colon x_1 = x_2 = x_3 = 0, \quad C_2\colon x_3 = x_4 = x_5 = 0$$

in $\mathbb{P}^4$. From here we construct a K3 surface X of codim 4 with $2A_1$ in $\mathbb{P}(1,1,1,1,1,2,2)$.

Define linear systems $\mathcal{L}_1$, $\mathcal{L}_2$ of all homogeneous polynomials of degrees $2,3$ respectively on $\mathbb{P}^4$, containing the lines C_1 and C_2 and take $f_1 \in \mathcal{L}$ and $f_2 \in \mathcal{L}$ sufficiently general, that is,

$$\begin{aligned} f_1 &= a_{11}x_1 + a_{12}x_2 + a_{13}x_3, \\ f_2 &= a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \end{aligned} \tag{5.8}$$

or, equivalently,

$$\begin{aligned} f_1 &= b_{11}x_3 + b_{12}x_4 + b_{13}x_5, \\ f_2 &= b_{21}x_3 + b_{22}x_4 + b_{23}x_5 \end{aligned} \tag{5.9}$$

where a_{ij} and b_{ij} are sufficiently general. Define

$$Y = X(2,3)\colon (f_1 = f_2 = 0) \subset \mathbb{P}^4.$$

The linear systems $\mathcal{L}_i$ have the base locus $C_1 \cup C_2$ and so by Bertini's theorem Y is nonsingular away from $C_1 \cup C_2$. Choosing a_{ij}, b_{ij} sufficiently general we can arrange Y nonsingular along $C_1 \cup C_2$. It is a K3 surface containing two lines C_1, C_2.

Now by using Cramer's rule we can solve (5.8) and (5.9) to get

$$y_1 \mathbf{x}^1 = \bigwedge^2 M \quad \text{and} \quad y_2 \mathbf{x}^2 = \bigwedge^2 N$$

where $M = (a_{ij})$, $N = (b_{ij})$ are 2×3 matrices and $\mathbf{x}^1 = (x_1, x_2, x_3)$, $\mathbf{x}^2 = (x_3, x_4, x_5)$. Let H be a hyperplane section of Y. Since Y is nonsingular $y_1 \in H^0(Y, 2H + C_1)$ and $y_2 \in H^0(Y, 2H + C_2)$ but y_1 and y_2 have zero along C_2 and C_1 respectively. Therefore $y_1 \in H^0(Y, 2H + C_1 - C_2)$ and $y_2 \in H^0(Y, 2H + C_2 - C_1)$ which imply that $y_1 y_2 \in H^0(Y, 4H)$. In other words, $y_1 y_2 = g(x_1, \dots, x_5)$ for some homogeneous polynomial g of degree 4. The variety X can be defined by the following nine equations:

$$f_1 = f_2 = 0, \quad y_1 \mathbf{x}^1 = \bigwedge^2 M, \quad y_2 \mathbf{x}^2 = \bigwedge^2 N \quad \text{and} \quad y_1 y_2 = g(x_1, \dots, x_5).$$

Define a rational map $\varphi \colon Y \subset \mathbb{P}^4 \to X \subset \mathbb{P}(1,1,1,1,1,2,2)$ by sending

$$(x_1, \dots, x_5) \longmapsto (x_1, \dots, x_5, y_1, y_2).$$

This is a birational map and an isomorphism

$$Y \setminus C_1 \cup C_2 \xrightarrow{\sim} X \setminus q_5 \cup q_6.$$

Hence it is enough to check that X is quasismooth at the vertices q_5, q_6. Locally at q_5 there are four linearly independent equations $\mathbf{x}^1 = \bigwedge^2 M$ and $y_2 = g(x_1, \dots, x_5)$. This gives a singularity A_1. Similarly, at q_6 there are four linearly independent equations $\mathbf{x}^2 = \bigwedge^2 N$ and $y_2 = g(x_1, \dots, x_5)$. Hence q_6 is of type A_1.

The following example is a beautiful construction of a K3 surface (X, D) of codim 4 via an $(X, 2D)$-model.

Example 5.14 We want to construct a codim 4 K3 surface (X, D) with $7A_1$ in $\mathbb{P}(1,2,2,2,2,3,3)$ via constructing an $(X, 2D)$-model with $7A_1$ in $\mathbb{P}^4$. Generally speaking, we construct a complete intersection K3 surface $X(2,3)$ in $\mathbb{P}^4$ such that $X(2,3)$ is tangent to the hyperplane $y_0 = 0$ along a twisted cubic curve Γ (by analogy of the famous Kummer surface: begin by taking a plane in $\mathbb{P}^3$ with a conic C and a quartic surface $X(4)$ in $\mathbb{P}^3$ which touches the plane along C then $X(4)$ in general has six nodes. This leads to a K3 surface $X(4,6) \subset \mathbb{P}(1,2,2,2,3)$ with six nodes, see the Appendix of this chapter), and $\operatorname{div} y_0$ is Cartier which is 2Γ with Γ twisted cubic Weil divisor. It turns out that $X(2,3)$ has seven nodes along Γ. This gives that $X(2,3)$ is an $(X, 2D)$-model with $7A_1$ in $\mathbb{P}^4$. Then the ring $R(X, 2D)$ can be written down quite simple so that we can figure out the details of the ring $R(X, D)$.

Let $y_1, \ldots, y_4$ be homogeneous coordinates on $\mathbb{P}^3$. The twisted cubic curve Γ' in $\mathbb{P}^3$ is given by the quadric equations

$$Q_1 \colon y_1y_3 - y_2^2 = 0, \quad Q_2 \colon y_1y_4 - y_2y_3 = 0, \quad Q_3 \colon y_2y_4 - y_3^2 = 0.$$

Let F be a cubic hypersurface in $\mathbb{P}^3$ given by the equation

$$y_1y_4^2 - 2y_2y_3y_4 + y_3^3 = 0.$$

Observe that $Q_1 \cap F = 2\Gamma'$ by multiplying the defining equation of F by y_1 and adding $-y_3^2$ times the defining equation of Q_1. Define a general surface $X = X(2,3)$ in terms of F, Q_1 as follows

$$X \colon (f_1 = f_2 = 0) \subset \mathbb{P}^4$$

where

$$f_1 = y_1y_3 - y_2^2 + y_0y_4, \quad f_2 = y_1y_4^2 - 2y_2y_3y_4 + y_3^3 + y_0G$$

for a general homogeneous polynomial G of degree 2 in the variables $y_0, \ldots, y_4$. Then the hyperplane section $X \cap (y_0 = 0)$ is 2Γ where $\Gamma = \Gamma' \cap (y_0 = 0)$. Here we set $2D = 2\Gamma$. Notice that since $2D$ is a hyperplane section, it is Cartier on X. We claim that X has seven nodes along Γ. By Bertini's theorem, the general X is nonsingular away from Γ. To view the singularities we write the Jacobian matrix $J(X)$ of X along Γ as follows

$$\left.\frac{\partial(f_1, f_2)}{\partial(y_0, \ldots, y_4)}\right|_{\Gamma} = \left.\begin{pmatrix} y_4 & y_3 & -2y_2 & y_1 & 0 \\ G & y_4^2 & -2y_3y_4 & y_3^2 & 0 \end{pmatrix}\right|_{\Gamma}.$$

Consider the twisted cubic $\Gamma' \subset \mathbb{P}^3$ given as the image of the map

$$t \longmapsto (1, t, t^2, t^3).$$

Substitute $y_1 = 1, y_2 = t, y_3 = t^2, y_4 = t^3$ into $J(X)$ to get

$$\begin{pmatrix} t^3 & t^2 & -2t & 1 & 0 \\ G_{|\Gamma} & t^6 & -2t^5 & t^4 & 0 \end{pmatrix}. \tag{5.10}$$

By multiplying the first row of the matrix in (5.10) by $-t^4$ and adding this to the second row it leads to the solution of the polynomial $G(0,1,t,t^2,t^3) - t^7$ of degree 7 in the variable t. This in general gives seven singularities. To see that these are actually nodes it is enough to consider only one singularity, and by a change of coordinates we assume that the singular point is $p = (0,1,0,0,0)$. This implies that $y_1^2 \notin G$. We can easily see that $(f_1 = 0)$ is nonsingular at p. Therefore $(f_1 = 0)$ on the affine piece $y_1 = 1$ of $\mathbb{P}^4$ is locally isomorphic to $\mathbb{C}^3$ and the local coordinates are y_0, y_2, y_4. Locally $(f_2 = 0)$ on $(f_1 = 0)$ is $(y_4 - y_2^3)^2 + y_0 A$ by substituting $y_3 = y_2^2 - y_0 y_4$ where $A = y_2 y_4^2 + y_0 y_2^4 y_4 + y_0^2 y_2^2 y_4 + y_0^3 y_4^3 + G$. Proving that the singular point p is a node is equivalent to showing that

$$(t^7 - G)(p) = 0 \quad \text{and} \quad \frac{\partial G}{\partial y_2}(p) \neq 0.$$

Since G is general it contains the term $y_1 y_2$. Therefore p is a node.

We can blow up the singularities of X along D since $2D$ is Cartier. Then the resolution $\sigma\colon S \to X$ along D, by Proposition 2.4, looks like

$$\sigma^*(D) = B + \frac{1}{2}\sum_{i=1}^{7} C_i.$$

where the C_i are -2-curves and B is the divisor meeting the C_i transversally. This implies together with $(2D)^2 = 6$ that $B^2 = -2$. Then $h^0(S, \sigma^*(D)) = h^0(X, D) = 1$ which implies that $H^0(X, D)$ is generated by one element, say x, of degree 1 such that x vanishes on D. As we know, by construction $H^0(X, 2D)$ is generated by $y_0, \ldots, y_4$ such that y_0 vanishes on Γ. So $x^2 = y_0$. Consider elements of $H^0(X, 3D)$ as elements of $H^0(X, 4D)$ vanishing on Γ. This implies the existence of two new generators z_1, z_2 in degree 3 such that

$$\begin{aligned} R_3\colon &\quad x z_2 - y_2 y_4 + y_3^2 = 0, \\ R_4\colon &\quad x z_1 - y_1 y_4 + y_2 y_3 = 0, \\ R_5\colon &\quad y_2^2 - y_1 y_3 - y_0 y_4 = 0. \end{aligned} \tag{5.11}$$

So far we have seven generators $x, y_1 \ldots, y_4, z_1, z_2$ of an (X, D)-model with three relations in degree 4. We want to prove that (X, D) is a codim 4 K3 surface in $\mathbb{P}(1,2,2,2,2,3,3)$ with nine equations. First of all, we find out explicitly these nine equations. Then we prove that we do not need any further ring generators and relations.

Now observe that Γ' has two syzygies, namely

$$\begin{aligned} y_4(y_1 y_3 - y_2^2) - y_3(y_1 y_4 - y_2 y_3) + y_2(y_2 y_4 - y_3^2) = 0, \\ y_3(y_1 y_3 - y_2^2) - y_2(y_1 y_4 - y_2 y_3) + y_1(y_2 y_4 - y_3^2) = 0. \end{aligned}$$

By inspecting these two syzygies of Γ' and the three relations R_3, R_4, R_5 of (X, D) we get two more relations in degree 5

$$\begin{aligned} R_1\colon &\quad y_2z_2 - z_1y_3 - xy_4^2 = 0, \\ R_2\colon &\quad y_1z_2 - z_1y_2 - xy_4y_3 = 0. \end{aligned} \tag{5.12}$$

For example, take this relation $xz_1 = y_1y_4 - y_2y_3$ and multiply by y_2 then replace $y_2(y_1y_4 - y_2y_3)$ by $y_3(y_1y_3 - y_2^2) + y_1(y_2y_4 - y_3^2)$. We end up $xz_1y_2 = y_3(y_1y_3 - y_2^2) + y_1(y_2y_4 - y_3^2)$. By using the other two relations $y_1y_3 - y_2^2 = -y_0y_4$ and $xz_2 = y_2y_4 - y_3^2$ of (X, D) we get $xz_1y_2 = y_3(-y_0y_4) + y_1xz_2$ which gives the desired relation $z_1y_2 = y_1z_2 - xy_3y_4$. Similarly we can do the same thing for the second relation in degree 5. Now by using the cubic relation of $(X, 2D)$ together with necessary relations of (X, D) we have so far, we can rewrite $f_2 = x(y_4z_1 - y_3z_2 + xG)$ which gives another new relation of (X, D) in degree 5

$$R_7\colon \quad -y_4z_1 + y_3z_2 - xG = 0. \tag{5.13}$$

There are three new relations of (X, D) in degree 6

$$\begin{aligned} R_6\colon &\quad -z_1z_2 + y_4^2y_3 - y_2G = 0, \\ R_8\colon &\quad -z_1^2 + y_4y_3^2 - y_1G = 0, \\ R_9\colon &\quad z_2^2 - y_4^3 + y_3G = 0 \end{aligned} \tag{5.14}$$

coming from multiplying $xz_1 = y_1y_4 - y_2y_3$ by either z_1 or z_2 and $xz_2 = y_2y_4 - y_3^2$ by z_2. The equations in (5.11), (5.12), (5.13) and (5.14) give 9 relations of (X, D) in degrees $4^3, 5^3, 6^3$.

We claim that there are enough generators. We prove this by contradiction. First, suppose that there is a new generator w in an even degree. Since $X(2,3)$ is an $(X, 2D)$-model w is generated by $y_0, \ldots, y_4$, a contradiction. Second, assume that there is a new generator w in an odd degree. Then consider xw, which is of even degree. It vanishes on Γ. Therefore

$$xw = \alpha_1(y_1y_3 - y_2^2) + \alpha_2(y_1y_4 - y_2y_3) + \alpha_3(y_2y_4 - y_3^2)$$

for some α_i in $k[y_0, \ldots, y_4]$ for $i = 1, 2, 3$. By using the equations in (5.11) w is not a new generator.

We claim that there are enough relations. We divide the proof into two parts. First, assume that there is a new relation f of (X, D) in an even degree. Observe that f is written in the variables $y_0, \ldots, y_4$. So set $g(y_0, \ldots, y_4) = f(x, y_1, \ldots, z_1, z_2)$. Since g is in the ideal of $(X, 2D)$ f is not a new relation. Finally, assume that there is a new relation f in an odd degree. Consider

xf in the variables $y_0, \ldots, y_4$ so that xf is in the ideal $I(X, 2D)$ of $(X, 2D)$. Therefore $xf = af_1 + bf_2$ for some homogeneous polynomials a, b. Since $f_2 = xR_7$ f is written as a combination of f_1 and R_7, a contradiction.

We conclude that (X, D) is a K3 surface with $7A_1$ in $\mathbb{P}(1,2,2,2,2,3,3)$ having 9 relations of degrees $4^3, 5^3, 6^3$ and 16 syzygies of degrees $6^2, 7^6, 8^6, 9^2$. To see the syzygies, we observe that the relations can be obtained by the submaximal Pfaffians of the following three 5×5 skew-symmetric matrices:

$$\begin{pmatrix} 0 & x & y_1 & y_2 & y_3 \\ & 0 & y_2 & y_3 & y_4 \\ & & 0 & -xy_4 & z_1 \\ & -\mathrm{sym} & & 0 & z_2 \\ & & & & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & y_1 & x & -y_3 & -z_1 \\ & 0 & y_2 & z_1 & y_3y_4 \\ & & 0 & y_4 & z_2 \\ & -\mathrm{sym} & & 0 & -G \\ & & & & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & y_3 & x & y_4 & z_2 \\ & 0 & y_2 & z_2 & y_4^2 \\ & & 0 & -y_3 & -z_1 \\ & -\mathrm{sym} & & 0 & G \\ & & & & 0 \end{pmatrix}.$$

These give 15 syzygies in degrees $6^2, 7^7, 8^4, 9^2$ but one of the syzygies in degree 7 is a combination of others. Moreover, there are two remaining ones in degree 8 which are:

$$\begin{aligned} &GR_5 + y_2R_6 - xy_4R_7 + y_1R_9 + z_2R_2 + y_4^2R_4 = 0, \\ &GR_4 - y_4R_8 + z_1R_7 + y_3R_6 = 0. \end{aligned}$$

5.3.1 Lists of possible K3 surfaces in codimension 4

Table 5.2: Weighted K3 surfaces in codimension 4

Nb.	Weighted K3 surface	Resolution Data
1	$X \subset \mathbb{P}(1,1,1,1,1,1,2)$	$8 + \frac{1}{2}$
2	$X \subset \mathbb{P}(1,1,1,1,1,2,2)$	$6 + 2\times\frac{1}{2}$
3	$X \subset \mathbb{P}(1,1,1,1,2,2,2)$	$4 + 3\times\frac{1}{2}$
4	$X \subset \mathbb{P}(1,1,1,1,2,3,4)$	$4 + \frac{3}{4}$
5	$X \subset \mathbb{P}(1,1,1,1,2,2,3)$	$4 + \frac{1}{2}+\frac{2}{3}$
6	$X \subset \mathbb{P}(1,1,1,2,2,2,2)$	$2 + 4\times\frac{1}{2}$
7	$X \subset \mathbb{P}(1,1,1,2,2,3,3)$	$2 + 2\times\frac{2}{3}$
8	$X \subset \mathbb{P}(1,1,1,2,3,4,5)$	$2 + \frac{4}{5}$
9	$X \subset \mathbb{P}(1,1,1,2,2,2,3)$	$2 + 2\times\frac{1}{2}+\frac{2}{3}$
10	$X \subset \mathbb{P}(1,1,1,2,2,3,4)$	$2 + \frac{1}{2}+\frac{3}{4}$
11	$X \subset \mathbb{P}(1,1,2,2,2,2,2)$	$0 + 6\times\frac{1}{2}$
12	$X \subset \mathbb{P}(1,1,2,3,3,4,4)$	$0 + 2\times\frac{3}{4}$
13	$X \subset \mathbb{P}(1,1,3,4,5,6,7)$	$0 + \frac{6}{7}$
14	$X \subset \mathbb{P}(1,1,2,2,3,3,3)$	$0 + 3\times\frac{2}{3}$
15	$X \subset \mathbb{P}(1,1,2,2,3,5,7)$	$0 + \frac{3\cdot 4}{7}$
16	$X \subset \mathbb{P}(1,1,2,3,3,4,7)$	$0 + \frac{2\cdot 5}{7}$
17	$X \subset \mathbb{P}(1,1,2,2,2,3,4)$	$0 + 3\times\frac{1}{2}+\frac{3}{4}$
18	$X \subset \mathbb{P}(1,1,2,2,3,4,5)$	$0 + 2\times\frac{1}{2}+\frac{4}{5}$
19	$X \subset \mathbb{P}(1,1,2,2,3,3,5)$	$0 + \frac{2}{3}+\frac{2\cdot 3}{5}$
20	$X \subset \mathbb{P}(1,1,2,2,2,3,5)$	$0 + 2\times\frac{1}{2}+\frac{2\cdot 3}{5}$
21	$X \subset \mathbb{P}(1,1,2,2,2,2,3)$	$0 + 4\times\frac{1}{2}+\frac{2}{3}$

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Table 5.2: Weighted K3 surfaces in codimension 4

Nb.	Weighted K3 surface	Resolution Data
22	$X \subset \mathbb{P}(1,1,2,3,4,5,6)$	$0 + \frac{1}{2} + \frac{5}{6}$
23	$X \subset \mathbb{P}(1,1,2,3,3,4,5)$	$0 + \frac{2}{3} + \frac{4}{5}$
24	$X \subset \mathbb{P}(1,1,2,2,3,3,4)$	$0 + \frac{1}{2} + \frac{2}{3} + \frac{3}{4}$
25	$X \subset \mathbb{P}(1,2,2,2,2,3,3)$	$-2 + 7 \times \frac{1}{2}$
26	$X \subset \mathbb{P}(1,2,3,3,5,8,11)$	$-2 + \frac{4\cdot 7}{11}$
27	$X \subset \mathbb{P}(1,3,4,4,5,5,5)$	$-2 + 3 \times \frac{4}{5}$
28	$X \subset \mathbb{P}(1,3,4,4,5,9,13)$	$-2 + \frac{3\cdot 10}{13}$
29	$X \subset \mathbb{P}(1,2,2,2,3,3,3)$	$-2 + 4 \times \frac{1}{2} + 2 \times \frac{2}{3}$
30	$X \subset \mathbb{P}(1,2,2,2,3,3,4)$	$-2 + 5 \times \frac{1}{2} + \frac{3}{4}$
31	$X \subset \mathbb{P}(1,2,2,2,3,3,5)$	$-2 + 4 \times \frac{1}{2} + \frac{2\cdot 3}{5}$
32	$X \subset \mathbb{P}(1,2,2,3,4,5,6)$	$-2 + 4 \times \frac{1}{2} + \frac{5}{6}$
33	$X \subset \mathbb{P}(1,2,2,3,3,4,4)$	$-2 + 3 \times \frac{1}{2} + 2 \times \frac{3}{4}$
34	$X \subset \mathbb{P}(1,2,2,3,3,3,3)$	$-2 + \frac{1}{2} + 4 \times \frac{2}{3}$
35	$X \subset \mathbb{P}(1,2,2,3,3,4,7)$	$-2 + 3 \times \frac{1}{2} + \frac{2\cdot 5}{7}$
36	$X \subset \mathbb{P}(1,2,2,3,3,5,8)$	$-2 + 2 \times \frac{1}{2} + \frac{3\cdot 5}{8}$
37	$X \subset \mathbb{P}(1,2,2,3,5,7,9)$	$-2 + \frac{1}{2} + \frac{4\cdot 5}{9}$
38	$X \subset \mathbb{P}(1,2,4,5,6,7,8)$	$-2 + 3 \times \frac{1}{2} + \frac{7}{8}$
39	$X \subset \mathbb{P}(1,2,3,3,3,4,5)$	$-2 + 3 \times \frac{2}{3} + \frac{4}{5}$
40	$X \subset \mathbb{P}(1,2,3,4,4,5,5)$	$-2 + 2 \times \frac{1}{2} + 2 \times \frac{4}{5}$
41	$X \subset \mathbb{P}(1,2,3,3,3,4,4)$	$-2 + 2 \times \frac{2}{3} + 2 \times \frac{3}{4}$
42	$X \subset \mathbb{P}(1,2,3,3,4,4,4)$	$-2 + \frac{1}{2} + 3 \times \frac{3}{4}$
43	$X \subset \mathbb{P}(1,2,3,3,4,5,7)$	$-2 + \frac{2\cdot 3}{5} + \frac{2\cdot 5}{7}$
44	$X \subset \mathbb{P}(1,2,3,4,4,5,9)$	$-2 + 2 \times \frac{1}{2} + \frac{2\cdot 7}{9}$

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Table 5.2: Weighted K3 surfaces in codimension 4

Nb.	Weighted K3 surface	Resolution Data
45	$X \subset \mathbb{P}(1,2,3,3,4,4,5)$	$-2 + 2\times\frac{3}{4}+\frac{2\cdot 3}{5}$
46	$X \subset \mathbb{P}(1,2,3,3,3,4,7)$	$-2 + 2\times\frac{2}{3}+\frac{2\cdot 5}{7}$
47	$X \subset \mathbb{P}(1,2,3,4,5,5,7)$	$-2 + \frac{4}{5}+\frac{3\cdot 4}{7}$
48	$X \subset \mathbb{P}(1,2,3,3,4,5,8)$	$-2 + \frac{3}{4}+\frac{3\cdot 5}{8}$
49	$X \subset \mathbb{P}(1,3,5,6,7,8,9)$	$-2 + 2\times\frac{2}{3}+\frac{8}{9}$
50	$X \subset \mathbb{P}(1,3,4,4,5,6,7)$	$-2 + 2\times\frac{3}{4}+\frac{6}{7}$
51	$X \subset \mathbb{P}(1,4,5,6,6,7,7)$	$-2 + \frac{1}{2}+2\times\frac{6}{7}$
52	$X \subset \mathbb{P}(1,3,4,5,5,6,6)$	$-2 + \frac{2}{3}+2\times\frac{5}{6}$
53	$X \subset \mathbb{P}(1,5,7,8,9,10,11)$	$-2 + \frac{2\cdot 3}{5}+\frac{10}{11}$
54	$X \subset \mathbb{P}(1,4,5,6,6,7,13)$	$-2 + \frac{1}{2}+\frac{2\cdot 11}{13}$
55	$X \subset \mathbb{P}(1,3,4,5,5,6,11)$	$-2 + \frac{2}{3}+\frac{2\cdot 9}{11}$
56	$X \subset \mathbb{P}(1,3,4,4,5,5,9)$	$-2 + \frac{4}{5}+\frac{2\cdot 7}{9}$
57	$X \subset \mathbb{P}(1,3,4,5,6,7,7)$	$-2 + \frac{6}{7}+\frac{2\cdot 5}{7}$
58	$X \subset \mathbb{P}(1,2,2,3,3,5,5)$	$-2 + \frac{1}{2}+2\times\frac{2\cdot 3}{5}$
59	$X \subset \mathbb{P}(1,2,3,3,4,7,10)$	$-2 + \frac{1}{2}+\frac{3\cdot 7}{10}$
60	$X \subset \mathbb{P}(1,2,2,3,3,5,7)$	$-2 + \frac{1}{2}+\frac{2}{3}+\frac{3\cdot 4}{7}$
61	$X \subset \mathbb{P}(1,6,8,9,10,11,12)$	$-2 + \frac{1}{2}+\frac{2}{3}+\frac{11}{12}$
62	$X \subset \mathbb{P}(1,4,6,7,8,9,10)$	$-2 + \frac{1}{2}+\frac{3}{4}+\frac{9}{10}$
63	$X \subset \mathbb{P}(1,4,5,6,6,7,8)$	$-2 + \frac{1}{2}+\frac{5}{6}+\frac{7}{8}$
64	$X \subset \mathbb{P}(1,4,5,6,7,8,9)$	$-2 + \frac{1}{2}+\frac{4}{5}+\frac{8}{9}$
65	$X \subset \mathbb{P}(1,3,4,5,6,7,8)$	$-2 + \frac{2}{3}+\frac{3}{4}+\frac{7}{8}$
66	$X \subset \mathbb{P}(1,3,4,5,5,6,7)$	$-2 + \frac{2}{3}+\frac{4}{5}+\frac{6}{7}$
67	$X \subset \mathbb{P}(1,2,3,3,4,5,6)$	$-2 + \frac{1}{2}+2\times\frac{2}{3}+\frac{5}{6}$

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Table 5.2: Weighted K3 surfaces in codimension 4

Nb.	Weighted K3 surface	Resolution Data
68	$X \subset \mathbb{P}(1,2,3,4,5,6,7)$	$-2 + 2\times\frac{1}{2}+\frac{2}{3}+\frac{6}{7}$
69	$X \subset \mathbb{P}(1,2,3,4,4,5,6)$	$-2 + 2\times\frac{1}{2}+\frac{3}{4}+\frac{5}{6}$
70	$X \subset \mathbb{P}(1,2,3,4,5,5,6)$	$-2 + \frac{1}{2}+\frac{2\cdot 3}{5}+\frac{5}{6}$
71	$X \subset \mathbb{P}(1,2,3,3,4,5,5)$	$-2 + \frac{2}{3}+\frac{4}{5}+\frac{2\cdot 3}{5}$
72	$X \subset \mathbb{P}(1,2,3,3,4,4,7)$	$-2 + \frac{1}{2}+\frac{3}{4}+\frac{2\cdot 5}{7}$
73	$X \subset \mathbb{P}(1,2,2,3,3,3,4)$	$-2 + 2\times\frac{1}{2}+2\times\frac{2}{3}+\frac{3}{4}$
74	$X \subset \mathbb{P}(1,2,2,3,3,4,5)$	$-2 + 3\times\frac{1}{2}+\frac{2}{3}+\frac{4}{5}$
75	$X \subset \mathbb{P}(1,2,2,3,3,4,5)$	$-2 + 2\times\frac{1}{2}+\frac{3}{4}+\frac{2\cdot 3}{5}$
76	$X \subset \mathbb{P}(1,2,2,3,3,5,7)$	$-2 + \frac{1}{2}+\frac{2}{3}+\frac{3\cdot 4}{7}$
77	$X \subset \mathbb{P}(1,2,2,3,3,3,5)$	$-2 + \frac{1}{2}+2\times\frac{2}{3}+\frac{2\cdot 3}{5}$
78	$X \subset \mathbb{P}(1,2,3,3,4,4,5)$	$-2 + \frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\frac{4}{5}$
79	$X \subset \mathbb{P}(2,3,5,5,7,12,17)$	$-4 + \frac{7\cdot 10}{17}$
80	$X \subset \mathbb{P}(2,2,3,5,7,9,11)$	$-4 + 3\times\frac{1}{2}+\frac{5\cdot 6}{11}$
81	$X \subset \mathbb{P}(2,3,4,5,7,7,11)$	$-4 + \frac{2\cdot 5}{7}+\frac{5\cdot 6}{11}$
82	$X \subset \mathbb{P}(3,3,4,5,7,10,13)$	$-4 + 2\times\frac{2}{3}+\frac{4\cdot 9}{13}$
83	$X \subset \mathbb{P}(3,5,6,7,8,11,14)$	$-4 + \frac{5}{6}+\frac{5\cdot 9}{14}$
84	$X \subset \mathbb{P}(5,5,6,7,8,9,11)$	$-4 + 2\times\frac{2\cdot 3}{5}+\frac{2\cdot 9}{11}$
85	$X \subset \mathbb{P}(2,3,4,4,5,7,11)$	$-4 + 4\times\frac{1}{2}+\frac{3\cdot 8}{11}$
86	$X \subset \mathbb{P}(2,3,3,4,5,7,9)$	$-4 + 3\times\frac{2}{3}+\frac{4\cdot 5}{9}$
87	$X \subset \mathbb{P}(2,3,3,4,5,5,5)$	$-4 + \frac{2}{3}+3\times\frac{2\cdot 3}{5}$
88	$X \subset \mathbb{P}(2,3,4,5,5,7,7)$	$-4 + \frac{3}{4}+2\times\frac{3\cdot 4}{7}$
89	$X \subset \mathbb{P}(2,5,6,7,9,11,13)$	$-4 + \frac{5}{6}+\frac{6\cdot 7}{13}$
90	$X \subset \mathbb{P}(2,2,3,3,4,5,5)$	$-4 + 4\times\frac{1}{2}+2\times\frac{2\cdot 3}{5}$

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Table 5.2: Weighted K3 surfaces in codimension 4

Nb.	Weighted K3 surface	Resolution Data
91	$X \subset \mathbb{P}(2,2,3,3,4,5,7)$	$-4 + 4\times\frac{1}{2}+\frac{2}{3}+\frac{3\cdot4}{7}$
92	$X \subset \mathbb{P}(2,3,3,4,4,5,7)$	$-4 + 3\times\frac{1}{2}+2\times\frac{2}{3}+\frac{2\cdot5}{7}$
93	$X \subset \mathbb{P}(2,3,3,4,5,8,11)$	$-4 + 2\times\frac{1}{2}+\frac{2}{3}+\frac{4\cdot7}{11}$
94	$X \subset \mathbb{P}(2,3,4,5,5,7,12)$	$-4 + \frac{1}{2}+\frac{3}{4}+\frac{5\cdot7}{12}$
95	$X \subset \mathbb{P}(2,3,4,5,5,5,6)$	$-4 + 2\times\frac{1}{2}+\frac{4}{5}+2\times\frac{2\cdot3}{5}$
96	$X \subset \mathbb{P}(2,3,4,5,5,6,8)$	$-4 + 3\times\frac{1}{2}+\frac{4}{5}+\frac{3\cdot5}{8}$
97	$X \subset \mathbb{P}(2,3,4,5,5,7,9)$	$-4 + \frac{3}{4}+\frac{2\cdot3}{5}+\frac{4\cdot5}{9}$
98	$X \subset \mathbb{P}(3,4,4,5,6,7,10)$	$-4 + \frac{1}{2}+2\times\frac{3}{4}+\frac{3\cdot7}{10}$
99	$X \subset \mathbb{P}(2,3,4,4,5,5,7)$	$-4 + 2\times\frac{1}{2}+2\times\frac{3}{4}+\frac{3\cdot4}{7}$
100	$X \subset \mathbb{P}(2,5,6,7,7,8,9)$	$-4 + 2\times\frac{1}{2}+\frac{6}{7}+\frac{4\cdot5}{9}$
101	$X \subset \mathbb{P}(2,3,4,5,7,7,9)$	$-4 + \frac{1}{2}+\frac{2\cdot5}{7}+\frac{4\cdot5}{9}$
102	$X \subset \mathbb{P}(2,3,4,5,7,9,11)$	$-4 + \frac{2}{3}+\frac{3}{4}+\frac{5\cdot6}{11}$
103	$X \subset \mathbb{P}(2,3,3,4,5,5,7)$	$-4 + 2\times\frac{2}{3}+\frac{2\cdot3}{5}+\frac{3\cdot4}{7}$
104	$X \subset \mathbb{P}(3,4,5,6,7,10,13)$	$-4 + \frac{1}{2}+\frac{4}{5}+\frac{4\cdot9}{13}$
105	$X \subset \mathbb{P}(3,4,5,5,6,7,9)$	$-4 + 2\times\frac{2}{3}+\frac{2\cdot3}{5}+\frac{2\cdot7}{9}$
106	$X \subset \mathbb{P}(4,4,5,6,7,9,13)$	$-4 + 2\times\frac{1}{2}+\frac{3}{4}+\frac{3\cdot10}{13}$
107	$X \subset \mathbb{P}(3,4,5,5,6,7,9)$	$-4 + 2\times\frac{2}{3}+\frac{2\cdot3}{5}+\frac{2\cdot7}{9}$
108	$X \subset \mathbb{P}(3,4,4,5,6,7,7)$	$-4 + \frac{1}{2}+\frac{3}{4}+2\times\frac{2\cdot5}{7}$
109	$X \subset \mathbb{P}(3,4,4,5,6,7,11)$	$-4 + \frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\frac{3\cdot8}{11}$
110	$X \subset \mathbb{P}(3,4,5,5,6,7,8)$	$-4 + \frac{2}{3}+\frac{3}{4}+\frac{4}{5}+\frac{3\cdot5}{8}$
111	$X \subset \mathbb{P}(4,5,5,6,7,8,9)$	$-4 + \frac{1}{2}+\frac{4}{5}+\frac{2\cdot3}{5}+\frac{2\cdot7}{9}$
112	$X \subset \mathbb{P}(3,4,5,5,6,7,7)$	$-4 + \frac{2}{3}+\frac{4}{5}+\frac{2\cdot3}{5}+\frac{2\cdot5}{7}$
113	$X \subset \mathbb{P}(2,3,3,4,5,5,8)$	$-4 + \frac{1}{2}+\frac{2}{3}+\frac{2\cdot3}{5}+\frac{3\cdot5}{8}$
		continued on next page

Table 5.2: Weighted K3 surfaces in codimension 4

Nb.	Weighted K3 surface	Resolution Data
114	$X \subset \mathbb{P}(2,3,4,5,5,6,7)$	$-4 + 2\times\frac{1}{2}+\frac{2}{3}+\frac{4}{5}+\frac{3\cdot4}{7}$
115	$X \subset \mathbb{P}(2,3,3,4,4,5,5)$	$-4 + 2\times\frac{1}{2}+2\times\frac{2}{3}+\frac{3}{4}+\frac{2\cdot3}{5}$

Examples in codim 4, except #25, #84, #111, in Table 5.2 can be obtained by unprojecting from successive constructions of K3 surfaces in codimensions 2, 3 as in Example 5.11, and #84, #111 can only be constructed by unprojecting from #61, #65 in the list of codim 3 respectively. On the other hand, to check quasismooth there is some work that needs to be done. For #25, see Example 5.14.

Additionally, there are twenty three candidates for K3 surfaces in codim 4 which we do not understand well how to get from unprojection or something else.

Table 5.3: Candidates for weighted K3 surfaces in codimension 4

Nb.	Weighted K3 surface	Formal Resolution Data
1	$X \subset \mathbb{P}(2,2,3,4,5,5,6)$	$-4 + 7\times\frac{1}{2}+\frac{4}{5}$
2	$X \subset \mathbb{P}(3,3,4,4,5,5,6)$	$-4 + 4\times\frac{2}{3}+2\times\frac{3}{4}$
3	$X \subset \mathbb{P}(3,3,4,5,5,6,7)$	$-4 + 5\times\frac{2}{3}+\frac{4}{5}$
4	$X \subset \mathbb{P}(2,3,3,3,4,4,5)$	$-4 + 2\times\frac{1}{2}+5\times\frac{2}{3}$
5	$X \subset \mathbb{P}(2,2,3,3,3,4,4)$	$-4 + 6\times\frac{1}{2}+2\times\frac{2}{3}$
6	$X \subset \mathbb{P}(2,2,2,3,3,3,4)$	$-4 + 8\times\frac{1}{2}+\frac{2}{3}$
7	$X \subset \mathbb{P}(2,3,4,4,4,5,5)$	$-4 + 4\times\frac{1}{2}+3\times\frac{3}{4}$
8	$X \subset \mathbb{P}(2,5,6,7,8,8,9)$	$-4 + 4\times\frac{1}{2}+\frac{2\cdot3}{5}+\frac{7}{8}$
9	$X \subset \mathbb{P}(2,2,3,3,4,4,5)$	$-4 + 6\times\frac{1}{2}+\frac{2}{3}+\frac{3}{4}$
10	$X \subset \mathbb{P}(2,6,7,8,9,9,10)$	$-4 + 5\times\frac{1}{2}+\frac{2}{3}+\frac{8}{9}$

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Table 5.3: Candidates for weighted K3 surfaces in codimension 4

Nb.	Weighted K3 surface	Formal Resolution Data
11	$X \subset \mathbb{P}(2,4,5,6,7,7,8)$	$-4 + 5 \times \frac{1}{2} + \frac{3}{4} + \frac{6}{7}$
12	$X \subset \mathbb{P}(2,4,5,5,6,6,7)$	$-4 + 5 \times \frac{1}{2} + \frac{4}{5} + \frac{5}{6}$
13	$X \subset \mathbb{P}(4,5,6,7,7,8,9)$	$-4 + 2 \times \frac{3}{4} + \frac{5}{6} + \frac{3 \cdot 4}{7}$
14	$X \subset \mathbb{P}(3,5,6,7,7,8,9)$	$-4 + 3 \times \frac{2}{3} + \frac{2 \cdot 3}{5} + \frac{6}{7}$
15	$X \subset \mathbb{P}(3,4,4,5,5,6,7)$	$-4 + \frac{2}{3} + 3 \times \frac{3}{4} + \frac{2 \cdot 3}{5}$
16	$X \subset \mathbb{P}(3,6,7,8,8,9,10)$	$-4 + \frac{1}{2} + 4 \times \frac{2}{3} + \frac{7}{8}$
17	$X \subset \mathbb{P}(2,3,4,5,6,6,7)$	$-4 + 4 \times \frac{1}{2} + 2 \times \frac{2}{3} + \frac{5}{6}$
18	$X \subset \mathbb{P}(4,6,7,7,8,9,10)$	$-4 + 2 \times \frac{1}{2} + \frac{3}{4} + \frac{6}{7} + \frac{2 \cdot 5}{7}$
19	$X \subset \mathbb{P}(6,6,7,8,9,10,11)$	$-4 + 2 \times \frac{1}{2} + 2 \times \frac{2}{3} + \frac{5}{6} + \frac{6}{7}$
20	$X \subset \mathbb{P}(3,4,5,6,6,7,8)$	$-4 + \frac{1}{2} + 3 \times \frac{2}{3} + \frac{3}{4} + \frac{4}{5}$
21	$X \subset \mathbb{P}(2,3,4,4,5,5,6)$	$-4 + 4 \times \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5}$
22	$X \subset \mathbb{P}(5,6,6,7,8,9,10)$	$-4 + \frac{1}{2} + \frac{2}{3} + \frac{2 \cdot 3}{5} + 2 \times \frac{5}{6}$
23	$X \subset \mathbb{P}(4,5,6,6,7,8,9)$	$-4 + 2 \times \frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \frac{4}{5} + \frac{5}{6}$

Appendix

In this appendix we describe a construction of a quartic hypersurface $Y = X(4) \subset \mathbb{P}^3$ which touches a plane along a conic C and in general has six nodes. Then we show that this leads to a K3 surface $X(4,6) \subset \mathbb{P}(1,2,2,2,3)$ with six nodes which is a (Y,C)-model. This means that $X(4,6) = \operatorname{Proj} R(Y,C)$ where $R(Y,C) = \bigoplus_{n \geq 0} H^0(Y, \mathcal{O}_Y(nC))$.

Let y_1, y_2, y_3 be homogeneous coordinates on $\mathbb{P}^2$. The cone C' is given by the zero locus of $f = y_1 y_3 - y_2^2$. Now define a quartic hypersurface Y

$$(f^2 + y_0 g = 0) \subset \mathbb{P}^3 = \mathbb{P}(y_0, \dots, y_3)$$

where g is a general homogeneous polynomial of degree 3 in $k[y_0, \dots, y_3]$. It is easy to see that Y is nonsingular away from $C\colon y_0 = f = 0$ by Bertini's theorem and in general has six nodes along C. The hyperplane section ($y_0 =$

0) on Y is $Y \cap (y_0 = 0) = 2C$ satisfying $(2C)^2 = 4$. So Y is a $(Y, 2C)$-model. By the Riemann–Roch formula on a singular surface we have $h^0(Y, C) = 1$. This implies that there exists a generator x of degree 1 such that $y_0 = x^2$. Since $h^0(Y, 4C - C) = h^0(Y, 3C) = 5$ there is a generator z of degree 3 such that $xz = f$, which gives a relation of degree 4. By substituting this relation into the defining equation of Y we get another relation, namely $(z^2 + g = 0)$, of degree 6. Define $X(4, 6)$ by

$$(xz - f = z^2 + g = 0) \subset \mathbb{P}(1, 2, 2, 2, 3).$$

One can easily show that we do not get any new generators and relations by inspecting $H^0(Y, nC)$ for $n > 3$ (or use the same strategy as in Example 5.14 assuming that there are new generators or relations in order to get a contradiction). Therefore $X(4, 6)$ is a (Y, C)-model.

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