A generic Torelli theorem for the quintic threefold

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0.1 Introduction

This paper proves the generic Torelli theorem for the family of quintic hypersurfaces $X_5 \subset \mathbb{P}^4$:

Theorem 1 *The period map $\mathcal{P}_3$ defined on the quotient $U/\operatorname{PGL}(4)$, where $U \subset \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^4}(5)))$ parametrizes smooth hypersurfaces, is of degree one onto its image.*

Recall that if B parametrizes a family $(X_b)_{b\in B}$ of compact Kähler varieties of dimension k, the period map

$$\mathcal{P}_k \colon B \to \mathcal{D}/\Gamma,$$

sends $b \in B$ to the Hodge filtration $F^\bullet H^k(X_b)$ on $H^k(X_b,\mathbb{C}) \cong H^k(X_0,\mathbb{C})$. Here $\mathcal{D}$ is the period domain parametrizing all filtrations on $H^k(X_0,\mathbb{C})$ of given ranks, and Γ the group of automorphisms of $H^k(X_0,\mathbb{Z})$ preserving the intersection form, so that the isomorphism $H^k(X_b,\mathbb{Z}) \cong H^k(X_0,\mathbb{Z})$ is well defined up to Γ. It is known (Griffiths [13]) that $\mathcal{P}_k$ is holomorphic and satisfies Griffiths transversality.

Theorem 1 says that if $X \subset \mathbb{P}^4$ is a generic quintic threefold, any isomorphism of its polarized Hodge structure

$$\Big(H^3(X,\mathbb{Z}), \langle\ ,\ \rangle, F^\bullet H^3(X)\Big) \cong \Big(H^3(X',\mathbb{Z}), \langle\ ,\ \rangle, F^\bullet H^3(X')\Big)$$

with that of another quintic threefold X' is induced by a (projective) isomorphism $X \cong X'$.

We mention that Torelli statements for Calabi–Yau varieties may have an important role to play in mirror symmetry: at present, the mirror symmetry

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conjecture can be formulated in Hodge theoretic terms, as identifying the quantum variation of Hodge structure parametrized by $H^2(X^*, \mathbb{C})$, where X^* is a mirror Calabi–Yau threefold, with the variation of Hodge structure on $H^3(X, \mathbb{C})$ parametrized by the Kuranishi family of X. In this formulation it has been proved by Givental [8] when X^* is the quintic threefold and X its mirror (see [1]). It seems likely that Givental's method should also prove it when X is the quintic threefold and X^* its mirror. This last statement, together with our results would in fact prove a stronger conjecture, namely the fact that a complexified Kähler parameter on X^* determines a complex structure on X: indeed, what we show is that the generic quintic X is determined by its infinitesimal variation of Hodge structure, and it turns out that for Calabi-Yau threefolds, the infinitesimal variation of Hodge structure is equivalent to the isomorphism class of the Yukawa cubic (see below the discussion of the derivative of the period map) on the vector space $H^1(T_X)$. Moreover, the mirror symmetry conjecture identifies this with the quantum product on $H^2(X^*)$, which is determined by the complexified Kähler parameter on X^*.

The Torelli theorem for K3 surfaces was used in a similar way in Voisin [19] to construct the mirror map in this stronger sense for a particular class of Calabi–Yau threefolds. It seems that a Torelli statement could also play an important role in the new construction of mirror symmetry proposed by Strominger, Yau and Zaslow [17], and studied in Morrison [15] and Gross and Wilson [14].

0.2 The methods of Griffiths and Donagi

In the case of a Calabi–Yau threefold X (of which the quintic threefold is the most standard example), it is known that the local period map $\mathcal{P}_3$ from the Kuranishi family B of X to the local period domain is immersive. This follows from Griffiths' general description of the derivative of the period map at $0 \in B$, where $X = X_0$: by transversality, this derivative is described by a series of maps

$$\mu_p \colon T_{B,0} \to \operatorname{Hom}(H^{p,q}(X), H^{p-1,q+1}(X)) \quad \text{for } p + q = 3.$$

Griffiths proved that μ_p is the composite of the Kodaira–Spencer isomorphism $T_{B,0} \cong H^1(T_X)$ with the natural cup product map

$$H^1(T_X) \to \operatorname{Hom}(H^{p,q}(X), H^{p-1,q+1}(X)).$$

Now consider the period map $\mathcal{P}^{3,0}$ for holomorphic $(3,0)$ forms, which sends $t \in B$ to the line $H^{3,0} \subset H^3(X_0, \mathbb{C})$ based by the holomorphic $(3,0)$ form on X_t. It is obviously determined by $\mathcal{P}_3$, and by Griffiths transversality, its derivative is a map

$$\mathcal{P}^{3,0}_* \colon T_{B,0} \to \operatorname{Hom}\bigl(H^{3,0}(X), H^3(X, \mathbb{C})/H^{3,0}(X)\bigr),$$

with image contained in $\mathrm{Hom}(H^{3,0}(X), H^{2,1}(X))$, which can then be identified with μ_3. Now the cup product map

$$H^1(T_X) \to \mathrm{Hom}(H^{3,0}(X), H^{2,1}(X))$$

is clearly an isomorphism, since K_X is trivial, which implies that $\mathcal{P}_*^{3,0}$ is injective; therefore $\mathcal{P}_3$ is an immersion.

Apart from this general fact, there are very few results on the Torelli problem for Calabi–Yau threefolds. On the other hand, Donagi proved the generic Torelli theorem for many families of hypersurfaces in projective space, with a series of possible exceptions including Calabi–Yau hypersurfaces:

Theorem 2 (Donagi [4]) *The generic Torelli theorem holds for hypersurfaces in $\mathbb{P}^n$ of degree d, with the possible exception of the following cases:*

(a) *d divides $n+1$;*

(b) *$d = 3$ and $n = 3$;*

(c) *$d = 4$ and $n \equiv 1 \mod 4$;*

(d) *$d = 6$ and $n \equiv 2 \mod 6$.*

(In fact, (d) has since been excluded, see Cox and Green [3].) In the remainder of this section, we sketch the ideas of Donagi's proof, and explain how his methods enable us to deduce Theorem 1 from the purely algebraic statement of Theorem 5.

Let X be a smooth hypersurface of degree d in $\mathbb{P}^n$, with equation $F = 0$. We write R_F for the *Jacobian ring* of F, that is, the quotient of the polynomial ring in $n+1$ variables $S = \bigoplus_{k\geq 0} S^k$ by the ideal J_F generated by the partial derivatives of F. We note for future use that $R_F = \sum_{d=0}^{\sigma} R_F^d$ is a graded complete intersection ring, with dth graded piece R_F^d, having element of top degree $\sigma(n+1)(d-2)$, and the multiplication $R_F^d \times R_F^{\sigma-d} \to R_F^{\sigma} = \mathbb{C}$ is a perfect pairing, defining a duality $(R_F^{\sigma-d})^* = R_F^d$ (see [2]).

Then if $n \geq 3$, and $d \geq 5$ if $n = 3$, the natural map

$$\bigl\{\text{infinitesimal deformations of } F\bigr\} \to \bigl\{\text{infinitesimal deformations of } X\bigr\}$$

defines an isomorphism

$$R_F^d \cong H^1(T_X).$$

The infinitesimal variation of Hodge structure of X, or the differential of the period map at $[X]$, is given as described above by a series of maps

$$\mu_p\colon H^1(T_X) \to \mathrm{Hom}(H^{p,q}_{\mathrm{prim}}(X), H^{p-1,q+1}_{\mathrm{prim}}(X)) \quad \text{for } p+q = n-1.$$

We have the following theorem:

Theorem 3 (Carlson and Griffiths [2]) *There are natural isomorphisms*

$$H^{p,q}_{\mathrm{prim}}(X) \cong R_F^{(n-p)d-n-1} \quad \textit{for } p+q=n-1,$$

which, up to universal coefficients, identify the μ_p with the multiplication maps

$$R_F^d \to \mathrm{Hom}(R_F^{(n-p)d-n-1}, R_F^{(n-p+1)d-n-1}). \tag{0.1}$$

It is known that, except if $d=3$ and $n=3$, the (polarized) period map for X is a local immersion. To prove that it is of degree 1 onto its image, it is enough to prove that if V and V' are simply connected open subsets of the quotient $U^0/\,\mathrm{PGL}(n+1)$ parametrizing smooth hypersurfaces without automorphisms up to projective equivalence, and $j\colon V \cong V'$ a given isomorphism such that $\mathcal{P}_{n-1} \circ j = \mathcal{P}_{n-1}$, where $\mathcal{P}_{n-1}$ is the local period map, then $V = V'$ (here we identify the local period domains for V and V' by the given isomorphism of polarized Hodge structures $H^{n-1}_{\mathrm{prim}}(X_{t_0}) \cong H^{n-1}_{\mathrm{prim}}(X_{j(t_0)})$ for some $t_0 \in V$). By translating the condition that $\mathcal{P}_{n-1*} \circ j_* = \mathcal{P}_{n-1*}$, it follows immediately that we have an induced isomorphism of infinitesimal variations of Hodge structure at t and $j(t)$.

Using Theorem 3, we see that this isomorphism of infinitesimal variations of Hodge structures gives a partial isomorphism between the Jacobian rings of F_t and $F_{j(t)}$:

$$\begin{array}{ccc} R^d_{F_t} & \longrightarrow & \mathrm{Hom}(R^{(n-p)d-n-1}_{F_t}, R^{(n-p+1)d-n-1}_{F_t}) \\ \downarrow & & \downarrow \\ R^d_{F_{j(t)}} & \longrightarrow & \mathrm{Hom}(R^{(n-p)d-n-1}_{F_{j(t)}}, R^{(n-p+1)d-n-1}_{F_{j(t)}}), \end{array} \tag{0.2}$$

where the second vertical arrow is an isomorphism induced by isomorphisms $R^{(n-p)d-n-1}_{F_t} \cong R^{(n-p)d-n-1}_{F_{j(t)}}$ for each p. In other words, knowing the Hodge structure and infinitesimal variation of Hodge structure of X_t determines *some graded pieces* of the multiplication map of the Jacobian ring R_{F_t}. The key ingredient in going from this to the proof of Theorem 2 is the following result, which allows us in many cases to recover the whole Jacobian ring from the above data:

Theorem 4 (Donagi–Green symmetrizer lemma, cf. [6]) *Let R be the quotient of S by a regular sequence of $n+1$ homogeneous elements of degree $d-1$; write $\sigma = (d-2)(n+1)$ for the top degree of R, and define*

$$T_{a,b} = \bigl\{\varphi \in \mathrm{Hom}(R^a, R^b) \bigm| w\varphi(v) = v\varphi(w) \textit{ for all } v, w \in R^a\bigr\}.$$

Then $T_{a,b} = R^{b-a}$ if $a+b \le \sigma$ and $b+d-1 \le \sigma$.

This means that multiplication by elements of R^{b-a} is uniquely determined by the multiplication map $R^a \times R^b \to R^{a+b}$. In the range considered in Theorem 2, Donagi and Green [6] apply Theorem 4 to diagram (0.2) to recover isomorphisms between other pieces of the Jacobian rings of F_t and $F_{j(t)}$, and eventually to obtain a whole isomorphism between the Jacobian rings of F_t and $F_{j(t)}$ themselves; the degree 1 piece of this isomorphism then gives an isomorphism $\alpha\colon S^1 \cong S^1$ inducing $\alpha^{d-1}\colon S^{d-1} \cong S^{d-1}$ such that $\alpha^{d-1}(J^{d-1}_{F_t}) = J^{d-1}_{F_{j(t)}}$. It is then immediate to conclude that F_t and $F_{j(t)}$ are projectively equivalent. Thus Donagi's result is that, in the range of Theorem 2, the existence of the commutative diagrams (0.2) implies that $X_t \cong X_{j(t)}$ for generic t. See Donagi's survey [5] for more details.

The argument does not work for the series (a) of possible exceptions to Theorem 2, because when d divides $n+1$, the symmetrizer lemma applied to the multiplication maps (0.1) can never give new pieces of the Jacobian ring of degree not divisible by d.

0.3 Start of our proof

From now on we consider the case of the quintic threefold. This is a Calabi–Yau threefold, and its infinitesimal variation of Hodge structure is described by the maps

$$\begin{aligned}
&\mu_3\colon H^1(T_X) \to \mathrm{Hom}(H^{3,0}(X), H^{2,1}(X)),\\
&\mu_2\colon H^1(T_X) \to \mathrm{Hom}(H^{2,1}(X), H^{1,2}(X)),\\
&\mu_1\colon H^1(T_X) \to \mathrm{Hom}(H^{1,2}(X), H^{0,3}(X)).
\end{aligned}$$

We proved above that the first map is an isomorphism, and the last map is the same isomorphism, in view of the duality isomorphisms

$$H^{0,3}(X) = H^{3,0}(X)^* \quad \text{and} \quad H^{1,2}(X) = H^{2,1}(X)^*.$$

Thus the only algebraic invariants of the infinitesimal variation of Hodge structure of X are contained in the map

$$\mu\colon H^1(T_X) \otimes H^1(T_X) \to H^1(T_X)^*,$$

well defined up to a coefficient depending on the choice of a generator of $H^{3,0}(X)$, which is obtained from the map

$$H^1(T_X) \otimes H^1(\Omega^2_X) \to H^2(\Omega_X)$$

given by μ_2, after using the isomorphism μ_3 to substitute $H^1(T_X)$ for $H^1(\Omega^2_X)$, and μ_1 to substitute $H^1(T_X)^*$ for $H^2(\Omega_X)$. The map μ is easily seen to come from a cubic form on $H^1(T_X)$, the *Yukawa coupling*. It follows from the above that this cubic determines the whole infinitesimal variation of Hodge structure. Theorem 3 identifies $\mu\colon S^2H^1(T_X) \to H^1(T_X)^*$ with the multiplication map

$$\mu\colon S^2R^5_F \to R^{10}_F, \tag{0.3}$$

(for simplicity, we write xy for $\mu(x,y) \in R^{10}_F$ in what follows) followed by the duality isomorphism $R^{10}_F \cong (R^5_F)^*$.

To extend Donagi's method to the case of the quintic, it is enough to show that, at least for generic F, the multiplication map

$$\mu_{1,4}\colon S^1 \otimes R^4_F \to R^5_F \tag{0.4}$$

is uniquely determined by the map μ (0.3) and the isomorphism $R^{10}_F \cong (R^5_F)^*$. In other words, we need to prove that an isomorphism $R^5_F \cong R^5_{F'}$ compatible with μ induces isomorphisms $S^1 \cong S^1$ and $R^4_F \cong R^4_{F'}$ compatible with $\mu_{1,4}$. Indeed, if we can do this iterated application of the symmetrizer lemma allows us to reconstruct the whole Jacobian ring R_F, and hence F itself.

We view $\mu_{1,4}$ as an inclusion $R^4_F \subset \operatorname{Hom}(S^1, R^5_F) \cong (R^5_F)^5$, where the last isomorphism involves choosing a basis of S^1, and write $W \subset (R^5_F)^5$ for the image of this inclusion. From now on, W is the main unknown in our proof; our first attempt is to try to determine W from the following properties (a–g), which are stated in terms of the known map μ and isomorphism $R^{10}_F \cong (R^5_F)^*$.

First, some notation. The typical element of W is $w = (w_1, \dots, w_5) = (x_1\varphi, \dots, x_5\varphi)$, where $\varphi \in R^4_F$ and $x_1, \dots, x_5$ are coordinates on $\mathbb{P}^4$. For $t = (t_1, \dots, t_5) \in \mathbb{C}^5$ we write $w_t = \sum_i t_i w_i \in R^5_F$, which corresponds simply to the product $A\varphi$ of φ with a linear form $A \in S^1$. We think of $w \mapsto w_t$ as defining a *projection* $p_t\colon W \to R^5_F$ of $W \subset (R^5_F)^5$ to one factor R^5_F, and its image $p_t(W)$. In Section 2, we refer to this image as a *component* of W; in particular when t is one of the basis vectors $t = e_i$ of $\mathbb{C}^5$, we call $p_i(W) = p_{e_i}(W)$ the *ith component* of $W \subset (R^5_F)^5$. In a similar way, two elements $t, t' \in \mathbb{C}^5$ give rise to a *2-term projection* $W_{t,t'} = \{(w_t, w_{t'}) | w \in W\} \subset (R^5_F)^2$.

Properties of W

(a) $\dim W = \dim R^4_F = 65 = \binom{8}{4} - 5$.

(b) *The symmetrizer property.* For $w, w' \in W$ we have

$$w_i w'_j = w_j w'_i \in R^{10}_F \quad \text{for } i,j = 1,2,\dots,5.$$

More generally, $w_t w'_{t'} = w_{t'} w'_t$ for any $t, t' \in \mathbb{C}^5$. Indeed, w, w' correspond to elements φ, φ' of R^4_F, and t, t' to elements A, A' of S^1. Then in R^5_F, we have the equalities: $w_t = A\varphi$, $w_{t'} = A'\varphi$, $w'_t = A\varphi'$ and $w'_{t'} = A'\varphi'$, so that the result is clear.

(b′) More precisely,

$$W = \{z \in (R_F^5)^5 \mid z_i w_j = z_j w_i \text{ for all } i, j \text{ and all } w \in W\}.$$

This follows from the symmetrizer lemma (Theorem 4).

(c) $\dim R_F^5 \cdot W = 135$, where $R_F^5 \cdot W \subset (R_F^{10})^5$ is defined using the natural map $R_F^5 \otimes (R_F^5)^5 \to (R_F^{10})^5$. This holds because $R_F^5 \cdot W$ is isomorphic to R_F^9, which by Koszul has dimension $\binom{13}{4} - 5\binom{9}{4} + 10\binom{5}{4} = 135$.

The next property is less obvious. Assume that X is generic, and for generic $t, t' \in \mathbb{C}^5$, define the *partial symmetrizer* of the 2-term projection $W_{t,t'}$ by

$$W'_{t,t'} = \{(z_1, z_2) \in (R_F^5)^2 \mid z_1 w_t = z_2 w_{t'} \in R_F^{10} \text{ for all } (w_t, w_{t'}) \in W_{t,t'}\};$$

then (b) says that $W_{t,t'} \subset W'_{t,t'}$. The following fact, while technical, is very helpful in the remainder of our calculations:

(d) $\operatorname{codim}(W_{t,t'} \subset W'_{t,t'}) = 2$ and $\operatorname{codim}(R_F^5 W_{t,t'} \subset R_F^5 W'_{t,t'}) = 2$; moreover, we can identify $R_F^5 W'_{t,t'}$ with the orthogonal complement of $W_{t,t'}$ in $(R_F^{10})^2$ and vice versa, using the pairing $\langle z, w\rangle = z_1 w_2 - z_2 w_1$ between $(R_F^5)^2$ and $(R_F^{10})^2$, so that $R_F^5 W'_{t,t'}/R_F^5 W_{t,t'}$ and $W'_{t,t'}/W_{t,t'}$ are dual to one another. Equivalently, for generic $\varphi \in R_F^5$, the skew pairing on $W'_{t,t'}/W_{t,t'}$ defined by $\langle z, w\rangle_\varphi = \varphi(z_1 w_2 - z_2 w_1) \in R_F^{15}$ is nondegenerate. This form vanishes when φ belongs to $p_t(W) + p_{t'}(W)$.

We sketch the proof of (d) later. The other properties that we use are the following:

(e) For generic $t \neq 0 \in \mathbb{C}^5$, the projection $p_t \colon W \to R_F^5$ given by $w \mapsto w_t$ is injective. Furthermore, the map $\mathbb{C}^5 \to \operatorname{Hom}(W, R_F^5)$ is injective.

Indeed, W is identified with R_F^4, and t with a generic element A of S^1; the map p_t is simply the multiplication by A from R_F^4 to R_F^5. Thus it is enough to exhibit one example where this multiplication map is injective: take X to be the Fermat hypersurface, and A the form $\sum_i X_i$. The second statement is simply Macaulay's theorem (see [20]).

(f) For any $z \in (R_F^5)^5$, the map $\nu_z \colon W \to R_F^{10}$ given by $w \mapsto \sum_i z_i w_i$ is zero or has rank ≥ 21.

Indeed, let x_i be the basis of S^1 giving the isomorphism $S^1 \cong \mathbb{C}^5$; using the isomorphism $W \cong R_F^4$, it is immediate to see that $\nu_z \colon R_F^4 \to R_F^{10}$ is identified with the multiplication map by $\sum_i x_i z_i \in R_F^6$. So the statement is that for generic X, and any nonzero element $P \in R_F^6$, the multiplication map $\mu_P \colon R_F^4 \to R_F^{10}$ has rank ≥ 21, which is not difficult to prove: in fact this bound, which we need for technical reasons, is not at all sharp. For example, for the Fermat quintic, the only polynomials not satisfying this bound are, up to a permutation of the coordinates, of the form $X_0^3X_1^2A$, for A a linear form. Then an infinitesimal computation shows that these polynomials P with $\operatorname{rank}\mu_P \leq 20$ will not continue to exist for a generic deformation of the Fermat.

(g) For any four linearly independent elements $t_1, \ldots, t_4 \in \mathbb{C}^5$, we have $\sum_i p_{t_i}(W) = R_F^5$.

Indeed, four independent elements of S^1 generate the ideal of a point in $\mathbb{P}^4$, and since J_F^5 has no base point, they generate S^5 modulo J_F^5, or equivalently, they generate R_F^5.

Our plan in the next sections is to prove the following

Theorem 5 *Suppose that F is generic, and let $W \subset (R_F^5)^5$ be a subspace satisfying (a–g) above. Then there exists an isomorphism $S^1 \cong \mathbb{C}^5$ which identifies W with $R_F^4 \subset \operatorname{Hom}(S^1, R_F^5)$.*

This shows how to reconstruct $\mu_{1,4}$ from μ and the isomorphism $R_F^{10} \cong (R_F^5)^*$: the properties above are stated using only this data: the theorem shows the uniqueness of $W \subset (R_F^5)^5$ satisfying these properties, up to $\operatorname{Aut}\mathbb{C}^5$: choose such a W; put $R_F^4 = W$, $S^1 = \mathbb{C}^5$ and define $\mu_{1,4}\colon S^1 \otimes R_F^4 \to R_F^5$ as the natural map $\mathbb{C}^5 \otimes W \to R_F^5$. Then, as in Donagi's arguments mentioned above, the symmetrizer lemma allows us to recover the multiplication maps

$$\mu_{1,k}\colon S^1 \otimes R_F^k \to R_F^{k+1} \quad \text{for } k \leq 4,$$

giving the whole Jacobian ring of F, and therefore allows us to recover F itself. Thus Theorem 5 implies our main result Theorem 1.

Proof of (d) t, t' are identified with generic elements A, B of S^1; for any integer i, define the space $H^i_{A,B}$ as the middle cohomology of the complex

$$R_F^{i-1} \xrightarrow{(A,B)} R_F^i \times R_F^i \xrightarrow{B-A} R_F^{i+1}.$$

In degree $i = 5$, clearly $W_{t,t'}$ is identified with $\operatorname{Im}(A,B)$ and $W'_{t,t'}$ with $\operatorname{Ker}(B-A)$, so that $W'_{t,t'}/W_{t,t'}$ is naturally isomorphic to $H^5_{A,B}$. It is easy to prove that for generic A, B and X, $(A,B)\colon R_F^4 \to R_F^5 \times R_F^5$ is injective and $B - A\colon R_F^5 \times R_F^5 \to R_F^6$ is surjective. (This last fact means that the 5 partial derivatives of a generic quintic F, together with two generic linear forms, generate an ideal containing all sextic polynomials; this can be checked easily, for example using Macaulay.) Thus, since $\dim R_F^4 = 65$, $\dim R_F^5 = 101$, $\dim R_F^6 = 135$, it follows that $\dim W'_{t,t'}/W_{t,t'} = 2$.

Now it is immediate to see using the selfduality $R_F^k \cong (R_F^{15-k})^*$ of the ring R_F that $H_{A,B}^{10}$ is naturally dual to $H_{A,B}^5$, the duality being given by the pairing $\langle\ ,\ \rangle$ between $R_F^5 \times R_F^5$ and $R_F^{10} \times R_F^{10}$. On the other hand, $H_{A,B}^{10} \cong W_{t,t'}^{\perp}/R_F^5 W_{t,t'}$ by definition, and $R_F^5 W'_{t,t'} \subset W_{t,t'}^{\perp}$. For $\varphi \in R_F^5$, the multiplication map by $\varphi\colon H_{A,B}^5 \to H_{A,B}^{10}$ is skewsymmetric, and since $\dim H_{A,B}^5 = 2$, it is either zero or an isomorphism; in this last case, $\langle\ ,\ \rangle_\varphi$ is nondegenerate. It is easy to show that it is generically nonzero, hence we conclude that $R_F^5 W'_{t,t'} = W_{t,t'}^{\perp}$ and that $\langle\ ,\ \rangle_\varphi$ is generically nondegenerate. $\square$

0.4 Overall plan of the proof

Although Theorem 5 is stated for generic F, for which W satisfies the above properties (a–g), we prove it via an appropriate specialization F_0, when (a–g) no longer hold for W_0. We now discuss the logic of this reduction. First, (a–g) are used in Section 1 in connection with the following definition:

Definition 1 Given a subspace $W \subset (R_F^5)^5$, the *conductor* of S^1 to W is the subspace $[W : S^1] \subset (R_F^4)^5$ defined by

$$[W : S^1] := \left\{ z \in (R_F^4)^5 \;\middle|\; Az \in W \text{ for all } A \in S^1 \right\}.$$

Section 1, Proposition 1 gives the lower bound $\dim[W : S^1] \geq 19$, assuming that W satisfies (a–g). The next definition collects together for use in Section 2 the properties of W_0 that survive the specialization from F to F_0.

Definition 2 Let F_0 be the equation of a nonsingular quintic threefold. We say that a subspace $W_0 \subset (R_{F_0})^5$ has *Property P* if it satisfies (a), (b) of 0.3, and the following supplementary conditions (c′), (e) and (h):

(c′) $\dim R_{F_0}^5 W_0 \leq 135$.

(e) The first projection $W_0 \to R_{F_0}^5$ and the map $\mathbb{C}^5 \to \operatorname{Hom}(W_0, R_{F_0}^5)$ are both injective.

(h) The conductor $Z_0 = [W_0 : S^1]$ has dimension ≥ 19 (see Section 1, Proposition 1).

We justify these properties. Let $U \subset \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^4}(5)))$ be the open set parametrizing smooth hypersurfaces; over U, we can define the bundles $\mathcal{R}^5$ and $\mathcal{R}^{10}$, with fibres R_F^5 and R_F^{10} over $F \in U$. The multiplication in the ring R_F for any F gives a map

$$\mu\colon S^2\mathcal{R}^5 \to \mathcal{R}^{10}.$$

In the Grassmannian of 65 dimensional subspaces of $(\mathcal{R}^5)^5$, the set of spaces W satisfying (b) and (c$'$) is Zariski closed. It follows that if $W \subset (R^5_F)^5$ (defined over some finite cover of U) satisfies (a), (b), (c) for generic F, when F specializes to F_0, the corresponding limit W_0 of W satisfies (a), (b) and (c$'$).

Now assume that the generic W satisfies the remaining properties. By specializing along a suitable curve, it can be seen that we can impose the following conditions on W_0. Since we may assume using the $\operatorname{Aut}\mathbb{C}^5$ action and (e) that the first projection $\mathrm{pr}_1 \colon W \to R^5_F$ is injective for generic (W, F), we may also assume this is true for (W_0, F_0): for this, we consider the spaces W_t as subspaces of $R^5_{F_t}$ for generic t via the first projection; we then define W_0 as the limiting subspace of $R^5_{F_0}$. Moreover, since the map $\mathbb{C}^5 \to \operatorname{Hom}(W, R^5_F)$ is injective for generic W, we may also assume this to be true for W_0. In fact, for this it suffices to define the subspace $\mathbb{C}^5 \subset \operatorname{Hom}(W_0, R^5_{F_0})$ as the limit of the subspaces $\mathbb{C}^5 \subset \operatorname{Hom}(W_t, R^5_{F_t})$

When we specialize W to W_0, the dimension of the conductor $[W : S^1]$ can of course only jump up, so our specialized W_0 will satisfy (h).

An important role in the proof is played by the following definition.

Definition 3 The subspace $W_0 \subset (R_{F_0})^5$ is *degenerate* for X_4 if one component of $W_0 \subset (R^5_{F_0})^5$ (in the sense explained in 0.3) is contained in the ideal (X_4), or in other words, the composite

$$\mathbb{C}^5 \to \operatorname{Hom}(W_0, R^5_{F_0}) \to \operatorname{Hom}(W_0, R^5_{F_0}/(X_4)) \tag{0.5}$$

is not injective.

Section 2, Propositions 2–3 describe all the possible degenerate subspaces $W_0 \subset (R^5_{F_0})^5$ satisfying property P. There turn out to be just two possibilities: *either* W_0 is the image of $R^4_{F_0}$ in $\operatorname{Hom}(S^1, R^5_{F_0})$ for some isomorphism $S^1 \cong \mathbb{C}^5$, so that W_0 satisfies the conclusion of Theorem 5; *or* W_0 is a certain explicitely described degenerate limit of these under the action of $\operatorname{Aut}\mathbb{C}^5$. In Section 3, we use an infinitesimal argument to prove that any deformation of W_0 to a subspace W_t satisfying property P for generic t satisfies the conclusion of Theorem 5, whenever W_0 is degenerate or not.

The specialization F_0 we use is of the form

$$F_0(X_0, \ldots, X_4) = X_4^5 + G(X_0, \ldots, X_3), \tag{0.6}$$

with G a generic polynomial of degree 5. In this case $R^5_{F_0}/(X_4)$ occuring in (0.5) is simply R^5_G, the degree 5 piece of the Jacobian ring of G. The final Section 4 proves that for F_0 of the form (0.6), any subspace $W_0 \subset (R^5_{F_0})^5$ satisfying (a–g) above is degenerate, which combined with the results above concludes the proof of Theorem 5.

Remark To keep the paper to a reasonable length, we have chosen to skip the proofs of a certain number of technical lemmas giving lower bounds for the ranks of the multiplication map by a nonzero homogeneous element of the Jacobian ring of a generic hypersurface of degree 5 in $\mathbb{P}^3$.

1 Dimension of the conductor $Z = [W : S^1]$

Proposition 1 *Let F be a generic quintic, and $W \subset (R_F^5)^5$ a subspace satisfying (a–g) of 0.3. Then* $\dim[W : S^1] \geq 19$ *(see Definition 1).*

The rest of this section is devoted to the proof of Proposition 1. We first show the following result.

Lemma 1.1 *For generic $A, B \in S^1$, we have $[W : S^1] = [W : (A, B)]$.*

Proof If $z \in [W : (A, B)]$ then $z_i w_j - z_j w_i \in R_F^9$ is annihilated by A, B for any $w \in W$; but the surjectivity of $B - A\colon R_F^5 \otimes R_F^5 \to R_F^6$ used above implies by duality the injectivity of the map $(A, B)\colon R_F^9 \to R_F^{10} \times R_F^{10}$. Hence $z_i w_j - z_j w_i = 0 \in R_F^9$, which by (b′) implies that $z \in [W : S^1]$. □

We use the notation $H^k_{A,B}$ introduced in 0.3 in the proof of (d). We have shown that $H^5_{A,B}$ is two dimensional and that a generic $\varphi \in R_F^5$ gives a skew linear isomorphism $H^5_{A,B} \to H^{10}_{A,B}$. Consider the map $B - A\colon W \times W \to S^1 W \subset (R_F^6)^5$. Its kernel $\mathrm{Ker}(B - A)$ admits a natural projection α to $(H^5_{A,B})^5$. We have the next result:

Lemma 1.2 *The map $\alpha\colon \mathrm{Ker}(B - A) \to (H^5_{A,B})^5$ is the zero map.*

Proof Suppose that $Bw - Aw' = 0$; then we set

$$\alpha((w, w')) = (\alpha_1, \ldots, \alpha_5) := \big((w_1, w_1'), \ldots, (w_5, w_5')\big).$$

Now we claim that $\langle \alpha_i, \alpha_j \rangle_\varphi = 0$ for all i, j, where $\langle\ ,\ \rangle_\varphi$ is the (generically nondegenerate) pairing introduced in (d): indeed $\langle \alpha_i, \alpha_j \rangle_\varphi = \varphi(w_i w_j' - w_j w_i') \in R_F^{15}$, which is zero by (b). It follows that the α_i generate a subspace of $H^5_{A,B}$ of dimension at most one; thus after a change of basis of $\mathbb{C}^5$, we may assume $\alpha_i = 0$ for $i \leq 4$. This means that for some $z_1, \ldots, z_4 \in R_F^4$, we have $w = (Az_1, \ldots, Az_4, w_5)$ and $w' = (Bz_1, \ldots, Bz_4, w_5')$. Now this implies that

$$\chi_i \alpha_5 = 0 \quad \text{in } R_F^5 \quad \text{for any } \chi = (\chi_1, \ldots, \chi_5) \in W \text{ and any } i \leq 4.$$

This is because by (b), $\chi_i w_5 = \chi_5 w_i = \chi_5 A z_i$ and $\chi_i w_5' = \chi_5 w_i' = \chi_5 B z_i$. But by (g), we know that $\sum_{i \leq 4} \mathrm{pr}_i(W) = R_F^5$. So α_5 is annihilated by R_F^5, hence is zero. □

It follows from Lemmas 1.1 and 1.2 that

$$\dim S^1W \geq 2\dim W - \dim[W : S^1]. \tag{1.1}$$

Let $W_{12} = \mathrm{pr}_{12}(W)$ and $W'_{12} = W'_{e_1,e_2}$ for a generic choice of basis of $\mathbb{C}^5$. We note first that $\mathrm{pr}_{12}\colon S^1W \to S^1W_{12}$ is injective; indeed by (b), $\dim R^5_FW = 135$, while $\dim R^5_FW_{12} = \dim W^{\perp}_{12} - 2 = 135$ by (d). Thus $\mathrm{pr}_{12}\colon R^5_FW \to R^5_FW_{12}$ must be injective, and so is $\mathrm{pr}_{12}\colon S^1W \to S^1W_{12}$.

Now we prove the following fact.

Lemma 1.3 $\dim S^1W'_{12}/S^1W_{12} \geq 6$.

Proof It is enough to show that for x, y independent in W'_{12}/W_{12}, we have $\dim S^1x/S^1y \geq 3$, where S^1x/S^1y is the image of S^1x in $(S^1W'_{12}/S^1W_{12})/S^1y$. Let K be the kernel of the natural map $S^1 \to S^1x/S^1y$; for $\alpha \in K$ and $P \in R^4_F$, we have $\langle P\alpha x, y\rangle = 0$, so that $P\alpha x = 0$ in $R^5_FW'_{12}/R^5_FW_{12}$, by (d), since obviously also $\langle P\alpha x, x\rangle = 0$ in $R^5_FW'_{12}/R^5_FW_{12}$. This implies that $P\alpha$ annihilates W'_{12}/W_{12}, since multiplication by $P\alpha$ is a skew morphism. (We use the notation and results of (d).) Now assume $\dim K \geq 3$. We know that $\mathrm{pr}_1(W) + \mathrm{pr}_2(W) + R^4_FK$ annihilates W'_{12}/W_{12}, so it must be a proper subspace of R^5_F. In particular R^4_FK must be a proper subspace of R^5_F.

Now we use the following fact which holds for generic X, as follows from an easy dimension count:

Fact 1.4 *The set of lines $\Delta \subset \mathbb{P}^4$ such that I_Δ does not generate R^5_F is one dimensional, and for such a line, I_Δ generates a hyperplane of R^5_F.*

So under the assumption $\dim K \geq 3$, we find that there is a line Δ as above, for which W_1 is contained in $I_\Delta(5)$. Since the set of such lines is one dimensional, and the basis of $\mathbb{C}^5$ is generic, for a fixed such Δ there is a hypersurface in $\mathbb{P}(\mathbb{C}^5)$ consisting of t such that $W_t \subset I_\Delta(5) \mod J^5_F$. This contradicts (g). $\square$

Now let $A, B \in S^1$ and $q_1, q_2 \in S^2$ be generic; we prove the following:

Lemma 1.5 (i) $[S^1W_{12} : (A, B)] = W_{12}$;

(ii) $[S^1W'_{12} : (A, B)] = W'_{12}$;

(iii) $[S^1W_{12} : (q_1, q_2)] = [W_{12} : S^1]$;

(iv) $[S^1W'_{12} : (q_1, q_2)] = [W'_{12} : S^1]$;

(v) $[W'_{12} : S^1] = [W_{12} : S^1] = \mathrm{pr}_{12}([W : S^1])$.

Proof (i) and (ii): let $(z_1, z_2) \in [S^1W'_{12} : (A, B)]$; then for $w \in W$ we have that $z_1w_2 - z_2w_1 \in R_F^{10}$ is annihilated by A and B. Since the map $(A, B)\colon R_F^{10} \to R_F^{11} \times R_F^{11}$ has kernel of dimension 6, it follows that the map $W \to R_F^{10}$ given by $w \mapsto z_1w_2 - z_2w_1$ has rank ≤ 6. By (f) it vanishes identically, hence $(z_1, z_2) \in [S^1W'_{12} : S^1]$, which proves (ii). Now if $(z_1, z_2) \in [S^1W_{12} : (A, B)]$ we know that $(z_1, z_2) \in W'_{12}$ and its image in W'_{12}/W_{12} is annihilated by A, B. Using (d) and the fact that A, B are generic, we conclude that $(z_1, z_2) \in W_{12}$. Indeed, if A annihilates a nonzero element of W'_{12}/W_{12}, $R_F^4 A$ annihilates W'_{12}/W_{12}, and clearly this is not true for generic A.

The proof of (iii) and (iv) are completely similar, so we only prove (v). If $z \in [W'_{12} : S^1]$, consider the map $S^1 \to W'_{12}/W_{12}$ given by $A \mapsto Az$: its kernel has dimension ≥ 3. But then for any $C \in S^1$, the annihilator of $Cz \in W'_{12}/W_{12}$ has dimension ≥ 3, which implies that $Cz = 0$, by the proof of Lemma 1.3. Hence $z \in [W_{12} : S^1]$. Finally let $z \in [W_{12} : S^1]$; for generic $A, B \in S^1$ we have $w = (Az_1, Az_2, w_3, w_4, w_5) \in [W_{12} : S^1]$ and $w'(Bz_1, Bz_2, w'_3, w'_4, w'_5) \in [W_{12} : S^1]$. So the first two components of $Bw - Aw' \in S^1W$ vanish. But we have already used the fact that $\mathrm{pr}_{12}\colon S^1W \to S^1W_{12}$ is injective; it follows that $Bw - Aw' = 0$ hence $w_i = Az_i$ and $w'_i = Bz_i$ for $i = 3, 4, 5$ for some $z_i \in R_F^4$, using Lemma 1.2. Then $(z_1, \ldots, z_5) \in [W : (A, B)]$, hence $(z_1, \ldots, z_5) \in [W : S^1]$ using (i) above. Therefore $z \in \mathrm{pr}_{12}([W : S^1])$. $\square$

Define $H^i_{q_1,q_2}$ as the middle cohomology group of the sequence

$$R_F^{i-2} \xrightarrow{(q_1,q_2)} R_F^i \times R_F^i \xrightarrow{q_2-q_1} R_F^{i+2}.$$

There are natural maps

$$\begin{aligned}
\alpha\colon\ & \mathrm{Ker}[B - A\colon S^1W_{12} \times S^1W_{12} \to S^2W_{12}] \to (H^6_{A,B})^2,\\
\alpha'\colon\ & \mathrm{Ker}[B - A\colon S^1W'_{12} \times S^1W'_{12} \to S^2W'_{12}] \to (H^6_{A,B})^2,\\
\beta\colon\ & \mathrm{Ker}[q_2 - q_1\colon S^1W_{12} \times S^1W_{12} \to S^3W_{12}] \to (H^6_{q_1,q_2})^2,\\
\beta'\colon\ & \mathrm{Ker}[q_2 - q_1\colon S^1W'_{12} \times S^1W'_{12} \to S^3W'_{12}] \to (H^6_{q_1,q_2})^2.
\end{aligned}$$

From Lemma 1.5 we deduce the following inequalities:

Corollary 1.6

$$\begin{aligned}
\dim S^2W_{12} &\geq 2\dim S^1W_{12} - \dim[S^1W_{12} : S^1] - \mathrm{rank}\,\alpha\\
&= 2\dim S^1W_{12} - 65 - \mathrm{rank}\,\alpha,\\
\dim S^2W'_{12} &\geq 2\dim S^1W'_{12} - \dim[S^1W'_{12} : S^1] - \mathrm{rank}\,\alpha'\\
&= 2\dim S^1W'_{12} - 67 - \mathrm{rank}\,\alpha',
\end{aligned}$$

$$\begin{aligned}
\dim S^3W_{12} &\geq 2\dim S^1W_{12} - \dim[S^1W_{12} : (q_1, q_2)] - \operatorname{rank}\beta \\
&= 2\dim S^1W_{12} - \dim[W_{12} : S^1] - \operatorname{rank}\beta, \\
\dim S^3W'_{12} &\geq 2\dim S^1W'_{12} - \dim[S^1W'_{12} : (q_1, q_2)] - \operatorname{rank}\beta' \\
&= 2\dim S^1W'_{12} - \dim[W_{12} : S^1] - \operatorname{rank}\beta'.
\end{aligned}$$

Using Lemma 1.3 and inequality (1.1), we get then the following inequalities

Corollary 1.7

$$\begin{aligned}
\dim S^2W_{12} &\geq 4\dim W - 65 - 2\dim[W : S^1] - \operatorname{rank}\alpha \\
&= 195 - 2\dim[W : S^1] - \operatorname{rank}\alpha, \\
\dim S^2W'_{12} &\geq 4\dim W - 67 + 12 - 2\dim[W : S^1] - \operatorname{rank}\alpha' \\
&= 205 - 2\dim[W : S^1] - \operatorname{rank}\alpha', \\
\dim S^3W_{12} &\geq 4\dim W - 3\dim[W : S^1] - \operatorname{rank}\beta \\
&= 260 - 3\dim[W : S^1], \\
\dim S^3W'_{12} &\geq 272 - 3\dim[W : S^1] - \operatorname{rank}\beta'.
\end{aligned}$$

Note that on $H^6_{A,B}$, there is a natural skew pairing with values in R^{12}_F, defined by $\langle (z_1, z_2), (z'_1, z'_2)\rangle = z_1 z'_2 - z_2 z'_1$. One can show that for generic $\varphi \in R^3_F$ (and generic A, B, X) the pairing $\langle\ ,\ \rangle_\varphi = \varphi\langle\ ,\ \rangle$ with value in $R^{15}_F = \mathbb{C}$ is nondegenerate. Similarly we can define a nondegenerate skew pairing $\langle\ ,\ \rangle_\varphi$ on $H^6_{q_1,q_2}$. Hence there is an induced symmetric pairing on $H^6_{A,B} \times H^6_{A,B}$, defined by

$$\Big((\alpha_1, \alpha_2), (\alpha'_1, \alpha'_2)\Big)_\varphi = \langle \alpha_1, \alpha'_2\rangle_\varphi + \langle \alpha'_1, \alpha_2\rangle_\varphi,$$

and similarly for $H^6_{q_1,q_2}$. We have the following result.

Lemma 1.8 $\operatorname{Im}\alpha$ *and* $\operatorname{Im}\alpha'$ *are orthogonal under* $\langle\ ,\ \rangle_\varphi$*, as are* $\operatorname{Im}\beta$ *and* $\operatorname{Im}\beta'$*.*

This is immediate using (b) and the definition of the pairing. □

Since we have $\operatorname{rank} H^6_{A,B} = 14$, and $\operatorname{rank} H^6_{q_1,q_2} = 50$, we deduce from Lemma 1.8 that

$$\operatorname{rank}\alpha + \operatorname{rank}\alpha' + \operatorname{rank}\beta + \operatorname{rank}\beta' \leq 128,$$

which together with Corollary 1.7 gives

Corollary 1.9

$$\dim S^2W_{12} + \dim S^2W'_{12} + \dim S^3W_{12} + \dim S^3W'_{12} \geq 804 - 10\dim[W : S^1].$$

But note that by definition $S^2W_{12} \subset R_F^7 \times R_F^7$ and $S^3W'_{12} \subset R_F^8 \times R_F^8$ are orthogonal with respect to the perfect pairing $\langle(\alpha_1, \alpha_2), (\beta_1, \beta_2)\rangle = \alpha_1\beta_2 - \alpha_2\beta_1 \in R_F^{15} \cong \mathbb{C}$. Similarly, $S^2W'_{12} \subset R_F^7 \times R_F^7$ and $S^3W_{12} \subset R_F^8 \times R_F^8$ are orthogonal with respect to the same pairing. Since $\dim R_F^7 = \dim R_F^8 = 155$, we conclude that

$$\dim S^2W_{12} + \dim S^2W'_{12} + \dim S^3W_{12} + \dim S^3W'_{12} \leq 4 \dim R_F^7 = 620,$$

Using Corollary 1.9, we conclude that

$$620 \geq 804 - 10 \dim[W : S^1],$$

hence $\dim[W : S^1] \geq 19$ and Proposition 1 is proved. $\square$

2 The degenerate case

In this section we assume that W_0 is degenerate in the sense of Definition 3 of 0.4. We first prove the following result.

Proposition 2 *If W_0 is degenerate and satisfies Property P of Definition 2, then one of the following cases holds:*

(i) *W_0 is the image of $R_{F_0}^4$ in $(R_{F_0}^5)^5$ for some isomorphism $S^1 \cong \mathbb{C}^5$.*

(ii) *Up to a change of basis of $\mathbb{C}^5$, W_0 contains the 5-tuples $(X_4z, 0, \ldots, 0)$ for any $z \in R_{F_0}^4$ and for some $T \in R_{F_0}^4$ (well defined modulo X_4), and for some A_i for $i = 1, \ldots, 4$ independent modulo X_4 in S^1, W_0 contains the 5-tuples $(AT, X_4^3A_iA)$ for any $A \in S^1$.*

Remark It is immediate to see that such a W_0 satisfies Property P. (b) holds because $X_4^4 = 0$ in R_{F_0}. In fact one can check that any W_0 as in the proposition can be realized as the limit of a family $W_t \subset (R_{F_t}^5)^5$ in Case (i).

By assumption, the first projection $\mathrm{pr}_1 \colon W_0 \to R_{F_0}^5$ is injective, thus of rank 65, and since $\operatorname{rank} X_4R_{F_0}^4 = 61$, the first component $\mathrm{pr}_1(W_0)$ of W_0 (see 0.3) is not contained in (X_4). We assume first that

there exists a component of W_0 contained in (X_4) but not in (X_4^2);

we may of course assume this is the second component $\mathrm{pr}_2(W_0)$ of W_0. We prove first the following proposition.

Proposition 3 *Under this supplementary assumption, W_0 is in Case (i) of Proposition 2.*

The proof of this proposition involves various technical steps, but the idea is the following: an element of W_0 is of the form $w = (w_1, X_4z, w_3, w_4, w_5)$, with $z \in R^4_{F_0}$. Now we have the relations given by property (b)

$$X_4(w_iz' - w_i'z) = 0 \quad \text{for } w, w' \in W_0, \tag{2.1}$$

which are part of the assumptions of the symmetrizer lemma (Theorem 4) for the correspondence $z \mapsto w_i$, and we would like to show that there are enough such relations to conclude that this correspondence is of the form $w_i = A_iz$ for some $A_i \in S^1$.

In doing this, we meet two problems: first, we need to show that enough elements z' are involved in the relations (2.1). Secondly, the factor X_4 in front of these relations makes them weakened symmetrizer relations; however, we show that they induce enough true symmetrizer relations in R_G.

This leads us to split the proof into two main steps. The first is Lemma 2.1, where we show that if w_i is divisible by X_4^k then z is also divisible by X_4^k. This shows that the graded pieces of W_0 for the filtration given by the order of divisibility by X_4 are also generated by elements of the form $X_4^k(w_1, X_4z, w_3, w_4, w_5)$, with $z \in R^{4-k}_{F_0}$ and $w_i \in R^{5-k}_{F_0}$. This step allows us to translate relations (2.1) into symmetrizer relations in R_G.

The second step is Lemma 2.7, where we show that there exists $A_i \in S^1$ such that $w_i = A_iz \bmod X_4$. This is done by showing that (2.1) gives enough symmetrized relations in R_G to apply the symmetrizer lemma. The conclusion of the proof of the Proposition is then comparatively easy.

We write W' for the 2-term projection $\mathrm{pr}_{12}(W_0) \subset (R^5_{F_0})^2$. An element of W' is of the form (w_1, x_4z_2). Using (h), we prove the following:

Lemma 2.1 *If $w_1 \in X_4^iR^{5-i}_{F_0}$ then $z_2 \in X_4^iR^{4-i}_{F_0}$.*

The proof proceeds through Lemmas 2.2–2.6.

Proof of Lemma 2.1 Set $Z_0 = [W_0 : S^1]$. We know that $\dim Z_0 \geq 19$, and of course the first projection is injective on Z_0, since it is on W_0. Thus $\dim \mathrm{pr}_1(Z_0) = \dim Z' \geq 19$, where $Z' = \mathrm{pr}_{12}(Z_0)$. We define a filtration on W' and Z' using the order w.r.t. X_4 of the first component; the successive ranks of the associated graded objects will be denoted by $\alpha, \beta, \gamma, \delta$ for W', and by $\alpha', \beta', \gamma', \delta'$ for Z'. So a basis of W' can be listed as

$$\left.\begin{array}{ll} \alpha \text{ elements} & (w_1, X_4z_2) \\ \beta \text{ elements} & (X_4z_1, X_4z_2') \\ \gamma \text{ elements} & (X_4^2t_1, X_4z_2'') \\ \delta \text{ elements} & (X_4^3q_1, X_4z_2''') \end{array}\right\}, \quad \text{where } w_1, z_1, t_1, q_1 \not\equiv 0 \bmod X_4.$$

Similarly, a suitable basis of Z' can be listed as

$$\left.\begin{array}{ll} \alpha' \text{ elements} & (z'_1, X_4 t_2) \\ \beta' \text{ elements} & (X_4 t'_1, X_4 t'_2) \\ \gamma' \text{ elements} & (X_4^2 q'_1, X_4 t''_2) \\ \delta' \text{ elements} & (X_4^3 A_1, X_4 t'''_2) \end{array}\right\}, \quad \text{where } z'_1, t'_1, q'_1, A_1 \not\equiv 0 \bmod X_4.$$

Of course, $\alpha+\beta+\gamma+\delta = 65$ and $\alpha'+\beta'+\gamma'+\delta' \geq 19$. We first assume $\alpha \geq 12$: (b) gives $w_1 t'_2 = 0$ in R_G^8 for any $(w_1, X_4 z_2) \in W'$ and $(X_4 t'_1, X_4 t'_2) \in Z'$, since $X_4 w_1 t'_2 - X_4^2 t'_1 z_2 = 0$ in $R_{F_0}^9$. Similarly $w_1 t''_2 = w_1 t'''_2 = 0$ in R_G^8 for $(X_4^2 q'_1, X_4 t''_2)$, $(X_4^3 A_1, X_4 t'''_2)$ elements of Z'. Next, we use the following result.

Lemma 2.2 *If G is generic, for any nonzero $t \in R_G^3$, the multiplication map $\mu_t\colon R_G^5 \to R_G^8$ by t has rank ≥ 29, that is, μ_t has kernel of dimension ≤ 11 (since* $\dim R_G^5 = 40$*). Equivalently, by duality, the map $\mu_t\colon R_G^4 \to R_G^7$ has kernel of dimension ≤ 2.* $\square$

It follows that if $\alpha \geq 12$, we have $t'_2 = t''_2 = t'''_2 = 0$ in R_G^3. We now write $t'_2 = X_4 q'_2, t''_2 = X_4 q''_2, t'''_2 = X_4 q'''_2$ and show exactly as above that $q''_2 = q'''_2 = 0$ in R_G^2, since $(X_4^2 q'_1, X_4^2 q''_2)$ and $(X_4^3 A_1, X_4^2 q'''_2) \in Z'$. Similarly we write $q'''_2 = X_4 A_2$ and find that $A_2 \equiv 0 \bmod X_4$ since $(X_4^3 A_1, X_4^3 A_2) \in Z'$. In other words the property stated in the lemma is true for Z', and we can list a suitable basis of Z' as

$$\left.\begin{array}{ll} \alpha' \text{ elements} & (z'_1, X_4 t_2) \\ \beta' \text{ elements} & (X_4 t'_1, X_4^2 q_2) \\ \gamma' \text{ elements} & (X_4^2 q'_1, X_4^3 A_2) \\ \delta' \text{ elements} & (X_4^3 A_1, 0) \end{array}\right\}, \quad \text{where } z'_1, t'_1, q'_1, A_1 \not\equiv 0 \bmod X_4.$$

We now note that, again by (b), $z'_2 t'_1 = 0$ in R_G^7 for $(X_4 z_1, X_4 z'_2) \in W'$ and $(X_4 t'_1, X_4^2 q_2) \in Z'$, since $X_4^2 z'_2 t'_1 - X_4^3 q_2 z_1 = 0$ in $R_{F_0}^9$. Now we use the following result.

Lemma 2.3 *If G is generic, for any nonzero $z \in R_G^4$ the multiplication map $\mu_z\colon R_G^3 \to R_G^7$ by z has kernel of dimension at most one. It follows by duality that $\mu_z\colon R_G^5 \to R_G^9$ has kernel of dimension ≤ 21.* $\square$

So if $\beta' \geq 2$, we have $z'_2 = 0$ in R_G^4, and similarly $z''_2 = z'''_2 = 0$ in R_G^4. We write then $z''_2 = X_4 t''_2$, $z'''_2 = X_4 t'''_2$ and using $\alpha \geq 12$ and Lemma 2.2, show that $t''_2 = t'''_2 = 0$ in R_G^3 since $(X_4^2 t_1, X_4^2 t''_2), (X_4^3 q_1, X_4^2 t'''_2) \in W'$. Then writing $t'''_2 = X_4 q'''_2$, we find again that $q'''_2 = 0$ in R_G^2, since $(X_4^3 q_1, X_4^3 q'''_2) \in W'$. So if $\alpha \geq 12$ and $\beta' \geq 2$, Lemma 2.1 is true. It also holds if $\alpha \geq 12$ and $\gamma' \geq 1$, using the following lemma.

Lemma 2.4 *If G is generic, for any nonzero $z \in R_G^4$, the multiplication map $\mu_z\colon R_G^2 \to R_G^6$ by z is injective.* $\square$

So assume $\alpha \geq 12$ but $\beta' \leq 1$, $\gamma' = 0$. In this case we have $\alpha' \geq 14$, since $\delta' \leq 4$. On the other hand, $z_2' z_1' = 0$ in R_G^8 for $(X_4 z_1, X_4 z_2') \in W'$ and $(z_1', X_4 t_2) \in Z'$, by (b), since $X_4^2 z_1 t_2 - X_4 z_1' z_2' = 0$ in $R_{F_0}^9$. Now we use the following result.

Lemma 2.5 *If G is generic, for any nonzero $z \in R_G^4$ the multiplication map $\mu_z\colon R_G^4 \to R_G^8$ by z has kernel of dimension ≤ 11.* $\square$

It follows that $z_2' = 0$ in R_G^4. Similarly we find that $z_2'' = z_2''' = 0$ in R_G^4. Exactly as above we conclude that Lemma 2.1 is true in this case.

It remains to consider the case where $\alpha \leq 11$. Note that $\gamma \leq 20 = \dim R_G^3$ and $\delta \leq 10 = \dim R_G^2$, so $\alpha \leq 11$ implies $\beta \geq 24$. Now for $(X_4 z_1, X_4 z_2') \in W'$ and $(w_1, X_4 z_2) \in W'$, we have $w_1 z_2' = 0$ in R_G^9, by (b), since $X_4 w_1 z_2' - X_4^2 z_1 z_2 = 0$ in $R_{F_0}^{10}$, hence the space spanned by the z_2' in R_G^4 has dimension ≤ 19 by the following lemma.

Lemma 2.6 *If G is generic, for any nonzero $w \in R_G^5$ the multiplication map $\mu_w\colon R_G^4 \to R_G^9$ by w has kernel of dimension ≤ 19. Equivalently, by duality, the map $\mu_w\colon R_G^3 \to R_G^8$ has kernel of dimension ≤ 8.* $\square$

End of proof of Lemma 2.1 It follows that there exists $(X_4 z_1', X_4 z_2'') \in W'$ with $z_2'' \equiv 0 \bmod X_4$ and $z_1' \not\equiv 0 \bmod X_4$. Then for any $(X_4 z_1, X_4 z_2') \in W'$, we have $z_1' z_2' = 0$ in R_G^8, by (b), hence the space generated by the z_2' in R_G^4 has dimension ≤ 11 by Lemma 2.5. But then the space generated by the z_1' in R_G^4 for $(X_4 z_1', X_4 z_2'') \in W'$, with $z_2' \equiv 0 \bmod X_4$ has dimension ≥ 13. Still using Lemma 2.5 and (b), we conclude that $z_2' = 0$ in R_G^4 for any $(X_4 z_1, X_4 z_2') \in W'$. It is then easy to conclude that W' satisfies the conclusion of Lemma 2.1. $\square$

By Lemma 2.1, a suitable basis of W' can be listed as

$$\left.\begin{array}{ll} \alpha \text{ elements} & (w_1, X_4 z_2) \\ \beta \text{ elements} & (X_4 z_1, X_4^2 t_2) \\ \gamma \text{ elements} & (X_4^2 t_1, X_4^3 q_2) \\ \delta \text{ elements} & (X_4^3 q_1, 0) \end{array}\right\}, \quad \text{where } w_1, z_1, t_1, q_1 \not\equiv 0 \bmod X_4.$$

We want to show the following result.

Lemma 2.7 *There exists $A \in S^1$, with $A \not\equiv 0 \bmod X_4$ such that*

$$w_1 \equiv A z_2, \quad z_1 \equiv A t_2 \quad \textit{and} \quad t_1 \equiv A q_2 \mod X_4.$$

The proof proceeds through Lemmas 2.8–2.11. We begin with the following statement.

Lemma 2.8 (i) $z_2 \equiv 0 \mod X_4 \implies w_1 \equiv 0 \mod X_4$.

(ii) $t_2 \equiv 0 \mod X_4 \implies z_1 \equiv 0 \mod X_4$.

(iii) $q_2 \equiv 0 \mod X_4 \implies t_1 \equiv 0 \mod X_4$.

Proof We only give the proof of (i), the others being similar, or in fact even easier: so assume $z_2 \equiv 0 \bmod X_4$ but $w_1 \not\equiv 0 \bmod X_4$; then for any $(X_4 z_1, X_4^2 t_2) \in W'$, we have $w_1 t_2 = 0$ in R_G^8, using (b), since $X_4^2 z_1 z_2 - X_4^2 w_1 t_2 = 0$ in $R_{F_0}^{10}$, and $z_2 \equiv 0 \bmod X_4$. Similarly, for any $(X_4^2 t_1, X_4^3 q_2) \in W'$, we have $w_1 q_2 = 0$ in R_G^7. Now we use Lemma 2.6 and the following result.

Lemma 2.9 *If G is generic, for any nonzero $w \in R_G^5$ the multiplication map $\mu_w\colon R_G^2 \to R_G^7$ by w has kernel of dimension at most one.* □

This implies that the space generated by the t_2 in R_G^3 (respectively, by the q_2 in R_G^2) has dimension ≤ 8 (respectively 1).

Suppose $\beta \geq 12$: then the second projection $\mathrm{pr}_2^1\colon W_1'/W_2' \to R_G^3$ (which is well defined as a consequence of Lemma 2.1) has kernel of dimension ≥ 4, where W_i' is the filtration defined at the beginning of the proof of Lemma 2.1. Now let $(X_4 z_1, X_4^2 t_2)$ with $t_2 \equiv 0 \bmod X_4$, be in this kernel; then using (b), we see that z_1 annihilates the image of pr_2^1: indeed, for $(X_4 z_1', X_4^2 t_2') \in W_1'$, we have $X_4^3 z_1 t_2' - X_4^3 z_1' t_2 = 0$ in $R_{F_0}^{10}$, and $t_2 \equiv 0 \bmod X_4$. By Lemma 2.2, we conclude that $\mathrm{pr}_2^1 = 0$. But then for any element $(X_4 z_1, X_4^2 t_2) \in W_1'$, and any $(w_1, X_4 z_2) \in W'$, we have $z_1 z_2 = 0$ in R_G^8 (by (b) and by $t_2 \equiv 0 \bmod X_4$). By Lemma 2.5 and $\beta \geq 12$ we conclude that $z_2 \equiv 0 \bmod X_4$, which contradicts our assumption that the second component of W' is not contained in X_4^2. Using the fact that the second projection $\mathrm{pr}_2^2\colon W_2'/W_3' \to R_G^2$ has rank at most one, we conclude similarly that $\gamma \geq 3$ is absurd (we use Lemma 2.3). So we must have $\beta \leq 11$ and $\gamma \leq 2$; but $\delta \leq 10 = \dim R_G^2$, hence $\alpha \geq 42 > \operatorname{rank} R_G^5$, which is absurd. □

If we use the notation $\mathrm{pr}_2^0\colon W'/W_1' \to R_G^4$, $\mathrm{pr}_2^1\colon W_1'/W_2' \to R_G^3$ and $\mathrm{pr}_2^2\colon W_2'/W_3' \to R_G^2$ for the second projections (modulo X_4), Lemma 2.8 gives the following

Corollary 2.10 $\operatorname{rank} \mathrm{pr}_2^0 + \operatorname{rank} \mathrm{pr}_2^1 + \operatorname{rank} \mathrm{pr}_2^2 \geq 55$.

This follows from $\alpha + \beta + \gamma + \delta = 65$ and $\delta \leq 10$. □

Since $\operatorname{rank} \mathrm{pr}_2^1 + \operatorname{rank} \mathrm{pr}_2^2 \leq 30 = \operatorname{rank} R_G^3 + \operatorname{rank} R_G^2$, we conclude that $\operatorname{rank} \mathrm{pr}_2^0 \geq 25$. This allows us to show the following:

Lemma 2.11 *For $(X_4^2 t_1, X_4^3 q_2), (X_4^2 t_1', X_4^3 q_2') \in W_2'$, we have*

$$t_1 q_2' - t_1' q_2 = 0 \quad \text{in } R_G^5.$$

Proof Since by (b), $w_1 q_2 - t_1 z_2 = 0$ in R_G^7 and $w_1 q_2' - t_1' z_2 = 0$ in R_G^7 for any $(w_1, X_4 z_2) \in W'$, we have $z_2(t_1 q_2' - t_1' q_2) = 0$ in R_G^9 for any $z_2 \in \operatorname{Im} \operatorname{pr}_2^0$. From $\operatorname{rank} \operatorname{pr}_2^0 \geq 25$, one concludes that $t_1 q_2' - t_1' q_2 = 0$ in R_G^5 using Lemma 2.6. □

Proof of Lemma 2.7 Now assume that $\operatorname{rank} \operatorname{pr}_2^1 \geq 18$ or $\operatorname{rank} \operatorname{pr}_2^2 \geq 8$. Then $\operatorname{Im} \operatorname{pr}_2^1$ generates R_G^5 (respectively, $\operatorname{Im} \operatorname{pr}_2^2$ generates R_G^5). Applying the symmetrizer lemma (Theorem 4) to G and to $R_G^2(W_1'/W_2') \subset R_G^6 \times R_G^5$ (respectively, $R_G^3(W_2'/W_3') \subset R_G^6 \times R_G^5$), which surject on the second factor and by Lemma 2.11 satisfy (b), that is, the assumptions of the symmetrizer lemma, we find that there exists $A \in S^1/(X_4)$ such that $z_1 \equiv A t_2 \bmod X_4$ for all $(X_4 z_1, X_4^2 t_2) \in W_1'/W_2'$ (respectively, there exists $A \in S^1/(X_4)$ such that $t_1 \equiv A q_2 \bmod X_4$ for all $(X_4^2 t_1, X_4^3 q_2) \in W_2'/W_3'$). In either case, we conclude that $w_1 \equiv A z_2 \bmod X_4$ for $(w_1, X_4 z_2) \in W'$, since by (b) $w_1 - A z_2 \in R_G^5$ is annihilated by a 18 dimensional subspace of R_G^3 (respectively, by an 8 dimensional subspace of R_G^2). Since z_2 varies in a 25 dimensional subspace of R_G^4, one concludes easily that $z_1 \equiv A t_2$ and $t_1 \equiv A q_2 \bmod X_4$.

Finally, if $\operatorname{rank} \operatorname{pr}_2^2 \leq 7$ and $\operatorname{rank} \operatorname{pr}_2^1 \leq 17$, we have $\operatorname{rank} \operatorname{pr}_2^0 \geq 31$ by Corollary 2.10, that is, pr_2^0 is surjective. But then we can apply the symmetrizer lemma to $W'/W_1' \subset R_G^5 \times R_G^4$ and we can conclude the proof as above. □

Proof of Proposition 3 It is easy to conclude from Lemma 2.7 that Proposition 3 holds if the composite map $\mathbb{C}^5 \to \operatorname{Hom}(W_0, R_G^5)$ has rank 4; indeed, applying Lemma 2.7 to a generic component of W' in place of the first, we conclude that after a change of basis of $\mathbb{C}^5$, there exists elements $A_1, \ldots, A_4$ of S^1 such that elements of W_0 have the form $w = (w_1, \ldots, w_4, X_4 z_5)$, with $w_i \equiv X_4^l A_i z_5' \bmod X_4^{l+1}$ when $z_5 = X_4^l z_5'$, except maybe if $l \geq 3$, in which case $w_i \in (X_4^3)$. In fact, this last restriction can easily be removed, because if $X_4^3(w_1', \ldots, w_4', 0) \in W_0$, by (b), $z_5(A_i w_j' - A_j w_i') \equiv 0 \bmod X_4$ for at least one nonzero $z_5 \in R_G^4$. By Lemma 2.3, this implies that the space generated by the $A_i w_j' - A_j w_i'$ in R_G^3 has dimension at most one. But clearly the A_i have to be independent modulo X_4, otherwise $\mathbb{C}^5 \to \operatorname{Hom}(W_0, R_G^5)$ will not have rank 4; it is then immediate to conclude that in fact $A_i w_j' - A_j w_i'$ all vanish. Hence by the symmetrizer lemma, we find that $w_i' \equiv A A_i \bmod X_4$ for some $A \in S^1$.

Now it follows from this that W_0 can be deformed to $W_0' = R_{F_0}^4 \subset \operatorname{Hom}(S^1, R_{F_0}^5)$, by a deformation preserving (b): for this we use the one parameter group of automorphisms of the ring R_F given by multiplication of

X_4 by $\lambda \in \mathbb{C}^*$ and note that the limit of $\lambda(W_0)$ as λ tends to 0 is equal to the W'_0 corresponding to the basis of S^1 given by $A'_1, \ldots, A'_i, X_4$, where A'_i is the unique element of S^1 such that $A'_i \equiv A_i \bmod X_4$, and not involving the coordinate X_4. We conclude then that $W_0 = W'_0$ for some choice of basis of S^1 using the following easy rigidity lemma:

Lemma 2.12 *Choose a basis of S^1; the only (small) deformations of $W'_0 = R^4_{F_0} \subset \mathrm{Hom}(S^1, R^5_{F_0}) = (R^5_{F_0})^5$ preserving (b) come from a change of basis of S^1.* $\square$

So to conclude the proof of Proposition 3, we have to exclude the existence of another component of W_0 contained in (X_4). We may assume such a component is the third one; now let W' be the projection $\mathrm{pr}_{123}(W_0) \subset (R^5_{F_0})^3$. As above, we put a filtration W'_i on W', where W'_i is the set of elements whose first component lies in (X_4^i), and denote by $\alpha, \beta, \gamma, \delta$ the ranks of the successive quotients; using Lemmas 2.1, 2.7, we find that a suitable basis of W' can be listed as follows:

$$\left.\begin{array}{ll} \alpha \text{ elements} & (w_1, X_4 z_2, X_4 z_3) \\ \beta \text{ elements} & (X_4 z_1, X_4^2 t_2, X_4^2 t_3) \\ \gamma \text{ elements} & (X_4^2 t_1, X_4^3 q_2, X_4^3 q_3) \\ \delta \text{ elements} & (X_4^3 q_1, 0, 0) \end{array}\right\}, \tag{2.2}$$

and there exists $A_2, A_3 \in S^1$ such that $w_1 \equiv A_2 z_2 \equiv A_3 z_3$, $z_1 \equiv A_2 t_2 \equiv A_3 t_3$ and $t_1 \equiv A_2 q_2 \equiv A_3 q_3 \bmod X_4$. It is easy to show that this implies that $A_2 \equiv A_3 \bmod X_4$, using the fact that $\alpha + \beta + \gamma \geq 55$. It follows that some combination of the second and the third component lies in (X_4^2). Of course we can assume the third component lies in (X_4^2). We use now Lemma 2.14 below to conclude that the third component is in fact contained in (X_4^3). We then list a suitable basis of W' as

$$\left.\begin{array}{ll} \alpha \text{ elements} & (w_1, X_4 z_2, X_4^3 q_3) \\ \beta \text{ elements} & (X_4 z_1, X_4^2 t_2, X_4^3 q'_3) \\ \gamma \text{ elements} & (X_4^2 t_1, X_4^3 q_2, X_4^3 q''_3) \\ \delta \text{ elements} & (X_4^3 q_1, 0, X_4^3 q'''_3) \end{array}\right\}, \tag{2.3}$$

with $w_1 \equiv A z_2 \bmod X_4$ etc. In particular, we find that $\alpha \geq 25$, since $\beta \leq 20$, $\gamma \leq 10$ and $\delta \leq 10$. Now clearly by (b), $w_1 q'_3 = w_1 q''_3 = w_1 q'''_3 = 0$ in R^7_G, in the notation of (2.3). We now use the following lemma:

Lemma 2.13 *If G is generic, for any nonzero $q \in R^2_G$ the multiplication map $\mu_q \colon R^5_G \to R^7_G$ by q has kernel of dimension ≤ 9.* $\square$

It follows that $q'_3 = q''_3 = q'''_3 = 0$ in R^2_G. Now it follows that the third projection $W'/W'_1 \to R^2_G$ is well defined and has kernel of dimension ≥ 15. If $(w'_1, X_4 z'_2, 0)$ belongs to this kernel, by (b), $w'_1 q_3 = 0$ in R^7_G for any q_3, and using Lemma 2.13 again, this implies that $q_3 = 0$ in R^2_G. But then the third component of W' vanishes identically, which contradicts the fact that the map $\mathbb{C}^5 \to \mathrm{Hom}(W_0, R^5_{F_0})$ is injective. Hence Proposition 3 is proved. □

Proof of Proposition 2 By Proposition 3, to prove Proposition 2, it remains only to study the case where any component of W_0 contained in (X_4) is contained in (X_4^2). We split the proof in several lemmas. We first show (Lemma 2.14) that any component contained in (X_4^2) is contained in (X_4^3). We then show (Lemma 2.15) that if some component is contained in (X_4^3), then at most one component is not contained in X_4. We then conclude that W_0 has the form (ii) of Proposition 2. We begin with the following result.

Lemma 2.14 *Any component of W_0 contained in (X_4^2) is contained in (X_4^3).*

Proof We may of course assume it is the second component. We again set $W' = \mathrm{pr}_{12}(W_0) \subset (R^5_{F_0})^2$ and introduce the filtration W'_i used above (by the order of vanishing in X_4 of the first component), with successive quotients of ranks $\alpha, \beta, \gamma, \delta$. Then by assumption, a suitable basis of W' can be listed as

$$\left.\begin{array}{ll} \alpha \text{ elements} & (w_1, X_4^2 t_2) \\ \beta \text{ elements} & (X_4 z_1, X_4^2 t'_2) \\ \gamma \text{ elements} & (X_4^2 t_1, X_4^2 t''_2) \\ \delta \text{ elements} & (X_4^3 q_1, X_4^2 t'''_2) \end{array}\right\}. \tag{2.4}$$

Assume $\alpha > 11$; using (b), in the notation of (2.4), we must have $w_1 t'_2 = w_1 t''_2 = w_1 t'''_2 = 0$ in R^8_G, and by Lemma 2.2, it follows that $t'_2 = t''_2 = t'''_2 = 0$ in R^3_G. Similarly, writing $t''_2 = X_4 q''_2, t'''_2 = X_4 q'''_2$, we find that $q''_2 = q'''_2 = 0$ in R^2_G. So a suitable basis of W' can in fact be represented as

$$\left.\begin{array}{ll} \alpha \text{ elements} & (w_1, X_4^2 t_2) \\ \beta \text{ elements} & (X_4 z_1, X_4^3 q_2) \\ \gamma \text{ elements} & (X_4^2 t_1, 0) \\ \delta \text{ elements} & (X_4^3 q_1, 0) \end{array}\right\}.$$

In this notation, if $q_2 \equiv 0 \bmod X_4$, then $z_1 t_2 = 0$ in R^7_G, since $X_4^3 w_1 q_2 - X_4^3 z_1 t_2 = 0$ in $R^{10}_{F_0}$. If there is at least one nonzero t_2, it follows by Lemma 2.2 that the space generated by such z_1 in R^4_G has dimension ≤ 2, which implies

$\beta \leq 12$. One shows similarly that $\alpha \leq 20$, and since $\gamma \leq 20 = \operatorname{rank} R_G^3$ and $\delta \leq 10 = \operatorname{rank} R_G^2$, we contradict $\alpha + \beta + \gamma + \delta = 65$.

Next consider the case where $\alpha \leq 11$; then $\beta \geq 24$. Using the notation of (2.4), $w_1 t_2' = 0$ in R_G^8, which by Lemma 2.6 implies that the space generated by the t_2' modulo X_4 has dimension ≤ 8. It follows that the space generated by the t_2' modulo X_4^2 has dimension ≤ 18, hence that there is a 6 dimensional space of z_1 modulo X_4 such that $(X_4 z_1, 0) \in W'$. But these z_1 annihilate $t_2 \in R_G^3$ for $(w_1, X_4^2 t_2) \in W'$. Hence $t_2 \equiv 0 \bmod X_4$ by Lemma 2.2 and we conclude again that the second component is contained in (X_4^3). $\square$

We now show the following:

Lemma 2.15 *Assume there is one component of W_0 contained in (X_4^3). Then the map $\mathbb{C}^5 \to \operatorname{Hom}(W_0, R_G^5)$ has rank 1.*

Proof It has rank at least 1, since the first component, being of rank 65, cannot be contained in (X_4). Now assume it has rank ≥ 2. Up to a change of basis of $\mathbb{C}^5$, we may assume that the two first components are independent modulo (X_4), and the third is contained in (X_4^3). Now we use a proposition proved in Section 4 (cf. Proposition 6).

Proposition 4 *Let $W \subset (R_{F_0}^5)^2$ be a 65 dimensional subset satisfying (b). Then, if the map $\mathbb{C}^2 \to \operatorname{Hom}(W, R_G^5)$ is injective, and the first projection $W \to R_{F_0}^5$ is injective, we have*

$$w_1 \in (X_4^i) \iff w_2 \in (X_4^i) \quad \textit{for } (w_1, w_2) \in W.$$

Let W' be the projection of W_0 on the first three factors. According to Proposition 4, a basis of W' can be represented as follows:

$$\left.\begin{array}{ll} \alpha \text{ elements} & (w_1, w_2, X_4^3 q_3) \\ \beta \text{ elements} & (X_4 z_1, X_4 z_2, X_4^3 q_3') \\ \gamma \text{ elements} & (X_4^2 t_1, X_4^2 t_2, X_4^3 q_3'') \\ \delta \text{ elements} & (X_4^3 q_1, X_4^3 q_2, X_4^3 q_3''') \end{array}\right\} . \tag{2.5}$$

with w_1, w_2 independent modulo X_4, z_1, z_2 independent modulo X_4, etc. Now it is easy to see that α must be strictly larger than 10. Furthermore, in the notation of (2.5), $w_1 q_3' = w_1 q_3'' = w_1 q_3''' = 0$ in R_G^7, and this implies that $q_3' = q_3'' = q_3''' = 0$ in R_G^2, by Lemma 2.13. Now, since $\alpha > 10 = \operatorname{rank} R_G^2$, there exists $(w_1', w_2', X_4^3 q_3') \in W'$, with $q_3' \equiv 0 \bmod X_4$ and $w_1' \not\equiv 0 \bmod X_4$. Then for any $(w_1, w_2, X_4^3 q_3) \in W'$, by (b), $w_1' q_3 = 0$ in R_G^7, and this implies that the space generated by the q_3 in R_G^2 has dimension at most one, by Lemma 2.9. But then there is set of $(w_1', w_2', 0) \in W'/W_1'$ of dimension ≥ 10

and by Lemma 2.13 the condition $w_1'q_3 = 0$ in R_G^7 implies that $q_3 = 0$. So the third component should vanish identically, which is absurd, since the map $\mathbb{C}^5 \to \mathrm{Hom}(W_0, R_{F_0}^5)$ is injective. $\square$

It now follows from Lemmas 2.15, 2.14 and Proposition 3 that if there is one component of W_0 contained in (X_4^3), there is in fact a 4 dimensional set of such components so that after a change of basis of $\mathbb{C}^5$, we may filter W_0 by the order in X_4 of the first component and represent elements of W_0 as follows

$$\left.\begin{array}{ll} \alpha \text{ elements} & (w_1, X_4^3q_2, \ldots, X_4^3q_5) \\ \beta \text{ elements} & (X_4z_1, X_4^3q_2', \ldots, X_4^3q_5') \\ \gamma \text{ elements} & (X_4^2t_1, X_4^3q_2'', \ldots, X_4^3q_5'') \\ \delta \text{ elements} & (X_4^3q_1, X_4^3q_2''', \ldots, X_4^3q_5''') \end{array}\right\}. \tag{2.6}$$

End of proof of Proposition 2 To conclude the proof of Proposition 2, we want to show now that $q_i' = q_i'' = q_i''' \equiv 0 \bmod X_4$ and that w_1, q_i are in Case (ii) of Proposition 2. First of all we have the result:

Lemma 2.16 $\alpha \leq 10$.

Proof If $\alpha > 10$ the second projection $W_0/(W_0)_1 \to R_G^2$ is not injective. So there exists $(w_1', 0, X_4^3q_i) \in W_0$ with $w_1' \neq 0$ in R_G^5. Then in the notation of (2.6), $w_1'q_2 = 0$ in R_G^7, and by Lemma 2.9 this implies that the space generated by the q_2 in R_G^2 has dimension at most one. Hence the space generated by the w_1' as above in R_G^5 has dimension ≥ 10, and since it annihilates q_2, we conclude that $q_2 = 0$ in R_G^2 by Lemma 2.13. Also since $w_1q_i' = w_1q_i'' = w_1q_i''' = 0$ in R_G^7, and $\alpha \geq 11$, we find that $q_i' = q_i'' = q_i''' = 0$ in R_G^2, so that the second component of W_0 vanishes, in contradiction with our assumptions. $\square$

Now note that in the notation of (2.6), $w_1q_i' = w_1q_i'' = w_1q_i''' = 0$ in R_G^7, which by Lemma 2.9 implies that the space generated by the q_i', q_i'', q_i''' in R_G^2 has dimension at most one, and by Lemma 2.13, it is zero if $\alpha = 10$.

From Lemma 2.16, it follows that $\beta + \gamma + \delta \geq 55$, with equality possible only if the q_i', q_i'', q_i''' vanish. This implies that for any i, j there is at least a 54 dimensional subspace of $R_{F_0}^4$ consisting of elements z such that $(X_4z, X_4^3q_1, \ldots, X_4^3q_4) \in W_0$, with $q_i \equiv q_j \equiv 0 \bmod X_4$. Then it is immediate to conclude

Corollary 2.17 *The subspace of $R_{F_0}^5W_0$ made of elements of the form*

$$(X_4\varphi, X_4^3\chi_1, \ldots, X_4^3\chi_4) \quad \textit{with} \quad \chi_i = \chi_j = 0 \quad \textit{in } R_G^7$$

maps surjectively by the first projection onto $X_4R_{F_0}^9$.

Note that $X_4R^9_{F_0}$ has dimension 91. We want now to use (c$'$) which says that $\dim R^5_{F_0}W_0 \leq 135$: using Corollary 2.17, it implies that the projection onto the product of the $(i+1)$st and $(j+1)$st factors of $R^5_{F_0}W_0$ has dimension ≤ 44. Now we have the following.

Lemma 2.18 *Let $(q_1, q_2), (q'_1, q'_2) \in R^2_G \times R^2_G$; assume that $\dim R^5_G(q_1, q_2) + R^5_G(q'_1, q'_2) \leq 44$. Then $q_1q'_2 - q_2q'_1 = 0$ in R^4_G.*

Proof We may assume that q'_1, q'_2 are not proportional. Let $Z \subset R^5_G \times R^5_G$ be the kernel of the map

$$(q_1, q_2) + (q'_1, q'_2)\colon R^5_G \times R^5_G \to R^7_G \times R^7_G.$$

The first projection of Z in R^5_G has dimension ≥ 36, using Lemma 2.9, and $\dim R^5_G = 40$. On the other hand, it is easy to see that it annihilates $q_1q'_2 - q_2q'_1$. Hence $q_1q'_2 - q_2q'_1 = 0$ in R^4_G by Lemma 2.3. $\square$

Combining Corollary 2.17 and Lemma 2.18, we find

Corollary 2.19 *Let $(w_1, X_4^3q_1, \ldots, X_4^3q_5), (w'_1, X_4^3q'_1, \ldots, X_4^3q'_5) \in W_0$; then $q_iq'_j - q'_iq_j = 0$ in R^4_G.*

It follows from the corollary that if $q_i \equiv 0 \bmod X_4$, but $(q_1, \ldots, q_4) \not\equiv 0 \bmod X_4$, any other q'_i vanishes, because for generic G, the product of two nonzero elements of R^2_G does not vanish in R^4_G. Since no component of W_0 vanishes, we find that if $(q_1, \ldots, q_4) \not\equiv 0 \bmod X_4$, the q_i are independent modulo X_4. It is then easy to conclude from Corollary 2.19, that if $\operatorname{rank}(\mathrm{pr}_{2345}(W_0)) \geq 2$ there exists $(A_1, \ldots, A_4) \in (S^1)^4$, well defined modulo X_4, such that any $X_4^3(q_1, \ldots, q_4) \in \mathrm{pr}_{2345}(W_0)$ is of the form $AX_4^3(A_1, \ldots, A_4)$ for some $A \in S^1$.

Now the rank of $\mathrm{pr}_{2345}(W_0)$ cannot be equal to 1; for otherwise there is

$$(w'_1, 0, \ldots, 0) \in W_0 \quad \text{with } w'_1 \neq 0 \text{ in } R^5_G;$$

then for $(w_1, X_4^3q_i) \in W_0$, we have $w'_1q_i = 0$ in R^7_G, which by Lemma 2.9 implies that all the q_i are proportional. But then the four last components of W_0 are not independent, since they have one dimensional projection in $(R^5_{F_0})^4$, generated by $(X_4^3q_i)$ and the q_i are not independent in $R^5_{F_0}$.

Now assume $(w'_1, 0, \ldots, 0) \in W_0$ and $(w_1, X_4^3AA_i) \in W_0$, with $A \not\equiv 0 \bmod X_4$: then $w'_1AA_i = 0$ in R^7_G, and using the fact that the A_i are independent modulo X_4, we conclude by Macaulay's theorem that $Aw'_1 = 0$ in R^6_G hence that $w'_1 \equiv 0 \bmod X_4$, using the following lemma.

Lemma 2.20 *If G is generic, for any $w \neq 0$ in R^5_G and $A \neq 0$ in $S^1/(X_4)$, we have $Aw \neq 0$ in R^6_G.* $\square$

Since $\operatorname{rank} \mathrm{pr}_1(W_0) = 65$ and $\operatorname{rank} X_4 R^4_G = 61$, we find by the above that the projection pr_{2345} has rank ≥ 4, hence that its image is made of the $(X_4^3 A A_i)$ for arbitrary $A \in S^1$. Finally, since $w_1 \not\equiv 0 \bmod X_4$ implies that $A \not\equiv 0 \bmod X_4$, we can consider w_1 modulo X_4 as a function of A, and (b) implies that $w_1(A)A' - w_1(A')A \equiv 0 \bmod X_4$, since it is annihilated by all the A_i. The symmetrizer lemma implies then that there exists $T \in R^4_G$, such that $w_1(A) = TA$ in R^5_G.

Finally, $ATq'_i = ATq''_i = ATq'''_i = 0$ in R^7_G for any $A \in S^1$, and this implies that $Tq'_i = Tq''_i = Tq'''_i = 0$ in R^6_G by Macaulay's theorem. But then $q'_i = q''_i = q'''_i = 0$ in R^2_G, by Lemma 2.4. Hence W_0 is in Case (ii) of Proposition 2, and this proves Proposition 2. □

3 Proof of Proposition 2 $\Longrightarrow$ Theorem 5

Proposition 5 *Assume W_0 is in Case (i) or (ii) of Proposition 2. Then for a generic deformation F of F_0, any deformation $W \subset (R^5_F)^5$ of $W_0 \subset (R^5_{F_0})^5$ as a 65 dimensional subspace satisfying (b) and (c′) must be identified with the subspace $R^4_F \subset \operatorname{Hom}(S^1, R^5_F)$, for some isomorphism $S^1 \cong \mathbb{C}^5$.*

Proof If W_0 is in Case (i) of Proposition 2, the result is easy. Indeed, we know Lemma 2.12 that W_0 is infinitesimally rigid modulo $\operatorname{Aut} \mathbb{C}^5$ as a subspace of $(R^5_{F_0})^5$ satisfying (b). Thus consider the variety $\mathcal{G} \to U$ parametrizing 65 dimensional subspaces of $(\mathcal{R}^5)^5$ satisfying (b) and (c′) with respect to $\mu\colon S^2\mathcal{R}^5 \to \mathcal{R}^{10}$. Its infinitesimal dimension at (W_0, F_0) is $\leq 24 + \dim U$. But this is the dimension of the variety $\mathcal{G}' \subset \mathcal{G}$ parametrizing $W \cong R^4_F \subset \operatorname{Hom}(S^1, R^5_F)$. Hence we must have $\mathcal{G}' = \mathcal{G}$ in a neighbourhood of (W_0, F_0).

Thus we only need to study Case (ii). We note first that any $W_0 \subset (R^5_{F_0})^5$ in Case (ii) of Proposition 2 is in the Zariski closure of $\mathcal{G}'$: it is enough to consider a family $W_t \cong R^4_{F_t} \subset (R^5_{F_t})^5$ for a suitable one parameter family F_t, and for a family of isomorphisms $S^1 \cong_t \mathbb{C}^5$ given by a basis $A_0(t), \dots, A_4(t)$ of S^1 such that $A_0(0) = X_4$, and to take the limit W_0 of W_t such that $\mathrm{pr}_1\colon W_0 \to R^5_{F_0}$ is injective. (This supposes of course making elementary transformations over 0 on the bundle $\mathcal{W}$ with fibre W_t.) Then one finds that the limiting W_0 has the expected form, with $T = \frac{\partial}{\partial X_4}\left(\frac{dF_t}{dt}\big|_{t=0}\right)$.

Furthermore it is easy to see that the map $\mathcal{G}' \to U$ has relative dimension 24, and proper over the open set of U where the following property is satisfied:

> *The multiplication map $\mu_X\colon R^4_F \to R^5_F$ by X is injective for any nonzero $X \in S^1$.*

So, to conclude that $\mathcal{G} \subset \overline{\mathcal{G}'}$ near (W_0, F_0) (the statement of Proposition 5), it is enough to prove the following.

Lemma 3.1 *The dimension of the Zariski tangent space $T_{\mathcal{G},(W_0,F_0)}$ to $\mathcal{G}$ at (W_0, F_0) equals* $\dim U + 24$.

Proof Choosing a lifting of $R^5_{F_0}$ to S^5 gives a trivialization of the bundle $\mathcal{R}^5$ near F_0. An infinitesimal deformation of (W_0, F_0) is then described by maps $h, h_i \colon S^1 \to R^5_{F_0}$ and $k, k_i \colon R^4_{F_0} \to R^5_{F_0}$, together with a deformation $F_0 + \varepsilon H$ of F_0. The corresponding deformation $W_\varepsilon \subset (R^5_{F_\varepsilon})^5 \cong (R^5_{F_0})^5$ is generated by the vectors

$$\begin{aligned}(TA + \varepsilon h(A), A_i X_4^3 A + \varepsilon h_i(A)) \quad \text{for } A \in S^1\\ \text{and} \quad (X_4 z + \varepsilon k(z), \varepsilon k_i(z)) \quad \text{for } z \in R^4_{F_0}.\end{aligned}$$

Of course the infinitesimal deformation is trivial if $H = 0$ and $(h(A), h_i(A))$, $(k(z), k_i(z)) \in W_0$. Differentiating the equations given by (b), we get the following equations

(i) $X_4 z k_i(z') = X_4 z' k_i(z)$.

(ii) $ATk_i(z) = X_4 z h_i(A) + A_i X_4^3 A k(z) - \frac{1}{5}\frac{\partial H}{\partial X_4} A_i A z$.

(iii) $A_i X_4^3 A k_j(z) = A_j X_4^3 A k_i(z)$.

(iv) $A_i X_4^3 A h_j(A') + A_j X_4^3 A' h_j(A) = A_i X_4^3 A' h_j(A) + A_j X_4^3 A h_j(A')$.

(v) $h(A) A_i X_4^3 A' + ATh_i(A') = h(A') A_i X_4^3 A + A'Th_i(A)$.

These equations have to hold in $R^{10}_{F_0}$ for any $A, A' \in S^1$ and for any $z, z' \in R^4_{F_0}$. Applying the symmetrizer lemma to (i), we conclude that $k_i(z) = Y_i z + X_4^3 k_i'(z)$ for some $Y_i \in S^1$. The condition (ii) modulo X_4 then gives

$$ATY_i z = -\frac{1}{5}\frac{\partial H}{\partial X_4} A_i A z \quad \text{in } R^{10}_G,$$

which implies that $TY_i = -\frac{1}{5}\frac{\partial H}{\partial X_4} A_i$ in R^5_G. Using the fact that the map $\mu_T \colon R^2_G \to R^6_G$ is injective (Lemma 2.4), we conclude that $A_i Y_j = A_j Y_i$ in R^2_G, hence that $Y_i = \alpha A_i + \alpha_i X_4$, by the symmetrizer lemma. Furthermore $-\frac{1}{5}\frac{\partial H}{\partial X_4} = \alpha T$ in R^4_G: we write $-\frac{1}{5}\frac{\partial H}{\partial X_4} = \alpha T + X_4 H'$.

Condition (ii) modulo X_4^3 then gives

$$AT\alpha_i X_4 z = X_4 z h_i(A) + X_4 H' A z \quad \text{in } R^{10}_{F_0}/(X_4^3),$$

i.e., $h_i(A) = \alpha_i TA - H'A$ in $R^5_{F_0}/(X_4^2)$. Hence $h_i(A) = \alpha_i TA - H'A + X_4^2 h_i'(A)$. Then Condition (ii) gives

$$ATk_i'(z) = zh_i'(A) + A_i A k(z) \quad \text{in } R^7_G \tag{3.1}$$

It follows that $z(Ah_i'(A') - A'h_i'(A)) = 0$ in R_G^8 for any $z \in R_G^4$. Hence by the symmetrizer lemma there exists $q_i \in R_G^2$ such that $h_i'(A) \equiv q_i A \bmod X_4$. But then (3.1) gives $z(A_j q_i - A_i q_j) = 0$ in R_G^7, whenever $k_i'(z) = k_j'(z) = 0$ in R_G^2. By Lemma 2.2, it follows that $(A_j q_i - A_i q_j) = 0$ in R_G^3, hence by the symmetrizer lemma that $q_i \equiv A_i B \bmod X_4$. Hence $h_i'(A) \equiv A_i BA \bmod X_4$, and (3.1) then gives

$$Tk_i'(z) = zBA_i + A_i k(z) \quad \text{in } R_G^6 \tag{3.2}$$

Then $A_i k_j'(z) - A_j k_i'(z) \in R_G^3$ is annihilated by T. But $\mu_T \colon R_G^3 \to R_G^7$ has kernel of dimension at most one. It is easy to conclude from this that $A_i k_j'(z) - A_j k_i'(z)$ must in fact vanish identically in R_G^3, so that by the symmetrizer lemma, $k_i'(z) \equiv A_i k''(z) \bmod X_4$, for some function $k'' \colon R_{F_0}^4 \to S^1$. Finally (3.2) gives $Tk''(z) \equiv zB + k(z) \bmod X_4$. Summarizing, we have found

$$\begin{aligned} k(z) &\equiv Tk''(z) - zB \mod X_4, \\ k_i(z) &= \alpha A_i z + X_4^3 A_i k''(z) + \alpha_i X_4 z, \\ h_i(A) &= \alpha_i TA - H'A + X_4^2 A_i AB + X_4^3 h_i''(A), \\ \text{and} \quad -\frac{1}{5}\frac{\partial H}{\partial X_4} &= \alpha T + X_4 H' \quad \text{in } R_{F_0}^4. \end{aligned}$$

Now $(Tk''(z), X_4^3 A_i k''(z)) \in W_0$, and $(X_4\varphi, 0, \dots 0) \in W_0$. So the deformation induced by $(k(z), k_i(z)) \in (R_{F_0}^5)^5$ depends only on $(k(z) \mod X_4, k_i(z))$, and not on $k''(z)$. It follows that $(k(z), k_i(z)) \in (R_{F_0}^5)^5/W_0$ depends only on B modulo X_4, α, α_i. Hence it varies in a 9 dimensional vector space. Let us now consider the deformations of (W_0, F_0) for which the corresponding infinitesimal deformation of F_0 is trivial, and $\alpha, \alpha_i, B \equiv 0 \bmod X_4$. We now use (c′) to describe them: notice that for W_0 in Case (ii) we have in fact $\dim R_{F_0}^5 W_0 = 135$. Hence the infinitesimal version of condition (c′) gives the following

Suppose that $\varphi, \psi \in R_{F_0}^5$ *and* $(h(A), h_i(A)), (h(A'), h_i(A'))$ *are infinitesimal deformations of* $(TA, A_i A X_4^3), (TA', A_i A' X_4^3) \in W_0$. *Assume that*

$$\varphi(TA, A_i A X_4^3) + \psi(TA', A_i A' X_4^3) = 0 \quad \textit{in } (R_{F_0}^{10})^5.$$

Then $\varphi(h(A), h_i(A)) + \psi(h(A'), h_i(A')) \in R_{F_0}^5 W_0$.

In particular, for any $z \in R_{F_0}^4$, we have

$$zA'(TA, X_i X_4^3 A) + (-zA)(TA', X_i X_4^3 A') = 0.$$

Hence we must have $zA'(h(A), h_i(A)) - zA(h(A'), h_i(A')) \in R_{F_0}^5 W_0$.

Now, since $\alpha = 0, B = 0, H' = 0$, we have $h_i(A) = X_4^3 h_i''(A)$, hence

$$zA'\big(h(A), X_4^3 h_i''(A)\big) - zA\big(h(A'), X_4^3 h_i''(A')\big) \in R^5_{F_0} W_0.$$

Clearly this is equivalent to: there exists $\varphi \in R^6_G$ depending on z, A, A', such that

$$z\big(A'h(A) - Ah(A')\big) = T\varphi, \quad z\big(A'h_i''(A) - Ah_i''(A')\big) = A_i\varphi \quad \text{in } R^7_G. \tag{3.3}$$

From the second equation, we deduce that

$$z\big(A_j(A'h_i''(A) - Ah_i''(A')) - A_i(A'h_j''(A) - Ah_j''(A'))\big) = 0 \quad \text{in } R^8_G,$$

for any $z \in R^4_G$, hence by the symmetrizer lemma, that

$$\big(A'h_i''(A) - Ah_i''(A')\big) = A_i h''(A, A'), \tag{3.4}$$

for some bilinear function $h''\colon S^1 \times S^1 \to R^2_G$. This function then satisfies the following condition

$$A_1 h''(A_2, A_3) - A_2 h''(A_1, A_3) + A_3 h''(A_1, A_2) = 0$$

in R^3_G for any $A_i \in S^1$ from which it is easy to conclude that

$$h''(A, A') = A'h'''(A) - Ah'''(A'),$$

for some function $h'''\colon S^1 \to S^1/(X_4)$. But then by (3.4) we get

$$A'\big(h_i''(A) - A_i h'''(A)\big) - A\big(h_i''(A') - A_i h'''(A')\big) = 0 \quad \text{in } R^3_G,$$

and applying the symmetrizer lemma once more, we conclude that $h_i''(A) \equiv A_i h'''(A) + B_i A \bmod X_4$ for some $B_i \in S^1$. The second equality in (3.3) then gives $\varphi(z, A, A') = z(A'h'''(A) - Ah'''(A'))$ and the first $A'h(A) - Ah(A') = T(A'h'''(A) - Ah'''(A'))$ in R^6_G, that is:

$$A'(h(A) - Th'''(A)) - A(h(A') - Th'''(A')) = 0 \quad \text{in } R^6_G,$$

which by the symmetrizer lemma gives $h(A) - Th'''(A) \equiv \chi A \bmod X_4$ for some $\chi \in R^4_G$. In conclusion, we get

$$h_i(A) = X_4^3(A_i h'''(A) + B_i A), \quad h(A) = \chi A + Th'''(A).$$

Working modulo W_0, this is also:

$$h_i(A) = X_4^3 B_i A, \quad h(A) = \chi A.$$

Thus these deformations depend on the 16 parameters $B_i \in S^1$ and the 31 parameters $\chi \in R^4_G$, with the relation: $B_i = \eta A_i$, $\chi = \eta T$.

Since the allowed deformations for F_0 vary in the space of codimension 31 defined by the condition $\frac{\partial H}{\partial X_4} = 0$ in R^4_G, when α, α_i, B vanish, we have found that the tangent space to $\mathcal{G}$ at (W_0, F_0) has dimension $31+15+\dim U - 31 + 9 = \dim U + 24$, and Lemma 3.1 is proved. $\square$

4 Proof of the degeneracy property

To complete the proof of Theorem 5, it remains to prove that if F_0 is of the form (0.6), its W_0 is degenerate in the sense of Definition 3 (see 0.4). We start by proving the following result.

Proposition 6 *Let $V \subset (R^5_{F_0})^2$ be a 65 dimensional subspace, and assume that*

1. *the first projection $\mathrm{pr}_1 \colon V \to R^5_{F_0}$ is injective;*
2. *$w_1 w'_2 - w'_1 w_2 = 0$ in $R^{10}_{F_0}$ for all $w, w' \in V$;*
3. *the map $\mathbb{C}^2 \to \mathrm{Hom}(V, R^5_G)$ has rank 2.*

Then for $w = (w_1, w_2) \in V$

$$w_1 \in X_4^i R^{5-i}_{F_0} \iff w_2 \in X_4^i R^{5-i}_{F_0}$$

In particular, assume that $W_0 \subset (R^5_{F_0})^5$ satisfies (b), (e) and is non-degenerate; then

$$X_4^i(z_1, \dots, z_5) \in W_0 \implies$$
$$z_1, \dots, z_5 \text{ are either linearly independent or all } 0 \text{ in } R^{5-i}_G.$$

Proof We filter V by the order w.r.t. X_4, so that $V_i = V \cap (X_4^i R^{5-i}_{F_0})^2$, and denote by $\alpha, \beta, \gamma, \delta$ the ranks of the successive quotients. We have to show that if $w = X_4^i(z_1, z_2) \in V_i \setminus V_{i+1}$, then z_1, z_2 are independent in R^{5-i}_G. We only prove the case $i = 0$, the others being similar. So assume $w_2 \equiv 0 \bmod X_4$, but $w_1 \not\equiv 0 \bmod X_4$.

First of all, $\delta \leq 2$; indeed if $(X_4^3 q_1, X_4^3 q_2) \in V$ then $w_1 q_2 = 0$ in R^7_G, since $X_4^3 w_1 q_2 - X_4^3 w_2 q_1 = 0$ in $R^{10}_{F_0}$ and $w_2 \equiv 0 \bmod X_4$, which by Lemma 2.9 implies that the image of the second projection $V_3 \to R^2_G$ has dimension ≤ 1. But if $X_4^3(q_1, 0) \in V$ is in the kernel of this map, $q_1 w'_2 = 0$ in R^7_G for any $(w'_1, w'_2) \in V$. Since there is at least one nonzero w'_2 by the nondegeneracy assumption, we conclude that this map has kernel of dimension at most one, so that $\delta \leq 2$.

Suppose now $\gamma \geq 9$; for $X_4^2(t_1, t_2) \in V_2$ we have $w_1 t_2 = 0$ in R^8_G, and by Lemma 2.6, this implies that the image of the second projection $V_2 \to R^3_G$ has dimension ≤ 8. Hence there exists $X_4^2(t'_1, t'_2) \in V_2$ with $t'_2 = 0$ in R^3_G and $t'_1 \neq 0$ in R^3_G. Now for $(X_4 z_1, X_4 z_2) \in V_1$ we have $t'_1 z_2 = 0$ in R^7_G, since $X_4^3(t'_1 z_2 - t'_2 z_1) = 0$ in $R^{10}_{F_0}$ and $t'_2 \equiv 0 \bmod X_4$, which by Lemma 2.2 implies that the image of the second projection $V_1 \to R^4_G$ has dimension ≤ 2. If $\beta \geq 5$, there exists $(X_4 z'_1, X_4 z'_2), (X_4 z''_1, X_4 z''_2), (X_4 z'''_1, X_4 z'''_2) \in V_1$, with

$z_2' \equiv z_2'' \equiv z_2''' \equiv 0 \bmod X_4$, and z_1', z_1'', z_1''' independent in R_G^4. Then for any $X_4^2(t_1, t_2) \in V_2$ we have $z_1' t_2 = z_1'' t_2 = z_1''' t_2 = 0$ in R_G^7, which by Lemma 2.2 implies that the second projection $V_2 \to R_G^3$ vanishes. But then from $\gamma \geq 9$, we conclude that there is a 9 dimensional subspace of R_G^3 made of elements t_1 such that $X_4^2(t_1, t_2) \in V_2$, with $t_2 \equiv 0 \bmod X_4$. But we have for any such t_1 and any $(w_1, w_2) \in V$, $t_1 w_2 = 0$ in R_G^8, which by Lemma 2.6 implies that $w_2 = 0$ in R_G^5, and this contradicts the nondegeneracy assumption. So we conclude that $\gamma \geq 9 \implies \beta \leq 4$. Also, it is easy to show using Lemma 2.6 that $\gamma \leq 16$. Hence $\gamma \geq 9 \implies \beta + \gamma \leq 20$, and since $\delta \leq 2$ we find $\alpha \geq 65 - 22 = 43 > \operatorname{rank} R_G^5$. But this is absurd because the image of V/V_1 in $R_G^5 \times R_G^5$ is totally isotropic for any of the pairings $\langle w, w' \rangle_\varphi = \varphi(w_1 w_2' - w_2 w_1') \in R_G^{12} \cong \mathbb{C}$, where $\varphi \in R_G^2$, and for generic G and φ, this pairing is nondegenerate.

Thus we must have $\gamma \leq 8$ and $\alpha + \beta \geq 55$. Now we claim that $\beta \leq 19$. Indeed we have $w_1 z_2 = 0$ in R_G^9 for $(X_4 z_1, X_4 z_2) \in V_1$, and by Lemma 2.6 this implies that the second projection $V_1 \to R_G^4$ has rank ≤ 19. So if $\beta > 19$, there is an element $(X_4 z_1', X_4 z_2') \in V_1$, with $z_2' \equiv 0 \bmod X_4$ but $z_1' \not\equiv 0 \bmod X_4$. But this implies easily that $\beta \leq 22$ and $\gamma = 0$, using Lemmas 2.5 and 2.2. Then $\alpha \geq 41$ and this is absurd, since V/V_1 is a totally isotropic subspace of $R_G^5 \times R_G^5$ for any of the generically nondegenerate pairings

$$((u,v),(u',v')) \mapsto q(uv' - u'v) \in R_G^{12} \cong \mathbb{C}, \quad \text{for } q \in R_G^2.$$

Thus we must have $\beta \leq 19$, hence $\alpha \geq 65 - 19 - 8 - 2 = 36$. This implies in fact that $\delta \leq 1$: indeed let $K \subset R_G^5$ be the image of the second projection map $\mathrm{pr}_2 \colon V \to R_G^5$, and let $L \subset R_G^5$ be the image by the first projection $\mathrm{pr}_1 \colon V \to R_G^5$ of $\operatorname{Ker} \mathrm{pr}_2$. Then by assumption, both K and L are nonzero and $KL = 0$ in R_G^{10}. Furthermore $\alpha = \dim K + \dim L$. We know that the second projection $\mathrm{pr}_2^3 \colon V_3 \to R_G^2$ has rank at most one, and is annihilated by the space L introduced above. Its kernel is annihilated by the space K so has dimension at most one by Lemma 2.9. If both the image and the kernel of pr_2^3 are nonzero, we find that $\dim K \leq 9$, and $\dim L \leq 9$ by Lemma 2.13. Hence $\alpha \leq 18$, a contradiction.

From $\delta \leq 1$, we conclude that $\alpha \geq 37$. But in the above notation this contradicts $\alpha = \dim K + \dim L$ with K, L nontrivial, and the following result.

Lemma 4.1 *Let G be generic and K, L two nontrivial subspaces of R_G^5 such that $KL = 0$ in R_G^{10}; then $\dim K + \dim L \leq 36$.* $\square$

Thus Proposition 6 is proved. $\square$

So what we have to show now is the following:

Proposition 7 *There does not exist a 65 dimensional subspace $W \subset (R_{F_0}^5)^5$, satisfying the following conditions:*

(i) $\mathrm{pr}_{1|W}\colon W \to R^5_{F_0}$ *is injective.*

(ii) *Filter W by $W_i = W \cap (X_4^i R^{5-i}_{F_0})^5$; then for $w = (X_4^i z_i) \in W_i \setminus W_{i+1}$, the z_i are independent in R^{5-i}_G.*

(iii) $w_i w'_j = w_j w'_i$ *in $R^{10}_{F_0}$ for any i, j and any $w, w' \in W$.*

The first step is the following

Lemma 4.2 *If such W exists, then*

$$t_i t'_j - t_j t'_i = 0 \text{ in } R^6_G \quad \text{for all } t = X_4^2(t_1, \dots, t_5),\ t' = X_4^2(t'_1, \dots, t'_5) \in W_2.$$

Proof We prove this by contradiction: assuming Lemma 4.2 is not true, we have first the following properties:

(a) $\dim(S^1 W_2/W_3 + W_1/W_2) \geq \dim W_1/W_2 + 6$.

(b) $\dim(S^2 W_2/W_3 + W/W_1) \geq \dim W/W_1 + 8$.

Indeed, let $X, Y \in S^1/(X_4)$ such that $Xt + Yt' = (z_1, \dots, z_5) \in W_1/W_2$; then $z_i t_j - z_j t_i = 0$ in R^7_G, hence $Y(t_i t'_j - t_j t'_i) = 0$ in R^7_G. Similarly $X(t_i t'_j - t_j t'_i) = 0$ in R^7_G. But the annihilator in $S^1/(X_4)$ of any nonzero element of R^6_G has dimension at most one, since G is generic, hence it follows that the set of such pairs (X, Y) has dimension ≤ 2, so that $\dim S^1 t + S^1 t' \mod W_1/W_2 \geq 6$. Property (b) is proved similarly using the fact that the annihilator in $S^2/(X_4)$ of any nonzero element of R^6_G has dimension ≤ 6.

Now we show the following:

For any nonzero $(z_1, \dots, z_5) \in S^1 W_2/W_3 + W_1/W_2 \subset (R^4_G)^5$, the z_i are independent in R^4_G.

Indeed, assume by contradiction that $z_1 \neq 0$ and $z_5 = 0$; then for any elements $(w_1, \dots, w_5)$ in W/W_1 and $X_4(z'_1, \dots, z'_5)$ in W_1/W_2, by (iii), we have $z_1 w_5 = 0$ in R^9_G and $z_1 z'_5 = 0$ in R^8_G; using Lemmas 2.3 and 2.5 and (ii), this implies $\alpha \leq 21$ and $\beta \leq 12$. But then $\gamma + \delta \geq 32$ and this is absurd because by (ii), $\gamma \leq \dim R^3_G - 4 = 16$ and $\delta \leq \dim R^2_G - 4 = 6$.

This last fact implies that $\dim(S^1 W_2/W_3 + W_1/W_2) \leq \dim R^4_G - 4 = 27$, hence by inequality (a), we conclude that $\beta \leq 21$. Next, we show:

For any nonzero $(w_1, \dots, w_5) \in S^2 W_2/W_3 + W/W_1 \subset (R^5_G)^5$, the w_i are independent in R^5_G, unless $\alpha \leq 31$ and $\beta \leq 19$.

This follows in fact from Lemmas 2.9 and 2.6 and (ii); indeed by (iii), if $(w'_1, \dots, w'_5) \in S^2W_2/W_3 + W/W_1$ with $w'_5 = 0$ in R^5_G and $w'_1 \neq 0$ in R^5_G, we have $w'_1 w_5 = 0$ in R^{10}_G and $w'_1 z_5 = 0$ in R^9_G for any $(w_1, \dots, w_5) \in W/W_1$ and $(z_1, \dots, z_5) \in W_1/W_2$.

In the first case $\dim S^2W_2/W_3 + W/W_1 \leq \dim R^5_G - 4 = 36$ hence by inequality (b) above, $\dim W/W_1 \leq 28$. Since $\beta \leq 21$, in either case $\alpha + \beta \leq 50$, which implies $\gamma + \delta \geq 15$. Now we show:

$$\delta \leq 1 \textit{ and } \dim[W_2/W_3 : (X, Y)] \leq 1 \textit{ for generic } X, Y \in S^1/(X_4).$$

This is proved as follows: let $(q_1, \dots, q_5), (q'_1, \dots, q'_5)$ be elements of W_3 or $[W_2/W_3 : (X, Y)]$. It is immediate to see that

$$q_i w_j - q_j w_i = q'_i w_j - q'_j w_i = 0 \text{ in } R^7_G \quad \text{for } (w_1, \dots, w_5) \in W/W_1,$$

for all i, j. (In the first case, this follows from (iii); in the second, (iii) implies that these elements of R^7_G are annihilated by X and Y.) This implies that $q_i q'_j - q_j q'_i \in R^4_G$ is annihilated by w_i, w_j. Thus if it is nonzero we have $\alpha \leq 21$ by Lemma 2.3, and then $\alpha + \beta \leq 42$. So $\gamma + \delta \geq 23$, which contradicts the inequality $\gamma + \delta \leq 22$ proved above. Now we conclude by the following easy result.

Lemma 4.3 *There do not exist independent elements $(q_1, \dots, q_5), (q'_1, \dots, q'_5)$ of $(R^2_G)^5$ satisfying: the q_i are independent in R^2_G and $q_i q'_j = q_j q'_i$ in R^4_G for all i, j.*

So we have now proved that $\gamma \geq 14$ and $\dim[W_2/W_3 : (X, Y)] \leq 1$, and this implies that $\dim XW_2/W_3 + YW_2/W_3 \geq 27$. But S^1W_2/W_3 cannot be equal to $XW_2/W_3 + YW_2/W_3$ for generic X, Y because this would imply that the composite map: $S^1W_2/W_3 \xrightarrow{\mathrm{pr}_1} R^4_G \to R^4_G/(X, Y)$ is not surjective, while the assumption $\gamma \geq 14$ implies that the composite map $W_2/W_3 \xrightarrow{\mathrm{pr}_1} R^3_G \to R^3_G/(X, Y)$ is surjective for generic X, Y (cf. Green [11]). It follows that $\dim S^1W_2/W_3 \geq 28$, which is absurd because we have already proved that $\dim S^1W_2/W_3 \leq 27$. Hence Lemma 4.2 is proved. $\square$

Next we prove the following result.

Lemma 4.4 *Suppose that G is generic, and that $Z \subset (R^3_G)^5$ satisfies the following:*

(i) *For any nonzero $(t_1, \dots, t_5) \in Z$, the t_i are independent in R^3_G.*

(ii) *For $(t_1, \dots, t_5), (t'_1, \dots, t'_5) \in Z$, we have $t_i t'_j - t_j t'_i = 0$ in R^6_G.*

Then $\dim Z \leq 7$.

Proof In this proof S^k denotes the degree k component of the polynomial ring $S/(X_4)$ in four variables. Let X be generic in S^1. One shows that for generic $P \in (R_G^6/(X))^*$, the pairing $\langle\ ,\ \rangle_P$ on $R_G^3/(X)$ defined by $\langle \alpha, \beta \rangle_P = P(\alpha\beta)$ is nondegenerate. Now assume that Z is as above, but $\dim Z \geq 8$. Since $\dim R_G^3/(X) = \dim S^3/(X) = 10$, there exists $(t_1, \ldots, t_5) \in Z$ such that the t_i are not independent in $R_G^3/(X)$. After a change of basis of $\mathbb{C}^5$ we may assume that $t_1 \equiv 0 \bmod X$. Let r be the dimension of the kernel of the composite

$$f\colon Z \xrightarrow{\mathrm{pr}_1} R_G^3 \to R_G^3/(X).$$

Then for any $t \in \operatorname{Ker} f$ and $t' \in Z$ we have $t_i t_1' = t_1 t_i' = 0$ in $R_G^6/(X)$, hence $\langle t_i, t_1' \rangle_P = 0$, which implies that the space generated by the t_i modulo X with $t \in \operatorname{Ker} f$ has dimension $\leq 10 - 8 + r = 2 + r$. Since $\dim \operatorname{Ker} f = r$ it follows that there exists $t \in \operatorname{Ker} f$ such that $t_2, \ldots, t_5$ are linearly dependent in $R_G^3/(X)$. After a change of basis of $\mathbb{C}^5$, we may assume that $t_1 \equiv t_2 \equiv 0 \bmod X$, that is, $t = (Xq_1, Xq_2, t_3, t_4, t_5)$ for some $q_2, q_3 \in S^2$.

Let us show that $q_1, q_2 \in S^2$ cannot have a common factor: indeed, if $q_1 = AY$ and $q_2 = BY$ for some $A, B, Y \in S^1$, then $At_2' = Bt_1'$ in R_G^4 for any $t' \in Z$, since this is annihilated by $0 \neq XY \in S^2$. But the map $(B - A)\colon R_G^3 \times R_G^3 \to R_G^4$ has kernel of dimension ≤ 11, and should contain $\mathrm{pr}_{12}(Z)$. Since it contains the 10 dimensional subspace $(A, B)R_G^2$, this would imply that Z contains a subspace Z' of dimension ≥ 7 such that $\mathrm{pr}_{12}(Z') = \{(A, B)q \text{ for } q \in Q \subset R_G^2\}$. For $t \in Z'$, we can view t_3, t_4, t_5 as functions of $q \in Q$, and $qt_i(q') = q't_i(q) \in R_G^5$. Since $\operatorname{rank} Q \geq 7$, $R_G^3 Q \subset R_G^5$ is at least a hyperplane, and it is not difficult to adapt the symmetrizer lemma to conclude that there must exist $A_i \in S^1$ for $i = 3, 4, 5$ such that $t_i(q) = A_i q \in R_G^3$. But then A, B, A_i are not independent in S^1, and this contradicts (i). So q_1, q_2 cannot have a common factor. Now we use the following result.

Lemma 4.5 *Suppose that G is generic, and that $q_1, q_2 \in R_G^2 = S^2$ have no common factor. Then the map $(q_2 - q_1)\colon R_G^3 \times R_G^3 \to R_G^5$ has kernel of dimension ≤ 10.* □

Since $t = (Xq_1, Xq_2, t_3, t_4, t_5) \in Z$, by (ii) and the fact that multiplication $\mu_X\colon R_G^5 \to R_G^6$ by X is injective, we now have $q_1 t_2' - q_2 t_1' = 0$ in R_G^5 for any $t' \in Z$. So $\mathrm{pr}_{12}(Z)$, of dimension ≥ 8, is contained in $\operatorname{Ker}(q_2 - q_1)$, which has dimension ≤ 10 and contains the 4 dimensional subspace $S^1 \cdot (q_1, q_2)$. It follows that there exists $Y \in S^1$ not proportional to X such that $Y(q_1, q_2) \in \mathrm{pr}_{12}(Z)$. Let $t' = (Yq_1, Yq_2, t_3', t_4', t_5') \in Z$. Then the first two components of $Yt - Xt'$ vanish, and it follows easily using (ii) that $Yt - Xt' = 0$ in $(R_G^4)^5$. But the space $H_{X,Y}^3$, defined as the middle cohomology group of the sequence

$$0 \to R_G^2 \xrightarrow{(X,Y)} R_G^3 \times R_G^3 \xrightarrow{Y-X} R_G^4$$

has dimension at most one; hence it follows that after a change of basis of $\mathbb{C}^5$, we have $t = (Xq_1, \dots, Xq_4, t_5)$ and $t' = (Yq_1, \dots, Yq_4, t'_5)$ for some $q_3, q_4 \in R^2_G$. Thus we have proved that for generic X there is $t \in Z$ such that at least four components of t vanish modulo (X). Hence for generic X and T in S^1, after a change of basis of $\mathbb{C}^5$ there exists $t = (Xq_1, Xq_2, Xq_3, t_4, t_5) \in Z$ and $t' = (Tq'_1, Tq'_2, Tq'_3, t'_4, t'_5) \in Z$. Then $q_i q'_j = q'_i q_j$ in R^4_G for $i, j \leq 3$, and since we know that q_i, q_j do not have a common factor, this implies easily that (q_1, q_2, q_3) and (q'_1, q'_2, q'_3) are proportional in $(R^2_G)^3$. Hence we may assume they are equal. But then $Xt' - Tt$ must vanish in $(R^4_G)^5$, and since X and T are generic the space $H^3_{X,T}$ is zero. It follows that in fact $t = X(q_1, \dots, q_5)$ and $t' = T(q_1, \dots, q_5)$ for some $(q_1, \dots, q_5) \in (R^2_G)^5$. Then for any $z = (z_1, \dots, z_5) \in Z$, we have $q_i z_j - q_j z_i = 0$ in R^5_G for any $i, j \leq 5$.

Now let Z' be a subspace of Z supplementary to $X(q_1, \dots, q_5)$, so that $\dim Z' \geq 7$. Since $\dim R^3_G/(X) = 10$, there is a $t' \in Z'$ such that the t'_i are not independent in $R^3_G/(X)$. We may assume $t'_1 \equiv 0 \bmod X$, and using the property $q_1 t'_i = q_i t'_1$ in R^5_G, we conclude that $q_1 t'_i = 0$ in $R^5_G/(X)$. Now we use the result:

Lemma 4.6 *Suppose that G and $X \in S^1$ are generic. Then for any nonzero $q \in R^2_G/(X)$ the multiplication map $\mu_q \colon R^3_G/(X) \to R^5_G/(X)$ has kernel of dimension ≤ 3.* □

So if $q_1 \not\equiv 0$ modulo (X), we see that the t'_i generate a subspace of $R^3_G/(X)$ of dimension ≤ 3, so that at least two components of t' are divisible by X. We then conclude as before that $t' = X(q'_1, \dots, q'_5)$; then $(q_1, \dots, q_5)$ and $(q'_1, \dots, q'_5) \in (R^2_G)^5$ are not proportional with independent components, and satisfy $q_i q'_j - q'_i q_j = 0$ in R^4_G, which contradicts Lemma 4.3. Hence we have $q_1 \equiv 0 \bmod X$.

Thus we have shown that for generic X, there exists nonzero $q = XA \in \langle q_1, \dots, q_5 \rangle \subset R^2_G = S^2$. We then conclude that $[\langle q_1, \dots, q_5 \rangle : S^1] \neq \{0\}$. So after a change of basis of $\mathbb{C}^5$, we have $(q_1, \dots, q_5) = (AA_1, \dots, AA_4, q_5)$ for some $A \in S^1$ and independent $A_i \in S^1$. Then for $z \in Z$ we have $A_i z_j = A_j z_i$ in R^4_G for $i, j \leq 4$, which by the symmetrizer lemma implies that there exists $q \in R^2_G$ such that $(z_1, \dots, z_4) = q(A_1, \dots, A_4)$. Since $\dim Z \geq 8$, q runs through a subspace Q of S^2 of dimension ≥ 8. Again we view z_5 as a function of $q \in Q$, satisfying $z_5(q')q = z_5(q)q'$ for any $q, q' \in Q$. Since Q generate R^5_G by [11], we can apply the symmetrizer lemma to conclude that $z_5(q) = A_5 q$ for some $A_5 \in S^1$. But then the A_i for $i = 1, \dots, 5$ are not independent in S^1, which contradicts (i). This concludes the proof of Lemma 4.4. □

Proof of Proposition 7 By Lemmas 4.2 and 4.4 we have $\gamma \leq 7$. We also have $\delta \leq 1$. Indeed if $\delta \geq 2$, there exist independent elements

$$(q_1, \dots, q_5), (q'_1, \dots, q'_5) \in W_3.$$

Then by Lemma 4.3, there exist i, j such that $q_i q'_j - q_j q'_i \neq 0$ in R^4_G. But it annihilates $w_i, w_j \in R^5_G$ for any $(w_1, \dots, w_5) \in W/W_1$. Hence by property (ii) and Lemma 2.3, we have $\alpha \leq 20$. From $\gamma \leq 7, \delta \leq 6$ we conclude that $\beta \geq 65 - 20 - 6 - 7 = 32$, which is absurd, since $\beta \leq 31 = \dim R^4_G$.

Hence we have $\alpha + \beta \geq 57$. This implies easily that for any nonzero $(w_1, \dots, w_5) \in S^1(W_1/W_2)$, the w_i are independent in R^5_G, so that

$$\dim S^1(W_1/W_2) \leq \dim R^5_G - 4 = 36.$$

Now if $\beta \geq 22$, one can adapt the proof of Lemma 1.2 to show that for generic $X, Y \in S^1/(X_4)$ the kernel of the map

$$(X+Y)\colon W_1/W_2 \times W_1/W_2 \to S^1(W_1/W_2)$$

is equal to $(-Y, X)[W_1/W_2 : (X, Y)]$. Since

$$\dim W_1/W_2 \geq 22, \quad \dim S^1(W_1/W_2) \leq 36,$$

this implies $\dim[W_1/W_2 : (X, Y)] \geq 8$. On the other hand, $[W_1/W_2 : (X, Y)] \subset (R^3_G)^5$ satisfies the assumptions of Lemma 4.4, because if

$$(t_1, \dots, t_5), (t'_1, \dots, t'_5) \in [W_1/W_2] : (X, Y),$$

then $t_i t'_j - t_j t'_i$ is annihilated by X^2, Y^2, XY, hence must be zero. Therefore $\dim[W_1/W_2 : (X, Y)] \geq 8$ is impossible, and we must have $\beta \leq 21$, hence $\alpha \geq 36$. We claim this is impossible: if $\alpha \geq 36$, the image of $\mathrm{pr}_1 : W/W_1 \to R^5_G$ has codimension ≤ 4, hence $\dim S^1 \mathrm{pr}_1(W/W_1) \geq \dim R^6_G - 3$ by [11]. It follows that there exists a nonzero $(z_1, \dots, z_5) \in S^1(W/W_1)$ for which the z_i are not independent in R^6_G. We may assume that $z_5 = 0$, and then we have $z_i w_5 = z_5 w_i = 0$ in R^{11}_G for any $w \in W$. So the z_i for $i \leq 4$ are orthogonal to $S^1 \mathrm{pr}_5(W/W_1)$ in R^6_G under the perfect pairing $R^6_G \times R^6_G \to R^{12}_G \cong \mathbb{C}$. Since $S^1 \mathrm{pr}_5(W/W_1)$ has codimension ≤ 3 in R^6_G, we conclude that the z_i for $i \leq 4$ are not independent in R^6_G. So we may assume that $z_4 = 0$. But then the z_i for $i \leq 3$ are orthogonal to $S^1(\mathrm{pr}_4(W/W_1) + \mathrm{pr}_4(W/W_1))$ which has codimension ≤ 2 in R^6_G, since by property (ii) and $\alpha \geq 36$, $\mathrm{pr}_4(W/W_1) + \mathrm{pr}_4(W/W_1)$ has codimension ≤ 3 in R^5_G. So the z_i for $i \leq 3$ are not independent in R^6_G. Continuing in this way, we show finally that $(z_1, \dots, z_5) = 0$, which is a contradiction. $\square$

References

[1] P. Candelas, X.C. de la Ossa, P.S. Green, L. Parkes: A pair of Calabi–Yau Manifolds as an exactly soluble superconformal field theory, Nucl. Phys. **B359** (1991), 21–74

[2] J. Carlson, P. Griffiths: Infinitesimal variations of Hodge structures and the global Torelli problem, in *Géométrie algébrique*, Angers 1979, A. Beauville Ed., Sijthoff and Noordhoff (1980), 51–76

[3] D. Cox, M. Green: Polynomial structures and generic Torelli for projective hypersurfaces, Comp. Math. **73** (1990), 121–124

[4] R. Donagi: Generic Torelli theorem for projective hypersurfaces, Comp. Math. **50** (1983), 325–353

[5] R. Donagi: Generic Torelli and variational Schottky, in Topics in transcendental algebraic geometry, Ann. of Math. Studies **106**, Princeton 1984, 239–258

[6] R. Donagi, M. Green: A new proof of the symmetrizer lemma and a stronger weak Torelli theorem, J. Diff. Geom. **20** (1984), 459–461

[7] R. Friedman: On threefolds with trivial canonical bundle, in *Complex Geometry and Lie Theory* (Sundance UT, 1989), Proc. Symp. Pure Math. **53**, 1

[8] A. B. Givental: Equivariant Gromov–Witten invariants, Internat. Math. Res. Notices **13** (1996), 613–663

[9] M. Green: Koszul cohomology and the geometry of projective varieties, J. Diff. Geom. **20** (1984), 279–289

[10] M. Green: The period map for hypersurface sections of high degree of an arbitrary variety, Comp. Math. **55** (1984), 135–156

[11] M. Green: Restrictions of linear series to hyperplanes, and some results of Macaulay and Golzmann, in *Algebraic curves and projective geometry*, Trento 1988, E. Ballico and C. Ciliberto Eds, Lecture Notes in Mathematics **1389**, 76–86

[12] P. Griffiths: On the periods of certain rational integrals. I and II, Ann. of Math. **90** (1969) 460–541

[13] P. Griffiths: Periods of integrals on algebraic manifolds. I and II, Amer. J. Math. **90** (1968), 568–626 and 805–865

[14] M. Gross and P. M. H. Wilson: Mirror symmetry via 3 tori for a class of Calabi–Yau threefolds, Math. Ann. **309** (1997), 505–531.

[15] D. Morrison: The geometry underlying mirror symmetry, this volume, pp. 251–278.

[16] C. Peters, J. Steenbrink: Infinitesimal variations of Hodge structure and the generic Torelli problem for projective hypersurfaces, in *Classification of algebraic and analytic manifolds*, K. Ueno Ed., Progress in Math. **39** Birkhäuser (1983), pp 399–463.

[17] A. Strominger, S. T. Yau and E. Zaslow: Mirror symmetry is T-duality, Nucl. Phys. **B479** (1996), 243–259.

[18] C. Voisin: Variations of Hodge structure of Calabi–Yau threefolds, "Lezioni Lagrange", Roma 1996, Publications of the Scuola Normale Superiore di Pisa.

[19] C. Voisin: Miroirs et involutions sur les surfaces K3, in *Journées de géométrie algébrique d'Orsay, juillet 1992*, A. Beauville, O. Debarre and Y. Laszlo Eds., Astérisque **218** (1993)

[20] IVHS: Infinitesimal variations of Hodge structure, Comp. Math. **50** (1983):

I. J. Carlson, M. Green, P. Griffiths, J. Harris: Infinitesimal variations of Hodge structure, 109–205

II. P. Griffiths, J. Harris: Infinitesimal invariants of Hodge classes, 207–265

III. P. Griffiths: Determinantal varieties and the infinitesimal invariant of normal functions, 267–324

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