Hilbert series and applications to graded rings

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1 Introduction

We work with graded rings $R = \bigoplus_{n \geq 0} R_n$ that are finitely generated over a base field k, and satisfy $R_0 = k$. The *Hilbert function* of R is the numerical function $P_n = \dim R_n$ for $n \geq 0$; the *Hilbert series* $P(t)$ or $P_R(t)$ of R is the formal power series defined by $P(t) = \sum P_n t^n$. It is elementary and well known that $P(t)$ is a rational function of t. In fact if $x_1, \ldots, x_d$ are homogeneous elements of weight $\operatorname{wt} x_i = a_i$ generating R (or more generally, generating a subring over which R is finite) then $\prod(1 - t^{a_i})P(t) = Q(t)$ is a polynomial.

This paper contains a number of practical remarks on Hilbert series that we expect to be useful in various contexts: we use the "fractional Riemann–Roch formula" of Fletcher and Reid [8, Chapter III] to write out explicit formulas for $P(t)$ in a number of cases of interest for singular surfaces and 3-folds. If X is a $\mathbb{Q}$-Fano 3-fold and $S \in |-K_X|$ a K3 surface in its anticanonical system, polarised with $D = \mathcal{O}_S(-K_X)$, we determine the relation between $P_X(t)$ and $P_{S,D}(t)$. We discuss the denominator $\prod(1 - t^{a_i})$ of $P(t)$, and in particular the question of how to choose a reasonably small denominator. This idea has applications to finding K3 surfaces and Fano 3-folds whose corresponding graded rings have small codimension.

This work derives mainly from my University of Warwick Ph.D. thesis [1], and I would like to thank both Miles Reid and Alessio Corti for their help and encouragement at different times.

2 Fractional Riemann–Roch formula

2.1 Surfaces with Du Val singularities

We use the definitions and notation of Reid [8] for singularities. If S is a projective surface with Du Val singularities and D a Weil divisor on S, then some multiple rD is Cartier, and there is a formula [8, Theorem 9.1]

$$\chi(S, \mathcal{O}_S(D)) = \chi(\mathcal{O}_S) + \frac{1}{2}(D^2 - DK_S) + \sum_P c_P(D), \tag{2.1}$$

where $c_P(D) \in \mathbb{Q}$ is a fractional contribution due to the singularity of S and $\mathcal{O}_S(D)$ at P. Here $DK_S \in \mathbb{Z}$ is the intersection number with the canonical class and $D^2 \in \mathbb{Q}$ the self-intersection of the $\mathbb{Q}$-Cartier divisor D. Moreover, $c_P(D)$ can be written as a sum

$$c_P(D) = - \sum_{(r,a)\in\mathcal{B}} \frac{a(r-a)}{2r} \tag{2.2}$$

over a *basket* $\mathcal{B} = \{(r,a)\}$ with $0 < a < r$ and each a coprime to r. Here the basket appears simply as a list of combinatorial data for computing the right-hand side of (2.2); Reid [8, §9] interprets (2.1) and (2.2) as saying that the singularity of S and $\mathcal{O}_S(D)$ at P has the same effect on $\chi(S, \mathcal{O}_S(D))$ as a basket $\mathcal{B}$ of virtual cyclic quotient singularities of type $\frac{1}{r}(1,-1)$, at which $\mathcal{O}_S(D)$ is locally isomorphic to the eigensheaf of ε^a.

It follows from this interpretation and the proof of [8, Theorem 9.1] that for $n \in \mathbb{Z}$,

$$c_P(nD) = - \sum_{(r,a)\in\mathcal{B}} \frac{\overline{na}(r-\overline{na})}{2r}, \tag{2.3}$$

where for each $(r,a) \in \mathcal{B}$ the bar $\overline{}$ stands for smallest positive residue modulo r.

Lemma 2.1 *Let S be a surface with Du Val singularities and D a Weil divisor. We assume that $H^i(S, \mathcal{O}_S(nD)) = 0$ for all $i > 0$ and for all $n \geq 1$.*

Then the graded ring $R(S,D) = \bigoplus_{n\geq 0} H^0(S, \mathcal{O}_S(nD))$ *has Hilbert series*

$$P_S(t) \;=\; \frac{1+(\chi-1)t}{1-t} + \frac{t+t^2}{2(1-t)^3}D^2 - \frac{t}{2(1-t)^2}DK_S - \sum_{(r,a)\in\mathcal{B}} \frac{\sum_{n=1}^{r-1} \overline{an}(r-\overline{an})t^n}{2r(1-t^r)}. \tag{2.4}$$

Here $\chi = \chi(\mathcal{O}_S)$.

Proof The first three terms of (2.4) expand out as

$$1+\chi t+\chi t^2+\cdots, \quad (t+2^2t^2+3^2t^3+\cdots)D^2/2$$
$$\text{and} \quad -(t+2t^2+3t^2+\cdots)DK_S/2$$

respectively, corresponding to the sum over nD of first three terms of (2.1). For each element of the basket, the denominator $1-t^r$ has the effect of repeating the contribution $-\overline{na}(r-\overline{na})/2r$ from (2.3) periodically over the intervals $[0,r]$, $[r,2r]$, ... (zero at the endpoints), giving the last term of (2.1). $\square$

2.2 3-folds with canonical singularities

Let X be a projective 3-fold with canonical singularities and A a Weil divisor. To use the formulas of [8, §10], assume that at every singular point $P\in X$, we have $\mathcal{O}_X(A) \cong \mathcal{O}_X(kK_X)$ for some k (possibly depending on P). Then [8, Theorem 10.2] says that

$$\chi(X,A) = \chi(\mathcal{O}_X) + \frac{1}{6}A^3 - \frac{1}{4}A^2K_X + \frac{1}{12}A(K_X^2+c_2) + \sum c_Q(A), \tag{2.5}$$

where $c_Q(A)$ is a sum $\sum_{\mathcal{B}} c(r,a,k)$ taken over a basket $\mathcal{B} = \{(r,a,k)\}$ with a and k coprime to r, and the contributions are

$$c(r,a,k) = -\frac{r^2-1}{12r}k + \sum_{i=1}^{k-1} \frac{\overline{ai}(r-\overline{ai})}{2r} \tag{2.6}$$

$$= \frac{r^2-1}{12r}(r-k) - \sum_{i=k}^{r-1} \frac{\overline{ai}(r-\overline{ai})}{2r}. \tag{2.7}$$

The interpretation here is that $c(r,a,k)$ is the contribution from a singularity of type $\frac{1}{r}(a,1,-1)$ at which A is locally the ε^{ak} eigensheaf.

One can write out the other terms in the Hilbert series by analogy with Lemma 2.1 (see the following section). The contribution made by each element $(r,a,k) \in \mathcal{B}$ to the Hilbert series is thus

$$\sum_{n=1}^{\infty} c(r,a,nk)t^n = \frac{1}{1-t^r}\sum_{n=1}^{r-1} c(r,a,nk)t^n. \tag{2.8}$$

Remark 2.2 The two formulas (2.6–2.7) are equal because

$$\sum_{i=1}^{r-1} \overline{ai}(r-\overline{ai}) = \sum_{i=1}^{r-1} i(r-i) = \frac{1}{6}r(r^2-1); \tag{2.9}$$

moreover, they also hold for k not in $[0,r]$ (the expression only depends on k modulo r).

In dealing with a single divisor D, we may assume that k is coprime to r; but the proof of [8, §10] is valid for any k. Thus we can use formulas (2.6–2.7) for the contribution $c(r,a,nk)$ for any n in calculating $\chi(nD)$.

3 Fano 3-folds and K3 surfaces

3.1 Fano 3-folds

Let X be a Fano 3-fold, that is, a 3-fold with canonical singularities and ample anticanonical class $A=-K_X$. Standard use of vanishing gives

$$P_n = h^0(X,nA) = \chi(\mathcal{O}(nA)) \quad \text{for } n \geq 0.$$

Now $\chi(\mathcal{O}_X)=1$ and the basket of X is $\mathcal{B}=\{(r,a,-1)\}$, and by [8, Cor. 10.3], we have

$$\frac{1}{12}Ac_2 = -\frac{1}{12}K_Xc_2 = 2 - \sum_{\mathcal{B}} \frac{r^2-1}{12r}. \tag{3.1}$$

Using all this, the formulas (2.5–2.6) specialise to give

$$P_n = 2n+1+\frac{1}{12}n(n+1)(2n+1)A^3 + \sum_{\mathcal{B}}\left(-\frac{r^2-1}{12r}n + c(r,a,-n)\right). \tag{3.2}$$

Now consider $P_n - P_{n-1}$. To handle the second term we use

$$n(n+1)(2n+1) - (n-1)(n)(2n-1) = 6n^2.$$

By (2.7), the bracketed expression inside the sum over $\mathcal{B}$ equals

$$-\sum_{i=-n}^{r-1} \frac{\overline{ai}(r-\overline{ai})}{2r}, \tag{3.3}$$

and the difference from n to $n-1$ is just one term of the sum. We obtain

$$P_n - P_{n-1} = 2 + \frac{1}{2}n^2A^3 - \sum_{(r,a,-1)\in\mathcal{B}} \frac{\overline{an}(r-\overline{an})}{2r}. \tag{3.4}$$

Since $P_n - P_{n-1}$ is the coefficient of t^n in $(1-t)P_X(t)$, arguing as in Lemma 2.1 gives the following result:

Corollary 3.1

$$P_X(t) = \frac{1+t}{(1-t)^2} + \frac{t(1+t)}{2(1-t)^4}A^3 - \sum_{(r,a,-1)\in\mathcal{B}} \frac{\sum_{n=1}^{r-1}\overline{an}(r-\overline{an})t^n}{2r(1-t)(1-t^r)}. \tag{3.5}$$

3.2 K3 surfaces

For a K3 surface S with Du Val singularities and a Weil divisor D, Lemma 2.1 specialises to give the Hilbert series of $P_n = h^0(S, nD)$ in the form

$$P_S(t) = \frac{1+t}{1-t} + \frac{t(1+t)}{2(1-t)^3}D^2 - \sum_{(r,a)\in\mathcal{B}} \frac{\sum_{n=1}^{r-1}\overline{an}(r-\overline{an})t^n}{2r(1-t^r)}. \tag{3.6}$$

Corollary 3.2 *If X is a Fano 3-fold polarised by $A = -K_X$ and $S \in |-K_X|$ a K3 surface polarised by $D = A_{|S}$ then*

$$P_S(t) = (1-t)P_X(t). \tag{3.7}$$

This result follows of course from the restriction exact sequence

$$0 \to \mathcal{O}_X((n-1)A) \to \mathcal{O}_X(nA) \to \mathcal{O}_S(nD) \to 0.$$

The point, however, is that the corollary gives a formula for the Hilbert series $P_X(t)$ of the Fano 3-fold in terms of simpler data for a K3 surface. This formula is valid even if there is no K3 surface $S \in |-K_X|$, for example, if $P_1(X) = 0$ so that $|-K_X| = \emptyset$. Compare Corti, Pukhlikov and Reid [4], Remark 7.2.3.

4 Applications

In [1], we studied polarised K3 surfaces S in terms of their graded rings $R(S, D)$. We also studied Fano 3-folds X in a similar way. The first step in our strategy consisted of finding suitable weights a_0, a_1, a_2 so that

$$(1-t^{a_0})(1-t^{a_1})(1-t^{a_2})P(t)$$

is a polynomial with positive coefficients, and a_0, a_1, a_2 "fairly small". This is a combinatorial analogue of finding a polynomial subring $k[x_0, x_1, x_2] \subset R(S, D)$ over which $R(S, D)$ is a finite free module of "fairly small rank". If $\mathcal{B} = \{(r, a)\}$ is the basket of S then each $(1-t^r)$ appears in the denominator, so that a first necessary condition for this is that each r divides some a_i.

We have the obvious bound $\sum n \leq 19$ for the number and types of Du Val singularities on a K3 surface S. This implies the bound $\sum (r-1) \leq 19$ for the basket $\mathcal{B} = \{(r, a)\}$ on a K3 surface. For a Fano 3-fold X, a similar bound on the basket $\{(r, a, -1)\}$ is provided by an argument of Kawamata [7]: he proves first that $Ac_2 > 0$; then (3.1) implies

$$\sum \left(r - \frac{1}{r} \right) < 24.$$

Thus there are only finitely many possibilities for the basket $\mathcal{B}$. An easy calculation shows that in either case at most 5 distinct values of r occur.

In [1] we develop a procedure based on these ideas to find all possible Hilbert series for K3 surfaces and Fano 3-folds with graded ring of given codimension. We give explicit lists [2] of codimension 3 and 4 cases (comparable to the lists of hypersurfaces and codimension 2 in Fletcher [6]), and in most cases settle the question of existence of the varieties.

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