Flops, Type III contractions and Gromov–Witten invariants on Calabi–Yau threefolds

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1 Introduction

In this paper, we investigate Gromov–Witten invariants associated to exceptional classes for primitive birational contractions on a Calabi–Yau threefold X. As already remarked in [18], these invariants are locally defined, in that they can be calculated from knowledge of an open neighbourhood of the exceptional locus of the contraction; intuitively, they are the numbers of rational curves in such a neighbourhood. In §2, we make this explicit in the case of Type I contractions, where the exceptional locus is by definition a finite set of rational curves. Associated to the contraction, we have a flop; we deduce furthermore in Proposition 2.1 that the changes to the basic invariants (the cubic form on $H^2(X,\mathbb{Z})$ given by cup product, and the linear form given by cup product with the second Chern class c_2) under the flop are explicitly determined by the Gromov–Witten invariants associated to the exceptional classes.

The main results of this paper concern the Gromov–Witten invariants associated to classes of curves contracted under a Type III primitive contraction. Recall [17] that a primitive contraction $\varphi\colon X \to \overline{X}$ is of Type III if it contracts down an irreducible divisor E to a curve of singularities C. For X a smooth Calabi–Yau threefold, such contractions were studied in [18]; in particular, it was shown there that the curve C is smooth and that E is a conic bundle over C. We denote by $2\eta \in H_2(X,\mathbb{Z})/\operatorname{Tors}$ the numerical class of a fibre of E over C. In the case when E is a $\mathbb{P}^1$-bundle over C, this may in fact be a primitive class, and so the notation is at slight variance with that adopted in §2, where η is assumed to be the primitive class. In the case when the class of a fibre is not primitive (for instance, when E is not a $\mathbb{P}^1$-bundle over C), the primitive class contracted by φ will be η. We denote the Gromov–Witten numbers associated to η and 2η by n_1 and n_2, with the convention that $n_1 = 0$ if 2η is the primitive class. The above conventions have been adopted so as to achieve consistency of notation for all Type III contractions.

If the genus g of the curve C is strictly positive, under a general holomorphic deformation of the complex structure on X, the divisor E disappears leaving only finitely many of its fibres, and (except in the case of elliptic quasiruled surfaces, where all the Gromov–Witten invariants vanish) we have a Type I contraction. The results of §2 may then be applied to deduce the Gromov–Witten invariants associated to the classes $m\eta$ for $m > 0$. These are all determined by the Gromov–Witten numbers n_1 and n_2, and explicit formulas for n_1 and n_2 are given in Proposition 3.3; in particular $n_2 = 2g - 2$.

The formulas for n_1 and n_2 remain valid also for $g = 0$, although the slick proof given in Proposition 3.3 for the case $g > 0$ no longer works. The formula for n_1 is proved for all values of $g(C)$ by local deformation arguments in Theorem 3.5. Verifying that $n_2 = -2$ in the case when $g(C) = 0$ is rather more difficult, and involves the technical machinery of moduli spaces of stable pseudoholomorphic maps and the virtual neighbourhood method, as used in [2, 9] in order to construct Gromov–Witten invariants for general symplectic manifolds. In particular, we shall need a cobordism result from [13], which we show in Theorem 4.1 applies directly in the case where no singular fibre of E is a double line. The general case may be reduced to this one by making a suitable almost complex small deformation of complex structure. In §5, we give an application of our calculations. In [18], it was shown that if X_1, X_2 are Calabi–Yau threefolds which are symplectic deformations of each other (and general in their complex moduli), then their Kähler cones are the same. Now we can deduce (Corollary 5.1) that corresponding codimension one faces of these cones have the same contraction type.

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2 Flops and Gromov–Witten invariants

If X is a smooth Calabi–Yau threefold with Kähler cone $\mathcal{K}$, then the nef cone $\overline{\mathcal{K}}$ is locally rational polyhedral away from the cubic cone

$$W^* = \left\{D \in H^2(X, \mathbb{R}) \ ; \ D^3 = 0\right\};$$

moreover, the codimension one faces of $\overline{\mathcal{K}}$ (not contained in W^*) correspond to primitive birational contractions $\varphi\colon X \to \overline{X}$ of one of three different types [17].

In the numbering of [17], Type I contractions are those where only a finite number of curves (in fact $\mathbb{P}^1$s) are contracted. The singular threefold $\overline{X}$ then has a finite number of cDV singularities. Whenever one has such a small contraction on X, there is a flop of X to a different birational model X', also admitting a birational contraction to $\overline{X}$; moreover, identifying $H^2(X', \mathbb{R})$ with $H^2(X, \mathbb{R})$, the nef cone of X' intersects the nef cone of X along the codimension one face which defines the contraction to $\overline{X}$ [6, 7]. It is well known [7] that X' is smooth, projective and has the same Hodge numbers as X, but that the finer invariants, such as the cubic form on $H^2(X, \mathbb{Z})$ given by cup product, and the linear form on $H^2(X, \mathbb{Z})$ given by cup product with $c_2(X) = p_1(X)$, will in general change. Recall that, when X is simply connected, these two forms along with $H^3(X, \mathbb{Z})$ determine the diffeomorphism class of X up to finitely many possibilities [14], and that if furthermore $H_2(X, \mathbb{Z})$ is torsion free, this information determines the diffeomorphism class precisely [16].

When the contraction $\varphi\colon X \to \overline{X}$, corresponding to such a *flopping face* of $\overline{\mathcal{K}}$, contracts only isolated $\mathbb{P}^1$s with normal bundle $(-1, -1)$ (that is, $\overline{X}$ has only simple nodes as singularities), then it is a standard calculation to see how the above cubic and linear forms (namely the cup product $\mu\colon H^2(X, \mathbb{Z}) \to \mathbb{Z}$, and the form $c_2\colon H^2(X, \mathbb{Z}) \to \mathbb{Z}$) change on passing to X' under the flop. Since any flop is an isomorphism in codimension one, we have natural identifications

$$H^2(X', \mathbb{R}) \cong \mathrm{Pic}_{\mathbb{R}}(X') \cong \mathrm{Pic}_{\mathbb{R}}(X) \cong H^2(X, \mathbb{R}).$$

If we are in the case where the exceptional curves $C_1, \ldots, C_N$ are isolated $\mathbb{P}^1$s with normal bundle $(-1, -1)$, and if we denote by D' the divisor on X' corresponding to D on X, then

$$(D')^3 = D^3 - \sum (D \cdot C_i)^3 \quad \text{and} \quad c_2(X') \cdot D' = c_2(X) \cdot D + 2 \sum D \cdot C_i \ .$$

This is an easy verification – see for instance [1].

Proposition 2.1 *Suppose that X is a smooth Calabi–Yau threefold, and $\varphi\colon X \to \overline{X}$ is any Type I contraction, with X' denoting the flopped Calabi–Yau threefold. The cubic and linear forms $(D')^3$ and $D' \cdot c_2(X')$ on X' are then explicitly determined by the cubic and linear forms D^3 and $D \cdot c_2(X)$ on X, and the 3-point Gromov–Witten invariants Φ_A on X, for $A \in H_2(X, \mathbb{Z})$ ranging over classes which vanish on the flopping face.*

Remark 2.2 This is essentially the statement from physics that the A-model 3-point correlation function on $\mathcal{K}(X)$ may be analytically continued to give the A-model 3-point correlation function on $\mathcal{K}(X')$.

Proof We use the ideas from [18]; in particular, we know that on a suitable open neighbourhood of the exceptional locus of φ, there exists a small holomorphic deformation of the complex structure for which the exceptional locus splits up into disjoint $(-1, -1)$-curves ([18], Proposition 1.1).

Let $A \in H_2(X, \mathbb{Z})$ be a class with $\varphi_* A = 0$. The argument from [18], Section 1 then shows how the Gromov–Witten invariants $\Phi_A(D, D, D)$ can be calculated from local information. Having fixed a Kähler form ω on X, a small deformation of the holomorphic structure on a neighbourhood of the exceptional locus may be patched together in a C^∞ way with the original complex structure to yield an almost complex structure tamed by ω, and the Gromov–Witten invariants can then be calculated in this almost complex structure. The Gromov Compactness Theorem is used in this argument to justify the fact that all of the pseudoholomorphic rational curves representing the class A have images which are $(-1, -1)$-curves in the deformed local holomorphic structure.

Here we also implicitly use the Aspinwall–Morrison formula for the contribution to Gromov–Witten invariants from multiple covers of infinitesimally rigid $\mathbb{P}^1$s, now proved mathematically by Voisin [15]. So if $n(B)$ denotes the number of $(-1, -1)$-curves representing a class given B, then

$$\Phi_A(D, D, D) = (D \cdot A)^3 \sum_{kB = A} n(B)/k^3,$$

where the sum is taken over all integers $k > 0$ and classes $B \in H_2(X, \mathbb{Z})$ such that $kB = A$. So if $H_2(X, \mathbb{Z})$ is torsion free and A is the primitive class vanishing on the flopping face, this says that

$$\Phi_{mA}(D, D, D) = (D \cdot A)^3 \sum_{d|m} n(dA) d^3.$$

Recall that the Gromov–Witten invariants used here are the ones (denoted $\widetilde{\Phi}$ in [12]) which count marked parametrized curves satisfying a perturbed pseudoholomorphicity condition. Knowledge of the numbers $n(A)$ for the classes A with $\varphi_* A = 0$ determines the Gromov–Witten invariants Φ_A for classes A with $\varphi_* A = 0$, and vice-versa.

If we can now show that the local contributions to $(D')^3$ and $D' \cdot c_2(X')$ are well-defined and invariant under the holomorphic deformations of complex structure we have made locally, then the obvious formulas for them will hold. Let $\eta \in H_2(X, \mathbb{Z})/\operatorname{Tors}$ be the primitive class with $\varphi_* \eta = 0$ and n_d denote the total number of $(-1, -1)$-curves on the deformation which have numerical class $d\eta$; the n_d are therefore nonnegative integers (cf. [10], Remark 7.3.6). Then

$$(D')^3 = D^3 - (D \cdot \eta)^3 \sum_{d > 0} n_d d^3, \tag{2.1.1}$$

$$D' \cdot c_2(X') = D \cdot c_2(X) + 2(D \cdot \eta) \sum_{d > 0} n_d d. \tag{2.1.2}$$

To justify the premise in the first sentence of the paragraph, the basic result needed is that of local conservation of number, as stated in [3], Theorem 10.2.

For calculating the change in D^3 for instance, let X now denote the neighbourhood of the exceptional locus of φ and $\pi\colon \mathcal{X} \to B$ the small deformation under which the exceptional locus splits up into $(-1,-1)$-curves. So we have a regular embedding (of codimension six)

$$\begin{array}{ccc} \mathcal{X} & \hookrightarrow & \mathcal{X} \times \mathcal{X} \times \mathcal{X} \ = \mathcal{Y} \\ \downarrow & & \downarrow \\ B & = & B \end{array}$$

In order to calculate the triple products $D'_1 \cdot D'_2 \cdot D'_3$ from $D_1 \cdot D_2 \cdot D_3$ and the numbers n_d, we may assume *wlog* that the D_i are very ample, and so in particular we get effective divisors $\mathcal{D}_1$, $\mathcal{D}_2$ and $\mathcal{D}_3$ on $\mathcal{X}/B$. Applying [8], Theorem 11.10, we can flop in the family $\mathcal{X} \to B$, hence obtaining a deformation $\mathcal{X}' \to B$ of the flopped neighbourhood X'. We wish to calculate the local contribution to $D'_1 \cdot D'_2 \cdot D'_3$; with the notation as in [3], Theorem 10.2, we have a fibre square

$$\begin{array}{ccc} \mathcal{W} & \longrightarrow & \mathcal{D}'_1 \times \mathcal{D}'_2 \times \mathcal{D}'_3 \\ \downarrow & & \downarrow \\ \mathcal{X}' & \longrightarrow & \mathcal{X}' \times \mathcal{X}' \times \mathcal{X}' \end{array}$$

with $\mathrm{Supp}(\mathcal{W}) = \bigcap \mathrm{Supp}(\mathcal{D}'_i)$. Furthermore, we may assume that the divisors $\mathcal{D}_i$ were chosen so that $\mathcal{D}_1 \cap \mathcal{D}_2 \cap \mathcal{D}_3$ has no points in $\mathcal{X}$, and so in particular $\mathcal{W}$ is proper over B. Letting $D'_i(t)$ denote the restriction of $\mathcal{D}'_i$ to the fibre X'_t, we therefore have a well-defined local contribution to $D'_1(t) \cdot D'_2(t) \cdot D'_3(t)$ (concentrated on the flopping locus of X'_t), which is moreover independent of $t \in B$. Thus by making the local calculation as in (7.4) of [1], we deduce that

$$D_1 \cdot D_2 \cdot D_3 - D'_1 \cdot D'_2 \cdot D'_3 = (D_1 \cdot \eta)(D_2 \cdot \eta)(D_3 \cdot \eta) \sum_{d>0} n_d d^3$$

as required.

The proof for $c_2 \cdot D$ is similar. Here we consider the graph $\widetilde{X} \subset X \times X'$ of the flop, with $\pi_1\colon \widetilde{X} \to X$ and $\pi_2\colon \widetilde{X} \to X'$ denoting the two projections, and $E \subset \widetilde{X}$ the exceptional divisor for both π_1 and π_2. Then $\pi_2^*(T_{X'})_{|\widetilde{X}\setminus E} = \pi_1^*(T_X)_{|\widetilde{X}\setminus E}$, and so in particular $c_2(\pi_2^* T_{X'}) - c_2(\pi_1^* T_X)$ is represented by a 1-cycle Z on E. Suppose *wlog* that D is very ample, and that D' denotes the corresponding divisor on X'. Set $\pi_1^* D = \widetilde{D}$ and $\pi_2^* D' = \widetilde{D} + F$, with F supported on E. Then $c_2(X') \cdot D' = c_2(\pi_2^* T_{X'}) \cdot (\widetilde{D} + F)$. Hence

$$c_2(X') \cdot D' - c_2(X) \cdot D = c_2(\pi_2^* T_{X'}) \cdot F + Z \cdot \widetilde{D} = c_2((\pi_2^* T_{X'})_{|F}) + (Z \cdot \widetilde{D})_E$$

where the right-hand side is purely local. Note the slight abuse of notation here that F denotes also the fixed *scheme* for the linear system $|\pi_2^* D'|$.

Now taking X to be a local neighbourhood of the flopping locus, and taking a small deformation $\mathcal{X} \to B$ as before, we obtain families $\mathcal{X}'$, $\widetilde{\mathcal{X}}$, $\mathcal{D}$, $\mathcal{E}$, $\mathcal{F}$ and $\mathcal{Z}$ over B (corresponding to X', $\widetilde{X}$, D, E, F and Z). For ease of notation, we shall use π_1 and π_2 also for the morphisms of families $\widetilde{\mathcal{X}} \to \mathcal{X}$, respectively $\widetilde{\mathcal{X}} \to \mathcal{X}'$. Applying [3], Theorem 10.2 to the family of vector bundles $(\pi_2^* T_{\mathcal{X}'/B})_{|\mathcal{F}}$ on the scheme $\mathcal{F}$ over B yields that $c_2((\pi_2^* T_{\mathcal{X}'/B})_{|F_t})$ is independent of $t \in B$. Noting that $\widetilde{\mathcal{D}} \hookrightarrow \widetilde{\mathcal{X}}$ is a regular embedding, we apply the same theorem to the fibre square

$$\begin{array}{ccc}
\widetilde{\mathcal{D}} \times_{\widetilde{\mathcal{X}}} \mathcal{E} & \longrightarrow & \mathcal{E} \\
\downarrow & & \downarrow \\
\widetilde{\mathcal{D}} & \longrightarrow & \widetilde{\mathcal{X}}
\end{array}$$

and the cycle $\mathcal{Z}$ on $\mathcal{E}$. This yields that $(Z_t \cdot \widetilde{D}_t)_{E_t}$ on E_t is independent of $t \in B$, where by definition

$$Z_t = c_2(\pi_2^* T_{\mathcal{X}'/B})_{|X_t} - c_2(\pi_1^* T_{\mathcal{X}/B})_{|X_t}.$$

Thus the local contribution to $D'(t) \cdot c_2(X'_t)$ is well-defined and independent of t, and so we need only make the local calculation for generic t (where the exceptional locus of the flop consists of disjoint $(-1,-1)$-curves). This calculation may be found in [1], (7.4).

Speculation 2.3 There are reasons for believing that only the numbers n_1 and n_2 are nonzero, and hence that the Gromov–Witten invariants associated to classes $m\eta$ for $m > 2$ all arise from multiple covers. If this speculation is true, then the changes under flopping to the cubic form and the linear form would be determined by these two integers, and conversely.

3 Type III contractions and Gromov–Witten invariants

The main results of this paper concern the Gromov–Witten invariants associated to classes of curves contracted under a Type III primitive contraction. Recall [17] that a primitive contraction $\varphi\colon X \to \overline{X}$ is of Type III if it contracts down an irreducible divisor E to a curve of singularities C. For X a smooth Calabi–Yau threefold, such contractions were studied in [18]; in particular, it was shown there that the curve C is smooth and that E is a conic bundle over C. We denote by $2\eta \in H_2(X, \mathbb{Z})/\operatorname{Tors}$ the numerical class of a fibre of E over C. As explained in the Introduction, we denote by n_1 and n_2 the Gromov–Witten numbers associated to the classes η and 2η, where $n_1 = 0$ if E is a $\mathbb{P}^1$-bundle over C. If the generic fibre of E over C is reducible (consisting of two lines, each with class η), then, except in two cases, it follows from the arguments of [18], §4 that, by making a global holomorphic deformation of the complex structure, we may reduce down to the case where the generic fibre of E over C is irreducible. The two exceptional cases are:

(a) $g(C) = 1$ and E has no double fibres.

(b) $g(C) = 0$ and E has two double fibres.

However, Case (a) is an *elliptic quasi-ruled* surface in the terminology of [18], and hence disappears completely under a generic global holomorphic deformation. In particular, we know that all the Gromov–Witten invariants Φ_A are zero, for $A \in H_2(X, \mathbb{Z})$ having numerical class $m\eta$ for any $m > 0$.

In Case (b), E is a nonnormal generalized del Pezzo surface $\overline{\mathbb{F}}_{3;2}$ of degree 7 (see [18]). As argued there however, we may make a holomorphic deformation in a neighbourhood of E so that E deforms to a *smooth* del Pezzo surface of degree 7, and where the class η is then represented by either of two 'lines' on the del Pezzo surface (which are $(-1,-1)$-curves on the threefold); hence $n_1 = 2$. In fact, the smooth del Pezzo surface is fibred over $\mathbb{P}^1$ with one singular (line pair) fibre. The arguments we give below may be applied locally (more precisely with the global almost complex structure obtained by suitably patching the local small holomorphic deformation on an open neighbourhood of E with the original complex structure), and the Gromov–Witten invariants may be calculated as if the original contraction φ had contracted such a smooth del Pezzo surface of degree 7. In particular, $n_1 = 2$ comes from the two components of the singular fibre (Theorem 3.5), and $n_2 = -2$ is proved in §4 (see also Remark 3.4).

Let us therefore assume that the generic fibre of E over C is irreducible, and so in particular $E \to C$ is obtained from a $\mathbb{P}^1$-bundle over C by means of blowups and blowdowns. Moreover E itself is a conic bundle over C, and so its singular fibres are either line pairs or double lines.

Lemma 3.1 *In the above notation, E has only singularities on the singular fibres of the map $E \to C$. When the singular fibre is a line pair, we have an A_n singularity at the point where the two components meet (we include here the possibility $n = 0$ when the point is a smooth point of E). When the singular fibre is a double line, we have a D_n singularity on the fibre (here we need to include the case $n = 2$, where we in fact have two A_1 singularities, and $n = 3$, where we have an A_3 singularity).*

Proof The proof is obvious, once the correct statement has been found. The statement of this result in [17] omits (for fibre a double line) the cases D_n for $n > 2$.

Lemma 3.2 *Suppose that $E \to C$ as above has a_r fibres which are line pairs with an A_r singularity and b_s fibres which are double lines with a D_s singularity (for $r \geq 0$ and $s \geq 2$), then*

$$K_E^2 = 8(1-g) - \sum_{r\geq 0} a_r(r+1) - \sum_{s\geq 2} b_s s,$$

where g denotes the genus of C.

This enables us to give a slick calculation of the Gromov–Witten invariants when the base curve has genus $g > 0$. In this case, it was shown in [17] that for a generic deformation of X, only finitely many fibres from E deform, and hence the Type III contraction deforms to a Type I contraction. Thus Gromov–Witten numbers n_1 and n_2 may be defined as in Section 1, and are nonnegative integers.

Proposition 3.3 *When $g > 0$, we have*

$$n_1 = 2\sum_{r\geq 0} a_r(r+1) + 2\sum_{s\geq 2} b_s s \quad \textit{and} \quad n_2 = 2g-2.$$

Proof We take a generic 1-parameter deformation of X, for which the Type III contraction deforms to a Type I contraction. We therefore have a diagram

$$\begin{array}{ccc} \mathcal{X} & \longrightarrow & \overline{\mathcal{X}} \\ \downarrow & & \downarrow \\ \Delta & = & \Delta \end{array}$$

where $\Delta \subset \mathbb{C}$ denotes a small disc. Since the singular locus of $\overline{\mathcal{X}}$ consists only of curves of cDV singularities, we may again apply [8], Theorem 11.10 to deduce the existence of a (smooth) flopped fourfold $\mathcal{X}' \to \overline{\mathcal{X}}$. The induced family $\mathcal{X}' \to \Delta$ is given generically by flopping the fibres, and at $t = 0$ it is easily checked that $X_0' \cong X_0$; this operation is often called an *elementary transformation* on the family. Identifying the groups $H^2(X_t, \mathbb{Z}) \cong H^2(X_t', \mathbb{Z})$ as before, this has the effect (at $t = 0$) of sending E to $-E$ (cf. the discussion in [5], §3.3). So if E' denotes the class in $H^2(X_t', \mathbb{Z})$ corresponding to the class E in $H^2(X_t, \mathbb{Z})$, we have $(E')^3 = -E^3$. For $t \neq 0$, we just have a flop, and so

$(E')^3$ can be calculated from equation (2.1.1), namely $(E')^3 = E^3 + n_1 + 8n_2$. Therefore, using Lemma 3.2

$$n_1 + 8n_2 = -2E^3 = 16(g-1) + 2\sum_{r\geq 0} a_r(r+1) + 2\sum_{s\geq 2} b_s s.$$

Similarly, we have $c_2(X') \cdot E' = -c_2(X) \cdot E$, and so from equation (2.1.2) it follows that $2n_1 + 4n_2 = 2c_2 \cdot E$. An easy calculation of the right-hand side then provides the second equation

$$2n_1 + 4n_2 = 8(g-1) + 4\sum_{r\geq 0} a_r(r+1) + 4\sum_{s\geq 2} b_s s.$$

Solving for n_1 and n_2 from these two equations gives the desired result.

Remark 3.4 This result remains true even when $g = 0$, although the slick proof given above is no longer valid. The formula for n_1 is checked in Theorem 3.5 by local deformation arguments (for which the genus g is irrelevant), showing that the contribution to n_1 from a line pair fibre with A_r singularity is $2(r+1)$, and from a double line fibre with D_s singularity is $2s$. Let $A \in H_2(X, \mathbb{Z})$ denote the class of a fibre of $E \to C$. Observe that any pseudo-holomorphic curve representing the numerical class η will be a component of a singular fibre of $E \to C$. Moreover, the components l of a singular fibre represent the same class in $H_2(X, \mathbb{Z})$, and so in particular twice this class is A. Thus the Aspinwall–Morrison formula (as proved in [15]) yields the contribution to the Gromov–Witten invariants $\Phi_A(D, D, D)$ from double covers, purely in terms of n_1 and $D \cdot A$. The difference may be regarded as the contribution to $\Phi_A(D, D, D)$ from simple maps, and taking this to be $n_2(D \cdot A)^3$ determines the number n_2 (in §4, we shall see how n_2 may be determined directly from the moduli space of simple stable holomorphic maps). If $g > 0$, the above argument shows that this is in agreement with our previous definition, and yields moreover the equality $n_2 = 2g - 2$. The fact that $n_2 = -2$ when $g = 0$ requires a rather more subtle argument involving technical machinery – see Theorem 4.1. I remark that the value $n_2 = -2$ is needed in physics, and that there is also a physics argument justifying it (see [4], §5.2 and [5], §3.3) – essentially, it comes down to a statement about the A-model 3-point correlation functions. In §4 below, we give a rigorous mathematical proof of the assertion.

Theorem 3.5 *The formula for n_1 in Proposition 3.3 is valid irrespective of the value of the genus $g = g(C)$.*

Proof By making a holomorphic deformation of the complex structure on an open neighbourhood U in X of the singular fibre Z of $E \to C$, we may calculate the contribution to n_1 from that singular fibre – see [18], (4.1). The deformation of complex structure is obtained as in [18] by considering the one dimensional family of Du Val singularities in $\overline{X}$, and deforming this family locally in a suitable neighbourhood $\overline{U}$ of the dissident point. Our assumption is that the family $\overline{U} \to \Delta$ has just an A_1 singularity on $\overline{U}_t$ for $t \neq 0$, and we may assume also that $\overline{U} \to \Delta$ is a good representative (in the sense explained in [18]). The open neighbourhood U is then the blowup of $\overline{U}$ in the smooth curve of Du Val singularities ([18], p. 569). The contribution to n_1 may be calculated locally, and will not change when we make small holomorphic deformations of the complex structure on U, which in turn corresponds to making small deformations to the family $\overline{U} \to \Delta$.

First we consider the case where the singular fibre Z is a line pair – from this, it will follow that the dissident singularity on $\overline{U}$ is a $c\mathrm{A}_n$ singularity with $n > 1$, and that $\overline{U}$ has a local analytic equation of the form

$$x^2 + y^2 + z^{n+1} + tg(x, y, z, t) = 0$$

in $\mathbb{C}^3 \times \Delta$ (here t is a local coordinate on Δ, and $x = y = z = 0$ the curve C of singularities). For $t \neq 0$, we have an A_1 surface singularity, which implies that g must contain a term of the form $t^r z^2$ for some $r \geq 0$. By an appropriate analytic change of coordinates, we may then assume that $\overline{U}$ has a local analytic equation of the form

$$x^2 + y^2 + z^{n+1} + t^{r+1} z^2 + th(x, y, z, t) = 0,$$

where h consists of terms which are at least cubic in x, y, z. By making a small deformation of the family $\overline{U} \to \Delta$, we may reduce to the case $n = 2$, that is, $\overline{U}$ having local equation $x^2 + y^2 + z^3 + t^{r+1} z^2 + th = 0$. At this stage, we could in fact also drop the term th (an easy check using the versal deformation family of an A_2 singularity), but this will not be needed.

We now make a further small deformation to get $\overline{U}_\varepsilon \subset \mathbb{C}^3 \times \Delta$ given by a polynomial

$$x^2 + y^2 + z^3 + t^{r+1} z^2 + \varepsilon z^2 + th = x^2 + y^2 + z^2(z + t^{r+1} + \varepsilon) + th\ .$$

This then has $r + 1$ values of t for which the singularity is an A_2 singularity – for other values of t, it is an A_1 singularity. If we blow up the singular locus of $\overline{U}_\varepsilon$, we therefore obtain a smooth exceptional divisor for which $r + 1$ of the fibres over Δ are line pairs. By the argument of [18], (4.1), this splitting of the singular fibre into $r+1$ line pair singular fibres of the simplest type can be achieved by a local holomorphic deformation on a suitable open neighbourhood of the fibre in the original threefold X.

It is however clear that a line pair coming from a dissident cA_2 singularity of the above type contributes precisely two to the Gromov–Witten number n_1 – one for each line in the fibre. In terms of equations, we have a local equation for $\overline{X}$ of the form $x^2 + y^2 + z^3 + wz^2 = 0$; deforming this to say $x^2 + y^2 + z^3 + wz^2 + \varepsilon w = 0$, we get two simple nodes, and hence two disjoint $(-1,-1)$-curves on the resolution.

The argument of [18], (4.1) shows that the Gromov–Witten number n_1 may be calculated purely from these local contributions, and so the total contribution to n_1 from the line pair singular fibre of E with A_r singularity is indeed $2(r+1)$, as claimed.

For the case of the singular fibre Z of E being a double line, the dissident singularity must be cE_6, cE_7, cE_8, or cD_n for $n \geq 4$. Thus $\overline{U}$ has a local analytic equation of the form $f(x,y,z) + tg(x,y,z,t)$ in $\mathbb{C}^3 \times \Delta$ for f a polynomial of the appropriate type (t a local coordinate on Δ, and $x = y = z = 0$ the curve of singularities). To simplify matters, we may deform f to a polynomial defining a D_4 singularity, and hence make a small deformation of the family to one in which the dissident singularity is of type cD_4. We then have a local analytic equation of the form

$$x^2 + y^2 z + z^3 + tg(x,y,z,t) = 0.$$

For $t \neq 0$, we have an A_1 singularity, and so the terms of g must be at least quadratic in x, y, z. Moreover, by changing the x-coordinate, we may take the equation to be of the form

$$x^2 + y^2 z + z^3 + t^a y^2 + t^b yz + t^c z^2 + th(x,y,z,t) = 0,$$

with a, b, c positive, and where the terms of h are at least cubic in x, y, z. The fact that the blowup U of $\overline{U}$ in C is smooth is easily checked to imply that $a = 1$. Since

$$ty^2 + 2t^b yz = t(y + t^{b-1}z)^2 - t^{2b-1}z^2,$$

we have an obvious change of y-coordinate which brings the equation into the form

$$x^2 + y^2 z + z^3 + ty^2 + t^r z^2 + th_1(x,y,z,t) = 0,$$

where $r = \min\{c, 2b-1\}$ and h_1 has the same property as h.

When we blow up $\overline{U}$ along the curve $x = y = z = 0$, we obtain an exceptional locus E with a double fibre over $t = 0$, on which we have a D_{r+1} singularity (including the case $r = 1$ of two A_1 singularities, and $r = 2$ of an A_3 singularity). Moreover, this was also true of our original family, since the small deformation of f we made did not affect the local equation of the exceptional locus.

Moreover, by adding a term $\varepsilon_1 y^2 + \varepsilon_2 z^2$, we may deform our previous equation to one of the form

$$x^2 + y^2(z + t + \varepsilon_1) + z^2(z + t^r + \varepsilon_2) + th_1(x, y, z, t) = 0.$$

When $t + \varepsilon_1 = 0$, we have an A_3 singularity, and when $t^r + \varepsilon_2 = 0$, an A_2 singularity. Moreover, when we blow up the singular locus of this deformed family, the resulting exceptional divisor is smooth and has line pair fibres for these $r + 1$ values of t. Thus, as seen above, the contribution to n_1 from the original singular fibre (a double line with a D_{r+1} singularity) is $2(r + 1)$ as claimed.

4 Calculation of n_2 for Type III contractions

Let $\varphi\colon X \to \overline{X}$ be a Type III contraction on a Calabi–Yau threefold X, which contracts a divisor E to a (smooth) curve C of genus g. When $g > 0$, it was proved in Proposition 3.3 that the Gromov–Witten number n_2 (defined for arbitrary genus via Remark 3.4) is $2g - 2$. The purpose of this Section is to extend this result to include the case $g = 0$ (C is isomorphic to $\mathbb{P}^1$), and to prove $n_2 = 2g - 2$ in general.

Arguing as in [18], it is clear that the desired result is a local one, depending only on a neighbourhood of the exceptional divisor E. As remarked in §3, we may then always reduce down to the case that the generic fibre of $E \to C$ is irreducible. If all the fibres of $E \to C$ are smooth (so E is a $\mathbb{P}^1$-bundle over C), the fact that $n_2 = 2g - 2$ was proved in Proposition 5.7 of [11], using a cobordism argument. This latter result was extended by Ruan in [13], Proposition 2.10, using the theory of moduli spaces of stable maps and the virtual neighbourhood technique (cf. [2, 9]). If the singular fibres of $E \to C$ are line pairs, Ruan's result applies directly. We prove below that the linearized Cauchy–Riemann operator has constant corank for the stable (unmarked) rational curves given by the fibres of E over C, and hence by Ruan's result that there is an obstruction bundle $\mathcal{H}$ on C, with n_2 determined by the Euler class of $\mathcal{H}$. By Dolbeault cohomology, there is a natural identification of $\mathcal{H}$ with the cotangent bundle T_C^* on C, and hence the formula for n_2 follows. We note however that for Ruan's result to hold, we do not need an integrable almost complex structure on X. Provided we have a natural identification between the cokernel of the linearized Cauchy–Riemann operator and the cotangent space at the corresponding point of C, we can still deduce that $n_2 = 2g - 2$. In the general case of a Type III contraction which has double fibres, we show below that we can make a small local deformation of the almost complex structure on X so that E deforms to a family of pseudoholomorphic rational curves over C with at worst line pair singular fibres, and for which Ruan's method applies.

Theorem 4.1 *For any Type III contraction* $\varphi\colon X \to \overline{X}$*, the Gromov–Witten number* $n_2 = 2g-2$.

Proof We saw above that we may assume that the generic fibre of $E \to C$ is irreducible. Furthermore, we initially assume also that the singular fibres are all line pairs, and later reduce the general case to this one.

We let J denote the almost complex structure on X, which we know is integrable (at least in a neighbourhood of E), and tamed by a symplectic form ω. Let $A \in H_2(X,\mathbb{Z})$ be the class of a fibre of $E \to C$. Adopting the notation from [13], we consider the moduli space $\overline{\mathcal{M}}_A(X,J) = \overline{\mathcal{M}}_A(X,0,0,J)$ of stable unmarked rational holomorphic maps, a compactification of the space of (rigidified) pseudoholomorphic maps $\mathbb{CP}^1 \to X$, representing the class A. The theory of stable maps, as explained in Section 3 of [13], goes through for unmarked stable maps, by taking each component of the domain as a bubble component, and adding marked points (in addition to the double points) as in [13] in order to stabilize the components (thus taking a local slice of the automorphism group).

In the case that all the singular fibres of $E \to C$ are line pairs, $\overline{\mathcal{M}}_A(X,J)$ has two components, one corresponding to simple maps and the other to double covers. It is now a simple application of Gromov compactness to see that these two components are disjoint, since a sequence of double cover maps cannot converge to a simple map. A similar argument will show that for all almost complex structures J_t in some neighbourhood of $J = J_0$, the moduli space $\overline{\mathcal{M}}_A(X,J_t)$ will consist of two disjoint components, one corresponding to the simple maps and the other to the double covers.

Since any stable unmarked rational holomorphic map must be an embedding, it is clear that the component $\overline{\mathcal{M}}'_A(X,J)$ corresponding to the simple maps can be identified naturally with the smooth base curve C, and that for all almost complex structures in some neighbourhood of $J = J_0$, the moduli space $\overline{\mathcal{M}}'_A(X,J_t)$ of simple unmarked stable holomorphic maps is compact. The Gromov–Witten invariant n_2 that we seek can then be defined via Ruan's virtual neighbourhood invariant $\mu_{\mathcal{S}}$, and may be evaluated on (X,J) by using [13], Proposition 2.10.

Let us now go into more details of this. We consider C^∞ stable maps $f \in \overline{B}_A(X) = \overline{B}_A(X,0,0)$ in the sense of [13], Definition 3.1, where Ruan shows later in the same Section that the naturally stratified space $\overline{B}_A(X)$ satisfies a property which he calls *virtual neighbourhood technique admissible* or *VNA*, and as he says, for the purposes of the virtual neighbourhood construction, behaves as if it were a Banach V-manifold. Since any simple marked holomorphic stable map f in $\overline{\mathcal{M}}'_A(X,J)$ is forced to be an embedding, we may restrict our attention to C^∞ stable maps whose domain Σ comprises at most two $\mathbb{P}^1$s. We stratify $\overline{B}_A(X)$ according to the combinatorial type D of the domain Σ. Thus any $f \in \overline{\mathcal{M}}'_A(X,J)$ belongs to one of two strata of $\overline{B}_A(X)$.

In general, for k-pointed C^∞ stable maps of genus g, Ruan shows that for any given combinatorial type D, the substratum $B_D(X,g,k)$ is a Hausdorff Frechet V-manifold ([13], Proposition 3.6). As mentioned above, he needs to add extra marked points in order to stabilize the nonstable components of the domain Σ, thus taking a local slice of the action of the automorphism group on the unstable marked components of Σ. Moreover, the tangent space $T_f B_D(X,g,k)$ is identified with $\Omega^0(f^*T_X)$, as defined in his equation [13], (3.29). The tangent space $T_f\overline{B}_A(X,g,k)$ can then be defined as $T_f B_D(X,g,k)\times\mathbb{C}_f$, where $\mathbb{C}_f$ is the space of gluing parameters (see [13], equation before (3.67)).

In our case, however, things are a bit simpler. Given $f \in \overline{\mathcal{M}}'_A(X,J)$ with domain Σ consisting of two $\mathbb{P}^1$s, the tangent space $T_f\overline{B}_A(X)$ is of the form $\Omega^0(f^*T_X)\times\mathbb{C}$, and we have a neighbourhood $\widetilde{U}_f$ of f in $\overline{B}_A(X)$ defined by [13], (3.43), consisting of stable maps $\overline{f}^{v,w}$ parametrized locally by

$$\left\{w \in \Omega^0(f^*T_X)\ ;\ \|w\|_{C^1} < \varepsilon'\right\}$$

(corresponding to deformations within the stratum $B_D(X)$), and by $v \in \mathbb{C}_f^\varepsilon$ (an ε-ball in $\mathbb{C}_f = \mathbb{C}$ giving the gluing parameter at the double point). This then corresponds to the above decomposition of $T_f\overline{B}_A(X)$ into two factors. On the first factor, the linearization $D_f\overline{\partial}_J$ of the Cauchy–Riemann operator restricts to

$$\overline{\partial}_{J,f}\colon \Omega^0(f^*T_X)\to\Omega^{0,1}(f^*T_X)$$

in the notation of [13]. The index of this operator may be calculated using Riemann–Roch on each component of Σ (cf. the proof of Lemma 3.16 in [13], suitably modified to take account of the extra marked points), and is seen to be -2.

Let us now consider the stable maps $f^v = f^{v,0}$ for $v \in \mathbb{C}_f^\varepsilon\setminus\{0\}$. These are stable maps $\mathbb{CP}^1 \to X$ which differ from f only in small discs around the double point, and in this sense are approximately holomorphic. Set $v = re^{i\theta}$; then the gluing to get $f^v\colon \Sigma^v \to X$ is only performed in discs around the double point of radius $2r^2/\rho$ in the two components (ρ a suitable constant). It can then be checked for any $2 < p < 4$ that $\|\overline{\partial}_J(f^v)\|_{L^p_1} \le Cr^{4/p}$ (see [13] Lemma 3.23, and [10] Lemma A.4.3), from which it follows that the linearization

$$L_A = D_f\overline{\partial}_J$$

of the Cauchy–Riemann operator should be taken as zero on the factor $\mathbb{C}_f$ in $T_f\overline{B}_A(X)$. Thus we deduce that the index of L_A is zero, and that coker L_A is same as the cokernel of $\overline{\partial}_{J,f}\colon \Omega^0(f^*T_X) \to \Omega^{0,1}(f^*T_X)$, which by Dolbeault cohomology may be identified as

$$H^1(f^*T_X) = H^1(Z, T_{X|Z}),$$

where Z is the fibre of $E \to C$ (over a point $x \in C$) corresponding to the image of f.

We note that these are exactly the same results as are obtained in the smooth case, when Σ consists of a single $\mathbb{P}^1$. Here, we need to add three marked points to stabilize Σ, and Riemann–Roch then gives immediately that the index of L_A is zero.

Observe that Z is a complete intersection in X, and so for our purposes is as good as a smooth curve. Via the obvious exact sequence, $H^1(T_{X|Z})$ may be naturally identified with $H^1(N_{Z/X})$, which in turn may be naturally identified with $H^0(N_{Z/X})^*$ (since $K_Z = \bigwedge^2 N_{Z/X}$, we have a perfect pairing $H^0(N_{Z/X}) \times H^1(N_{Z/X}) \to H^1(K_Z) \cong \mathbb{C}$). Observing that $H^0(N_{Z/X}) = H^0(\mathcal{O}_Z \oplus \mathcal{O}_Z(E)) \cong \mathbb{C}$, we know that coker L_A has complex dimension one and is naturally identified with $T^*_{C,x}$, the dual of the tangent space at x to the Hilbert scheme component C. This we have seen is true for all $f \in \overline{\mathcal{M}}'_A(X,J)$.

We now apply [13], Proposition 2.10, (2) to our set-up, where $C = \overline{\mathcal{M}}'_A(X,J) = \mathcal{M}_{\mathcal{S}} = \mathcal{S}^{-1}(0)$ for $\mathcal{S}$ the Cauchy–Riemann section of $\overline{\mathcal{F}}_A(X)$ (as constructed in [13], §3) over a suitable neighbourhood of $\mathcal{M}_{\mathcal{S}}$ in $\overline{B}_A(X)$. The above calculations verify that the conditions of Proposition 2.10, (2) are satisfied, with $\mathrm{ind}(L_A) = 0$, $\dim(\mathrm{coker}\, L_A) = 2$ and $\dim(\mathcal{M}_{\mathcal{S}}) = 2$. Moreover, we deduce that the obstruction bundle $\mathcal{H}$ on $\mathcal{M}_{\mathcal{S}}$ is just the cotangent bundle T^*_C on C.

The Gromov–Witten number n_2 may then be defined to be $\mu_{\mathcal{S}}(1)$. It follows from the basic Theorem 4.2 from [13] that this is independent of any choice of tamed almost complex structure and is a symplectic deformation invariant. Thus by considering a small deformation of the almost complex structure and using [13], Proposition 2.10, (1), it is the invariant n_2 that we seek. Applying Ruan's crucial Proposition 2.10, (2), the invariant can be expressed as

$$\mu_{\mathcal{S}}(1) = \int_{\mathcal{M}'_A(X,J)} e(T^*_C),$$

from which it follows that $n_2 = 2g-2$ as claimed.

The general case (where $E \to C$ also has double fibres) can now be reduced to the case considered above. Suppose we have a point $Q \in C$ for which the corresponding fibre is a double line. We choose an open disc $\Delta \subset C$ with centre Q, and a neighbourhood U of Z in X, with U fibred over Δ, the fibre U_0 over Q containing the fibre Z. Letting $\overline{U} \to \Delta$ denote the image of U under φ, a family of surface Du Val singularities, we make a small deformation $\overline{\mathcal{U}} \to \Delta'$ of $\overline{U}$, as in the proof of Theorem 3.5 of this paper, and in this way obtain a holomorphic deformation $\mathcal{U} \to \Delta'$ of U under which $E_0 = E_{|\Delta}$ deforms to a family of surfaces E_t ($t \in \Delta'$), all fibred over Δ, and with at worst line pair singular fibres for $t \neq 0$. Considering $\overline{\mathcal{U}} \to \Delta \times \Delta'$ as a two parameter deformation of the surface singularity $\overline{U}_0$, we may take a good representative and apply Ehresmann's fibration theorem (with boundary) to the corresponding resolution $\mathcal{U} \to \Delta \times \Delta'$ (cf. [18], proof of Lemma 4.1). In this way, we may assume that $\mathcal{U} \to \Delta \times \Delta'$ is differentiably trivial over the base. In particular, the family $\mathcal{U} \to \Delta'$ is also differentiably trivial, and hence determines a holomorphic deformation of the complex structure on a fixed neighbourhood U of Z, where $U \to \Delta$ is also differentiably trivial.

We perform this procedure for each singular fibre $Z_1, \ldots, Z_N$ of $E \to C$, obtaining, for each i, an open neighbourhood U_i of Z_i fibred over $\Delta_i \subset C$, and a holomorphic complex structure J_i on U_i with the properties explained above (of course, if Z_i is a line pair, we may take J_i to be the original complex structure J). Let $\frac{1}{2}\overline{\Delta}_i$ denote the closed subdisc of Δ_i with half the radius, $C^* = C \setminus \bigcup_{i=1}^N \frac{1}{2}\overline{\Delta}_i$, and $E^* = E_{|C^*} \to C^*$ the corresponding open subset of E. We then take a tubular neighbourhood $U^* \to C^*$ of $E^* \to C^*$, equipped with the original complex structure J. By taking deformations to be sufficiently small and shrinking radii of tubular neighbourhoods if necessary, all these different complex structures may be patched together in a C^∞ way (tamed by the symplectic form) over the overlaps in C. In this manner, we obtain an open neighbourhood W of E in X, and a tamed almost complex structure J' on W, which is a small deformation of the original complex structure J and which satisfies the following properties:

(a) Each singular fibre Z_i of $E \to C$ has an open neighbourhood $U_i \subset W$ fibred over $\Delta_i \subset C$ with J' inducing an integrable complex structure on each fibre (thus $U_i \to \Delta_i$ is a C^∞ family of holomorphic surface neighbourhoods).

(b) The almost complex structure J' is integrable in a smaller neighbourhood $U_i' \subset U_i$ of each singular fibre, with the corresponding family $U_i' \to \Delta_i'$ being holomorphic.

(c) On the complement of $\bigcup U_i$ in W, the almost complex structure J' coincides with the original complex structure J.

(d) E deforms to a C^∞ family of pseudoholomorphic rational curves $E' \to C$ in (W, J'), with generic fibre $\mathbb{CP}^1$ and the only singular fibres being line pairs. Moreover, we may assume that any such singular fibre is contained in one of the above open sets U_i'.

Of course, we may now patch J' on W with the original complex structure J on X to get a global tamed almost complex structure on X, which we shall also denote by J'. Provided we have taken our deformations sufficiently small, the standard argument via Gromov compactness ensures that any pseudo-holomorphic stable map representing the class A has image contained in a fibre of $E' \to C$.

The theory of [13] applies equally well to almost complex structures, and hence to our almost complex manifold X' with complex structure J'. Clearly, all the calculations remain unchanged for stable maps whose image (a fibre of $E' \to C$) has a neighbourhood on which J' is integrable, and in particular this includes all the singular fibres. Suppose therefore that $f\colon \mathbb{CP}^1 \to X'$ is a pseudoholomorphic rational curve whose image Z is contained in an overlap $U_i \setminus U_i'$ (where J' may be nonintegrable). The linearized Cauchy–Riemann operator L_A still has index zero, since by the argument of [10], p. 24, the calculation via Riemann–Roch continues to give the correct value. We therefore need to show that $\operatorname{coker} L_A$ is still identified naturally as $T^*_{C,x}$, and hence that the obstruction bundle is $\mathcal{H} = T^*_C$ as before.

Setting $U = U_i$ and $\Delta = \Delta_i$, we know that $U \to \Delta$ is locally (around the image Z of f) a C^∞ family of holomorphic surface neighbourhoods. Moreover, the linearized Cauchy–Riemann operator $L_A = D_f\colon C^\infty(f^*T_U) \to \Omega^{0,1}(f^*T_U)$ fits into the following commutative diagram (with exact rows)

$$\begin{array}{ccccccccc}
0 & \to & C^\infty(f^*T_{U/\Delta}) & \to & C^\infty(f^*T_U) & \to & C^\infty(g^*T_\Delta) & \to & 0 \\
 & & \Big\downarrow \overline{\partial}_f & & \Big\downarrow D_f & & \Big\downarrow & & \\
0 & \to & \Omega^{0,1}(f^*T_{U/\Delta}) & \to & \Omega^{0,1}(f^*T_U) & \to & \Omega^{0,1}(g^*T_\Delta) & \to & 0
\end{array}$$

where g is the constant map on $\mathbb{CP}^1$ with image the point $x \in \Delta$, and where the fibre of E' over x is Z. Let us denote by U_x the corresponding holomorphic surface neighbourhood, the fibre of U over x. The cokernel of

$$\overline{\partial}_f\colon C^\infty(f^*T_{U/\Delta}) \to \Omega^{0,1}(f^*T_{U/\Delta})$$

is then naturally identified via Dolbeault cohomology with $H^1(T_{U_x}|_Z) \cong H^1(N_{Z/U_x})$. This latter space is in turn naturally identified with $H^1(N_f) \cong H^0(N_f)^* \cong T^*_{C,x}$.

I claim now that J' may be found as above for which $\operatorname{coker} L_A$ has the correct dimension (namely real dimension two) for all fibres of $E' \to C$. Since

L_A has index zero and ker L_A has dimension at least two, we need to show that the dimension of coker L_A is not more than two. This follows by a Gromov compactness argument. Suppose that, however close we take J' to J, the dimension is too big for some fibre of $E' \to C$. We can then find sequences of almost complex structures J'_ν (with the properties (a)–(d) described above) converging to $J = J_0$, and pseudoholomorphic rational curves $f_\nu\colon \mathbb{CP}^1 \to (X, J'_\nu)$ at which coker L_A has real dimension > 2. By construction, the image of such a map is not contained in any U'_i (since J'_ν would then be integrable on some neighbourhood of the image, and then we know that coker L_A has the correct dimension). Thus the image of f_ν has nontrivial intersection with the compact set $X \setminus \bigcup U'_i$. By Gromov compactness, the f_ν may be assumed to converge to a pseudoholomorphic rational curve on (X, J) whose image is not contained in any U'_i. This is therefore just an embedding $f\colon \mathbb{CP}^1 \to (X, J)$ of some smooth fibre of $E \to C$, at which we know that coker L_A has real dimension precisely two; this then is a contradiction. A similar argument, via Gromov compactness, then yields the fact that J' may be found as above such that the linear map $\mathrm{coker}(\overline{\partial}_f) \to \mathrm{coker}(D_f)$ is an isomorphism for all smooth fibres of $E' \to C$, since this is true for all the smooth fibres of $E \to C$ on (X, J).

For such a J', we deduce that coker L_A is naturally identified with $T^*_{C,x}$ for all fibres, and hence the obstruction bundle identified as T^*_C. The previous argument may then be applied directly with the almost complex structure J', showing that the symplectic invariant n_2 is $2g - 2$ in general. The proof of Theorem 4.1 is now complete.

5 An application to symplectic deformations of Calabi–Yaus

If X is a Calabi–Yau threefold which is general in moduli, we know that any codimension one face of its nef cone $\overline{\mathcal{K}}(X)$ (not contained in the cubic cone W^*) corresponds to a primitive birational contraction $\varphi\colon X \to \overline{X}$ of Type I, II or III_0, where Type III_0 denotes a Type III contraction for which the genus of the curve C of singularities on $\overline{X}$ is zero.

In [18], we studied Calabi–Yau threefolds which are symplectic deformations of each other. One of the results proved there (Theorem 2) said that if X_1 and X_2 are Calabi–Yau threefolds, general in their complex moduli, which are symplectic deformations of each other, then their Kähler cones are the same. The proof of this essentially came down to showing that certain Gromov–Witten invariants associated to exceptional classes were nonzero. Using the much more precise information obtained in this paper, we are able to make a stronger statement.

Corollary 5.1 *With the notation as above, any codimension one face (not contained in W^*) of $\overline{\mathcal{K}}(X_1) = \overline{\mathcal{K}}(X_2)$ has the same contraction type (Type I, II or III_0) on X_1 as on X_2.*

Proof The fact that Type II faces correspond is easy, since for D in the interior of such a face, the quadratic form $q(L) = D \cdot L^2$ is degenerate, which is not the case for D in the interior of a Type I or Type III_0 face. Stating it another way, if we consider the Hessian form associated to the topological cubic form μ, then h is a form of degree $\rho = b_2$ which has a linear factor corresponding to each Type II face. Thus the condition that a face is of Type II is topologically determined.

The result will therefore follow if we can show that a face of the nef cone which is Type I for one of the Calabi–Yau threefolds is not of Type III_0 for the other. However, for a Type I face, we saw in §2 that n_d is always nonnegative; for a Type III_0 face, we proved in Theorem 4.1 that $n_2 = -2$. Since Gromov–Witten invariants are invariant under symplectic deformations, the result is proved.

Remark 5.2 It is still an open question whether there exist examples of Calabi–Yau threefolds X_1 and X_2 which are symplectic deformations of each other but not in the same algebraic family.

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