Symplectic Gromov–Witten invariants

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Introduction

Gromov–Witten (GW) invariants "count" algebraic curves on a smooth projective variety X with certain incidence conditions, but in a rather refined way. They originated in the realm of symplectic rather than algebraic geometry. Salient features are

(1) in unobstructed situations, i.e., if the relevant moduli spaces of algebraic curves are smooth of the expected dimension ("expected" by looking at the Riemann–Roch theorem), one obtains the number that one would naively expect from algebraic geometry. A typical such example is the number of rational curves of degree d in the plane passing through $3d-1$ generic closed points, which is in fact a finite number.

(2) GW invariants are constant under (smooth projective) deformations of the variety.

For the original definition, one deforms X as an *almost complex* manifold and replaces algebraic by pseudoholomorphic curves (i.e., holomorphic with respect to the almost complex structure). For a generic choice of almost complex structure on X, the relevant moduli spaces of pseudoholomorphic curves are oriented manifolds of the expected dimension, and GW invariants can be defined by naive counting. Not every almost complex structure J is admitted though, but (for compactness results) only those *tamed* by a symplectic form ω, which by definition means that $\omega(v, Jv) > 0$ for any nonzero tangent vector $v \in T_X$. In the algebraic case, if J is sufficiently close to the integrable structure, ω may be chosen as the pullback of the Fubini–Study form. It turns out that GW invariants really depend only on the symplectic structure (or its deformation class), hence are symplectic in nature. Since the original definition basically neglects singular curves, GW invariants were restricted to projective manifolds with numerically effective anticanonical bundle.

More recently the situation has changed with the advent of a beautiful, purely algebraic and completely general theory of GW invariants based on an idea of Li and Tian [Be1], [BeFa], [LiTi1]. This development is surveyed in [Be2]. Due to the independent effort of many people there is now also a completely general definition of symplectic GW invariants available [FkOn], [LiTi2], [Ru2], [Si1]. The purpose of the present paper is to supplement Behrend's contribution to this volume by the symplectic point of view. We also sketch our own more recent proof of equivalence of symplectic and algebraic GW invariants for projective manifolds.

While it is perfect to have a purely algebraic theory, I believe that the symplectic point of view is still rewarding, even if one is not interested in symplectic questions: apart from the aesthetic appeal of the interplay between geometric and algebraic methods, symplectic techniques are sometimes easier and more instructive to use (if only as a preparation for an algebraic treatment). In [Si3], Proposition 1.1, I gave an example of GW invariants of certain projective bundles that are much more readily accessible by symplectic techniques. I also find that the properties of GW invariants, especially deformation invariance, are more intuitively obvious from the symplectic point of view, cf. also Section 4.2; this may just be a matter of taste. More philosophically, the symplectic nature of enumerative invariants in algebraic geometry should mean something, especially in view of their appearance in mirror symmetry. Finally, it is important to establish algebraic techniques to compute symplectic invariants. In fact, a closed formula for GW invariants, holding in even the most degenerate situations, can easily be derived from the definition, cf. [Si2]. The formula involves only Fulton's canonical class of the moduli space and the Chern class of a virtual bundle.

Gromov–Witten invariants have a rather interesting and involved history, with connections to gauge theory, quantum field theory, symplectic geometry and algebraic geometry. I include some remarks on this, following the referee's suggestion; however, I should point out that I concentrate only on the history of how these invariants were *defined*, rather than their computation and their many interesting applications.

The story begins with Gromov's seminal paper of 1985 [Gv]. In this paper Gromov laid the foundations for a theory of (pseudo-) holomorphic curves in almost complex manifolds. Of course, a notion of holomorphic maps between almost complex manifolds existed already for a long time. However, Gromov's points were that

(1) while higher dimensional almost complex submanifolds or holomorphic functions might not exist even locally, there are always many local holomorphic curves;

(2) the local theory of curves in almost complex manifolds largely parallels the theory in the integrable case, i.e., on $\mathbb{C}^n$ with the standard complex structure (Riemann removable singularities theorem, isolatedness of singular points and intersections, identity theorem);

(3) to get good global properties one should require the existence of a "*taming*" symplectic form ω (a closed, non-degenerate two-form) with $\omega(v, Jv) > 0$ for any nonzero tangent vector v (where J is the almost complex structure).

In fact, in the tamed setting, Gromov proves a compactness result for spaces of pseudoholomorphic curves in a fixed homology class. At first sight the requirement of a taming symplectic form seems to be merely technical. However, Gromov turned this around and observed that given a symplectic manifold (M, ω), the space of almost complex structures tamed by ω is always nonempty and connected. With the then recent ideas of gauge theory, Gromov studied moduli spaces of pseudoholomorphic curves in some simple cases for generic tamed almost complex structures. One such case was pseudoholomorphic curves homologous to $\mathbb{P}^1 \times \{\mathrm{pt}\}$ on $\mathbb{P}^1 \times T$ with T an (compact) n-dimensional complex torus. He shows that for any almost complex structure on $\mathbb{P}^1 \times T$ tamed by the product symplectic structure, such pseudoholomorphic curves exist. In modern terms, he shows that the associated GW invariant is nonzero. This can then be used to prove his famous squeezing theorem: the symplectic ball of radius r cannot be symplectically embedded into the cylinder $B_R^2 \times \mathbb{C}^n$ for $R < r$.

Several more applications of pseudoholomorphic curves to the global structure of symplectic manifolds were given already in Gromov's paper, and many more have appeared since. Probably the most striking of these is due to Floer [Fl]. He interpreted the Cauchy–Riemann equation of pseudoholomorphic curves as flow lines of a functional on a space of maps from the circle S^1 to the manifold. He can then do Morse theory on this space of maps. The homology of the associated Morse complex is the celebrated (symplectic) *Floer homology*, which has been used to solve the Arnol'd conjecture on fixed points of nondegenerate Hamiltonian symplectomorphisms. I mention Floer's work also because it is in the (rather extended) introduction to [Fl] that a (quantum) product structure on the cohomology of a symplectic manifold makes its first appearance (and is worked out for $\mathbb{P}^n$). As we now (almost) know [RuTi2], [PSaSc] this agrees with the product structure defined via GW invariants, i.e., quantum cohomology.

An entirely different, albeit related, development took place in physics. Starting from Floer's *instanton homology*, a homology theory developed by Floer in analogy to the symplectic case for gauge theory on three-manifolds, Witten [Wi1] observed that one can formulate supersymmetric gauge theory on closed four-manifolds, provided one changes the definition of the fields in an appropriate way ("twisting procedure"). The result is a physical theory that reproduces Donaldson's polynomial invariants as correlation functions. Because the latter are (differential) topological invariants, the twisted theory is referred to as a *topological quantum field theory.* In [Wi2] Witten applied the twisting procedure to nonlinear sigma models instead of gauge theory. Such a theory is modelled on maps from a Riemann surface to a closed, almost complex manifold. The classical extrema of the action functional are then pseudoholomorphic maps. The correlation functions of the theory are physical analogues of GW invariants. Witten was the first to observe much of the rich algebraic structure that one expects for these correlation functions from degenerations of Riemann surfaces [Wi3].

It is a curious fact that while simple versions of GW invariants were used as a tool in symplectic topology, and the technical prerequisites for a systematic treatment along the lines of Donaldson theory were all available (notably through the work of McDuff, the compactness theorem by Gromov, Pansu, Parker–Wolfson and Ye), it was only in 1993 that Ruan tied up the loose ends [Ru1] and defined symplectic invariants based on moduli spaces of pseudoholomorphic (rational) curves, mostly for positive symplectic manifolds. It was quickly pointed out to him that one of his invariants was the mathematical analogue of correlation functions in Witten's topological sigma model. At the end of 1993 the breakthrough in the mathematical development of GW invariants and their relations was achieved by Ruan and Tian in the important paper [RuTi1]. Apart from special cases (complex homogeneous manifolds), until recently, the results of Ruan and Tian were the only methods available to make precise sense of GW invariants for a large class of (semipositive) manifolds, including Fano and Calabi–Yau manifolds, and to establish relations between them, notably the associativity of the quantum product and the WDVV equation. Many of the deeper developments in GW theory used these methods, including Taubes' relationship between GW invariants and Seiberg–Witten invariants of symplectic four-manifolds [Ta], as well as Givental's proof of the mirror conjecture for the quintic via equivariant GW invariants [Gi]. For the case of positive symplectic manifolds, proofs for the glueing theorem for two rational pseudoholomorphic curves, which is the reason for associativity of the quantum product, were also given by different methods in the Ph.D. thesis of G. Liu [Liu] and in the lecture notes [McSa].

Early in 1994, Kontsevich and Manin [KoMa] advanced the theory in a different direction: rather than proving the relations among GW invariants, they formulated them as axioms and investigated their formal behaviour. They introduced a rather big compactification of the moduli space of maps from a Riemann surface by "stable maps" (cf. Definition 1.1 below). With this choice, all relations coming from degenerations of domains can be formulated in a rather regular and neat way. In the algebraic setting, spaces of stable maps have projective algebraic coarse moduli spaces [FuPa]; fine moduli spaces exist in the category of Deligne–Mumford stacks [BeMa]. Another plus is the regular combinatorial structure that allows the use of methods of graph theory to compute GW invariants in certain cases. No suggestion was made, however, of how to address the problem of degeneracy of moduli spaces, that in Ruan and Tian's approach, applied to projective algebraic manifolds, forces the use of general almost complex structures rather than the integrable one.

This problem was only solved in the more recent references given above, first in the algebraic and finally in the symplectic category, by constructing virtual fundamental classes on spaces of stable maps.

Here is an outline of the paper: Chapter 1 starts with a simple model case to discuss both the traditional approach and the basic ideas of [Si1]. Chapter 2 is devoted to the most technical part of my approach, the construction of a Banach orbifold containing the moduli space of pseudoholomorphic curves. The ambient Banach orbifold is used in Chapter 3 to construct the virtual fundamental class on the moduli space. Chapter 4 discusses the properties of GW invariants that one obtains easily from the virtual fundamental class. We follow here the same framework as in [Be2], so a comparison is easily possible. A fairly detailed sketch of the equivalence with the algebraic definition is given in the last chapter. The proof shows that the obstruction theory chosen in the algebraic context is natural also from the symplectic point of view. For this chapter we assume some understanding of the algebraic definition.

After finishing this survey, I received a similar survey by Li and Tian [LiTi3], in which they also announce a proof of equivalence of symplectic and algebraic Gromov–Witten invariants.[1]

A little warning is in order: the symplectic definition of GW invariants is more involved than the algebraic one. Modulo checking the axioms and the formal apparatus needed to do things properly, the latter can be given a rather concise treatment, cf. [Si2]. But as long as symplectic GW invariants are based on pseudoholomorphic curves, even to find local embeddings of the moduli space into finite dimensional manifolds ("Kuranishi model") means a considerable amount of technical work. In this survey I tried to emphasize ideas and the reasons for doing things in a particular way, but at the same time keep the presentation as nontechnical as possible. While we do not assume any knowledge of symplectic geometry or GW theory, the ideal reader would have some basic acquaintance with the traditional approach, e.g., from [McSa]. The reader who feels uneasy with symplectic manifolds is invited to replace the word "symplectic" by "Kähler".

I thank the referee and Miles Reid for pointing out some unclear points and (M.R.) for carefully editing the whole text.

1 Setting up the problem

1.1 The traditional approach

The purpose of this section is to present Ruan's approach to GW invariants on a simple example. We refer to the lecture notes [McSa] for background information and a more careful exposition. Let Σ be a closed Riemann surface with complex structure j, and (M, J) an almost complex manifold. We also fix some $k \geq 1$ and $\alpha \in (0,1)$. The space $\mathcal{B} := C^{k,\alpha}(\Sigma; M)$ of k-times

[1] Note added in proof: Both papers are meanwhile available as [LiTi4], [Si4].

differentiable maps from Σ to M with kth derivative of Hölder class α is a Banach manifold. Charts at $\varphi\colon \Sigma \to M$ are given, for example, by

$$\begin{array}{rcl} C^{k,\alpha}(\Sigma;\varphi^*T_M)\supset V & \longrightarrow & C^{k,\alpha}(\Sigma;M)\\ v & \longmapsto & \Big(z\mapsto \exp_{\varphi(z)} v(z)\Big). \end{array}$$

Here the exponential map is with respect to some fixed connection on M, and V is a sufficiently small open neighbourhood of 0 where the map is injective. The equation for φ to be J-holomorphic is

$$\overline{\partial}_J\varphi = 0, \quad \overline{\partial}_J\varphi = \frac{1}{2}(D\varphi + J\circ D\varphi\circ j) \in C^{k-1,\alpha}(\Sigma;\varphi^*T_M\otimes_{\mathbb{C}}\overline{\Omega}).$$

Here we write $\overline{\Omega} = \Lambda^{0,1}T^*_\Sigma$ for the bundle of $(0,1)$-forms on Σ. The equation $\overline{\partial}_J\varphi\circ j = -\varphi^*J\circ\overline{\partial}_J\varphi$ in the space of homomorphisms between the complex vector bundles (T_Σ, j) and $(\varphi^*T_M, \varphi^*J)$ shows that, viewed as section of φ^*T_M, $\overline{\partial}_J\varphi$ is indeed $(0,1)$-form valued. Intrinsically, these equations fit together to a section $s_{\overline{\partial},J}$ of the Banach bundle $\mathcal{E}\downarrow\mathcal{B}$ with fibers

$$\mathcal{E}_\varphi \;=\; C^{k-1,\alpha}(\Sigma;\varphi^*T_M\otimes\overline{\Omega}).$$

Local trivializations of $\mathcal{E}$ over the above charts can easily be constructed by (the complex linear part of) parallel transport of vector fields along the family of closed geodesics $\gamma_z(t)$ with $\gamma_z(0)=\varphi(z)$, $\dot\gamma_z(0)=v(z)$.

Obviously, $s_{\overline{\partial},J}$ is differentiable. As for any differentiable section of a vector bundle, its linearization *at a point of the zero locus*, which is a linear map $T_{\mathcal{B},\varphi}\to\mathcal{E}_\varphi$, is independent of the choice of local trivialization. Thus over the zero locus Z of $s_{\overline{\partial},J}$ the linearization induces a section σ of $\mathrm{Hom}(T_\mathcal{B},\mathcal{E})$. One computes for the linearization at J-holomorphic φ:

$$\begin{array}{rcl} \sigma_\varphi := D_\varphi s_{\overline{\partial},J}:\ T_{\mathcal{B},\varphi} = C^{k,\alpha}(\Sigma;\varphi^*T_M) & \longrightarrow & \mathcal{E}_\varphi = C^{k-1,\alpha}(\Sigma;\varphi^*T_M\otimes\overline{\Omega})\\ v & \longmapsto & \overline{\partial}{}^\varphi_J v + N_J(v, D\varphi). \end{array}$$

Here N_J is the Nijenhuis tensor or almost complex torsion of (M,J), a certain tensor on M depending only on J that vanishes identically if and only if J is integrable, see, e.g., [KoNo], Chap. IX, §2; and $\overline{\partial}{}^\varphi_J$ is the $\overline{\partial}$ operator belonging to a natural holomorphic structure on φ^*T_M. Concerning the latter, one actually *defines* $\overline{\partial}{}^\varphi_J$ as the $(0,1)$-part of the linearization of $s_{\overline{\partial},J}$. The integrability condition being void in dimension one, this complex linear partial connection defines a holomorphic structure on φ^*T_M, cf. Section 2.4.

We see that up to a zeroth order differential operator, σ_φ is just the Cauchy–Riemann operator of a holomorphic vector bundle over Σ of rank $n=\dim_{\mathbb{C}} M$. But $\overline{\partial}{}^\varphi_J$ is a Fredholm operator on appropriate spaces of sections, i.e., it has finite dimensional kernel and cokernel. It is crucial at this point to work with Hölder spaces: the Fredholm property is false for $\alpha = 0$. Alternatively, as below, one may work with Sobolev spaces. Now lower order perturbations are compact operators by the Arzela–Ascoli compactness theorem, and the Fredholm property and the index (the dimension of the kernel minus that of the cokernel) do not change under adding a compact operator. This shows that σ_φ is a Fredholm operator whose index is given by

$$\frac{1}{2}\operatorname{ind}\sigma_\varphi \;=\; \operatorname{ind}_{\mathbb{C}}\overline{\partial}^{\varphi}_J \;=\; \chi(\varphi^* T_M, \overline{\partial}^{\varphi}_J),$$

the holomorphic Euler characteristic of $\varphi^* T_M$. The latter can be computed by the ordinary Riemann–Roch theorem to be

$$\deg(\varphi^* T_M, \varphi^* J) + (1-g)\dim_{\mathbb{C}} M \;=\; c_1(M,J)\cdot\varphi_*[\Sigma] + (1-g)n.$$

If $s_{\overline{\partial},J}$ is transverse at φ, which by definition means that σ_φ is surjective, then an application of the implicit function theorem shows that in a neighbourhood of φ, the space $\mathcal{C}^{\mathrm{hol}}(\Sigma, M, J)$ of J-holomorphic maps $\Sigma \to M$ is a differentiable manifold of dimension $\operatorname{ind}\sigma_\varphi$. Moreover, near such points, $\mathcal{C}^{\mathrm{hol}}(\Sigma, M, J)$ is naturally oriented by complex linearity of $\overline{\partial}^{\varphi}_J$. Ignoring questions of compactness for the moment, if transversality is true everywhere, $\mathcal{C}^{\mathrm{hol}}(\Sigma, M, J)$ is a good moduli space for enumerative purposes involving J-holomorphic curves, i.e., GW invariants. In the integrable situation (i.e., M a complex manifold), transversality at φ means that the deformation theory of φ is unobstructed: φ deforms both under deformations of J and j. By the same token we see that the same statement holds even under deformations of J as almost complex structure.

Using the Sard–Smale theorem one can make $s_{\overline{\partial},J}$ transverse everywhere except possibly at so-called *multiple cover curves*, simply by a generic choice of J. A multiple cover curve is a $\varphi\colon \Sigma \to M$ that factors over a holomorphic map of Riemann surfaces of higher degree. For J-holomorphic curves this is equivalent to saying that φ is not generically injective. The reason is that the Sard–Smale theorem requires that perturbations of J generate the tangent space of the ambient space, and this may fail if φ is not generically injective. For certain (positive) manifolds the bad locus of compactifying and multiple cover curves can be proved to be of lower dimension for generic J and thus to be ignorable for enumerative questions. This is the original, very successful, approach of Ruan to GW theory [Ru1], following a similar scheme in gauge theory.

In the general case there are two ways of proceeding. If one wants to stick to manifolds one can introduce an *abstract perturbation term*, which is just a section $\nu \in C^1(\mathcal{B};\mathcal{E})$, and consider solutions of the perturbed equation

$$\overline{\partial}_J\varphi \;=\; \nu(\varphi),$$

i.e., look at the zero locus of the perturbed section $s_{\overline{\partial},J} - \nu$. Again by the Sard–Smale theorem, for generic choice of ν, $\mathcal{C}^{\text{hol}}(\Sigma, M, J, \nu) := Z(s_{\overline{\partial},J} - \nu)$ is a canonically oriented manifold of dimension $\operatorname{ind} \sigma_\varphi$. So one replaces the possibly singular, wrong dimensional $\mathcal{C}^{\text{hol}}(\Sigma, M, J)$ by an approximating manifold inside $\mathcal{B}$. In GW theory this has again been pioneered by Ruan to extend the range of the previous approach to semipositive manifolds by removing nongenericity of multiple cover curves (the perturbation term unfortunately vanishes at "bubbling components" of the domain, cf. Section 1.3, that have to be included on the compactifying part; so this method still does not work generally). The idea of associating (cobordism classes of) finite dimensional submanifolds to Fredholm maps by perturbations goes back (at least) to Smale [Sm].

The other approach, that we follow for the most part, is to replace the manifold by a homology class located on $\mathcal{C}^{\text{hol}}(\Sigma, M, J)$. The homology class should be thought of as the limit of the fundamental classes of the perturbed manifolds $Z(s_{\overline{\partial},J} - \nu)$ as ν tends to zero. Because its image in $H_*(\mathcal{B})$ represents any of these fundamental classes, the homology class is called the *virtual fundamental class* of $\mathcal{C}^{\text{hol}}(\Sigma, M, J)$. For conceptual clarity let us discuss this topic in an abstract setting.

1.2 Localized Euler classes in finite and infinite dimensions

First a note on homology theories. While cohomology has good properties on a large class of spaces making it essentially unique, (singular) homology behaves well only on compact spaces. Several extensions to noncompact spaces are possible. Since we need fundamental classes of noncompact oriented manifolds, the natural choice is singular homology of the second kind, i.e., with only *locally finite* singular chains, or Borel–Moore homology with coefficients in the ring $\mathbb{Z}$ (or later also $\mathbb{Q}$). These two homology theories are isomorphic under fairly mild conditions on the spaces, that are fulfilled in cases of our interest, cf. [Sk]. Note that this homology theory has restriction homomorphisms to open sets, obeys invariance only under *proper* homotopy, and pushes forward only under *proper* morphisms. General references are [Br],[Iv] and [Sk].

Given a Hausdorff space X with a closed subspace Z the localized cap products are homomorphisms

$$\cap \colon H_n(X) \otimes H^k_Z(X) \longrightarrow H_{n-k}(Z),$$

where, as mostly in the sequel, we suppressed coefficient rings. If s is a section of an oriented topological vector bundle E of finite rank r over X, the Euler class of E can be localized on the zero locus Z of s. Namely, let $\Theta_E \in H^r_X(E)$ be the Thom class of E. Locally, Θ_E is of the form $\pi^*\delta_0$, where $\pi\colon E|_U \to \mathbb{R}^r$ is a local trivialization and $\delta_0 \in H^r_{\{0\}}(\mathbb{R}^r, \mathbb{Z})$ is the unique generator compatible with the orientation. Then $s^*\Theta_E \in H^r_Z(X)$ represents the Euler class of E. And if X is an oriented topological manifold one may pair $s^*\Theta_E$ with the fundamental class $[X]$ using the localized cap product to arrive at a homology class on Z which is Poincaré dual on X to the Euler class of E.

In a differentiable situation, i.e., E a differentiable vector bundle over an oriented differentiable manifold X, let s_i be (differentiable) transversal sections converging to s. Then $[X] \cap s_i^*\Theta_E = [Z(s_i)]$, the fundamental class of the naturally oriented manifold $Z(s_i)$. These converge to $[X] \cap s^*\Theta_E \in H_{n-r}(Z)$, $n = \dim X$, which may thus be viewed as natural homological replacement for the zero loci of generic perturbations of s.

In the infinite dimensional setting of the previous section, neither $\Theta_{\mathcal{E}}$ nor $[\mathcal{B}]$ make sense. But if s is differentiable with Fredholm linearizations, if $Z = Z(s)$ is compact and if $\mathcal{B}$ admits differentiable bump functions, we may do the following: by hypothesis it is possible to construct a homomorphism from a trivial vector bundle $\tau\colon F = \underline{\mathbb{R}}^r \to \mathcal{E}$ so that for any $x \in Z(s)$, $\tau_x + D_x s\colon \mathbb{R}^r \oplus T_{\mathcal{B},x} \to \mathcal{E}_x$ is surjective. Replace $\mathcal{B}$ by the total space of F (which is just $\mathcal{B} \times \mathbb{R}^r$, but we will need nontrivial bundles later), and consider the section $\widetilde{s} := q^*s + \tau$ of $q^*\mathcal{E}$, where $q\colon F \to \mathcal{B}$ is the bundle projection. Note that if we identify $\mathcal{B}$ with the zero section of F then $\widetilde{s}|_{\mathcal{B}} = s$. $\tau_x + D_x s$ to be surjective means that $\widetilde{s}$ is a transverse section, at least in a neighbourhood of $Z = \widetilde{Z} \cap \mathcal{B}$, $\widetilde{Z} = Z(\widetilde{s})$. So $\widetilde{Z} \subset F$ is a manifold near the zero section of F. And F being of finite rank it has a Thom class, no matter the base is infinite dimensional. Ignoring questions of orientation we may then define

$$[\mathcal{E}, s] \;:=\; [\widetilde{Z}] \cap \Theta_F \;\in\; H_*(Z).$$

It is not hard to check independence of choices and coincidence with $[Z]$ in transverse situations. The dimension of $[\mathcal{E}, s]$ is locally constant and equals the index of the linearization of s.

Similar ideas have been applied in certain cases to compute both Donaldson and GW invariants, notably if the dimension of the cokernel is constant. In the presented generality this is due to Brussee who used it to study Seiberg–Witten theory in degenerate situations [Bs].

I should also point out that, locally, $[\mathcal{E}, s]$ is uniquely determined by a *Kuranishi model* for Z: let s be given locally by a differentiable Fredholm map $f\colon U \to E = \mathcal{E}_x$, $U \subset \mathcal{B}$ open. Let $q\colon E \to Q$ be a projection with finite dimensional kernel C ("C" for cokernel) such that $D(q \circ f)\colon T_{\mathcal{B},x} \to Q$ is surjective. Possibly after shrinking U, $\widetilde{Z} = (q \circ f)^{-1}(0)$ is a manifold of dimension $\operatorname{ind} f + r$, $r = \dim C$. Then $f|_{\widetilde{Z}}\colon \widetilde{Z} \to C$ is a differentiable map between finite dimensional manifolds. Again ignoring questions of orientation, we observe

$$[\mathcal{E}, s]\Big|_U \;=\; [\widetilde{Z}] \cap (f|_{\widetilde{Z}})^*\delta_0 \;\in\; H_*(Z \cap U),$$

$\delta_0 \in H^r_{\{0\}}(C, \mathbb{Z})$ the positive generator. However, unless Z is of *expected dimension* $\operatorname{ind} s$, the glueing of the local classes may not be unique. So the knowledge of Kuranishi models covering $Z(s)$ may not be enough to determine the class $[\mathcal{E}, s]$. It would be interesting to understand precisely what additional datum is needed to globalize these classes. In a sense this is the question how topological a theory of localized Euler classes of *differentiable* Fredholm sections can be made.

1.3 The compactification problem: stable J-curves

Since we used compactness of Z for the construction of $[\mathcal{E}, s]$, the method of the last section applies directly to our model in 1.1 only on compact components of the space of J-holomorphic maps from Σ to M. This never holds for the important case of $\Sigma = \mathbb{P}^1$ because of noncompactness of $\operatorname{Aut}(\mathbb{P}^1) = \mathrm{PGL}(2)$, which acts on these spaces by reparametrization. This trivial cause of noncompactness could be avoided by factoring out the connected component of the identity $\operatorname{Aut}^0(\Sigma)$. More fundamentally though, the space of J-holomorphic maps is not compact if so-called *bubbles* appear in limits of sequences of such maps. If $\varphi_i \colon \Sigma \to M$ is a sequence of J-holomorphic maps, a bubble is a J-holomorphic rational curve $\psi \colon \mathbb{P}^1 \to M$ obtained as the limit of rescalings of φ_i near a sequence of points $P_i \to P \in \Sigma$ with $|D\varphi_i(P_i)|$ unbounded. A simple example of bubbling off in algebraic geometry is the degeneration of a family of conics in the plane to a line pair. The content of the Gromov compactness theorem is that this phenomenon is the precise reason for noncompactness of moduli spaces of J-holomorphic curves of bounded volume, cf. Theorem 1.2. As one knows from examples in algebraic geometry this happens quite often unless $(\varphi_i)_*[\Sigma]$ (constant on connected components of $\mathcal{C}^{\mathrm{hol}}(\Sigma, M, J)$) is indecomposable in the cone in $H_2(M, \mathbb{Z})$ spanned by classes representable by J-holomorphic curves.

We are thus lead to the problem of introducing an appropriate compactification of $\mathcal{C}^{\mathrm{hol}}(\Sigma, M, J)$.[2] This is due to Gromov [Gv], and Parker and Wolfson from a different point of view [PrWo], but has been put into its final form by Kontsevich through the notion of stable map [KoMa]. It is convenient to also incorporate marked points on Σ now.

Definition 1.1 $(C, \mathbf{x}, \varphi)$ is called a *stable J-holomorphic curve* if

[2]Another reason compactification is essential is of course that we need a degree map to extract well-defined numbers out of the homology class, cf. 4.1

- C is a connected, reduced, complete complex algebraic curve with at most ordinary double points (a "Riemann surface with nodes");
- $\mathbf{x} = (x_1, \ldots, x_k)$ with pairwise distinct $x_i \in C_{\mathrm{reg}}$;
- for any irreducible component $D \subset C$, $\varphi|_C$ is J-holomorphic;
- $\mathrm{Aut}(C, \mathbf{x}, \varphi) := \{\sigma \colon C \to C \text{ biregular} \mid \varphi \circ \sigma = \varphi\}$ is finite. □

The first two conditions are sometimes summarized by saying that $(C, \mathbf{x})$ is a *pre-stable (marked) curve.* The condition on finiteness of the automorphism group is the *stability condition.* It can be rephrased by saying that the normalization (=desingularization) D of any rational component of C contracted by φ contains at least three special points (marked points or preimages of nodes under $D \to C$). Note that by putting $M = \{\mathrm{pt}\}$ one retrieves the definition of stable algebraic curves with marked points due to Deligne and Mumford [DeMu] and Knudson [Kn]. So the notion of stable J-holomorphic curve should be viewed as natural extension of the concept of Deligne–Mumford stable curve to the situation relative to M rather than the spectrum of a field. The *genus* $g(C)$ of $(C, \mathbf{x}, \varphi)$ is by definition the arithmetic genus $h^1(C, \mathcal{O}_C)$ of C. Since $h^1(C, \mathcal{O}_C) = 1 - \chi(C, \mathcal{O}_C)$ is constant in flat families, $g(C)$ could alternatively be defined as the genus of a smooth fiber of a deformation of C, i.e., the genus of the closed surface obtained from C by replacing each double point $x \cdot y = 0$ by a cylinder $x \cdot y = \varepsilon$.

How does this concept incorporate bubbling phenomena, say in our model of maps $\varphi \colon \Sigma \to M$? After rescaling at P_i in such a way that the differentials become uniformly bounded at P_i, there might be another sequence of points with unbounded differentials. So we may end up in the limit with a *tree* $\psi \colon B \to M$ of J-holomorphic rational curves at P. To be a tree means that B is simply connected and has at worst ordinary double points. To achieve the latter, one might have to introduce more rational components than necessary to make sense of a limiting map, i.e., ψ might be trivial on some irreducible components $D \subset B$, but only if D contains at least three double points. Because the only marked Riemann surface with infinitesimal automorphisms fixing one more point (a double point making the whole curve connected) is $\mathbb{P}^1$ with fewer than two marked points, this is the stability condition! So in this case, the domain of the limiting map will be $(C, \mathbf{x}) = (\Sigma \cup_P B, \emptyset)$, or more generally, several trees $B_1, \ldots, B_b$ of rational curves attached to Σ at several points.

Conversely, if $(C, \mathbf{x}, \varphi)$ is a stable J-holomorphic curve, there is always an associated stable curve $(C, \mathbf{x})^{\mathrm{st}} = (C_{\mathrm{st}}, \mathbf{x}_{\mathrm{st}})$ so that $(C, \mathbf{x}, \varphi)$ looks like the curve obtained starting from a sequence of J-holomorphic curves by bubbling off (in reality this deformation problem might be obstructed). $(C, \mathbf{x})^{\mathrm{st}}$

is just the *stabilization* of the abstract curve $(C, \mathbf{x})$ obtained by successive contraction of (absolutely) unstable components. The latter are by definition rational components $D \subset C$ with $\#\{x_i \in D\} + \#C_{\text{sing}} \cap D < 3$. Equivalently, $\operatorname{Aut}^0(C, \mathbf{x})$ acts nontrivially on D. After contracting all unstable components of $(C, \mathbf{x})$, some previously stable components may become unstable. The process is then repeated until the result is a Deligne–Mumford stable curve $(C, \mathbf{x})^{\text{st}}$. Note that the genus does not change under this process.

By this picture it is natural to distinguish *bubbling* and *principal* components of the domain of stable J-holomorphic curves $(C, \mathbf{x}, \varphi)$, depending on whether or not the component gets contracted under the *stabilization map* $(C, \mathbf{x}) \to (C, \mathbf{x})^{\text{st}}$. Note that if $(C, \mathbf{x})^{\text{st}}$ is singular, there is another possible type of bubbling off, different from that discussed above, introducing a *chain* of $\mathbb{P}^1$s joining the two branches over an ordinary double point.

One subtlety in the discussion of stable J-curves is that their domains are only pre-stable curves, which do not in general possess decent moduli spaces. To explain this, recall the local description of $\mathcal{M}_{g,k}$, the coarse moduli space of (Deligne–Mumford) stable curves of genus g with k marked points. For later use, it is better to work complex analytically now. If $(C, \mathbf{x}) \in \mathcal{M}_{g,k}$ there is an open subset $S \subset \operatorname{Ext}^1(\Omega_C(x_1 + \cdots + x_k), \mathcal{O}_C) \simeq \mathbb{C}^{3g-3+k}$, a flat family $q\colon \mathcal{C} \to S$ (with $\mathcal{C}$ smooth) of pre-stable curves with k sections $\underline{\mathbf{x}}\colon S \to \mathcal{C} \times_S \cdots \times_S \mathcal{C}$, such that the germ of $(\mathcal{C} \to S, \underline{\mathbf{x}})$ at $0 \in S$ is an analytically universal deformation of $(C, \mathbf{x})$. This means that the germ of any flat family of marked stable curves with central fiber $(C, \mathbf{x})$ is (canonically isomorphic to) the pullback of $(\mathcal{C} \to S, \underline{\mathbf{x}})$ under a map from the parameter space to S. If $(C, \mathbf{x})$ has nontrivial automorphisms, the action on the central fiber extends to an action on (the germ at $0 \in S$ of) $\mathcal{C}$ and S making q and $\underline{\mathbf{x}}$ equivariant. After possibly shrinking S, one may also assume that s and $s' \in S$ parametrize isomorphic marked stable curves if and only if there exists an automorphism of $(C, \mathbf{x})$ carrying s to s'. Since $\operatorname{Aut}(C, \mathbf{x})$ is finite we may assume that the action is in fact well defined on all of $\mathcal{C}$ and S. The quotient $S/\operatorname{Aut}(C, \mathbf{x})$ exists as complex space and is a neighbourhood of $(C, \mathbf{x})$ in $\mathcal{M}_{g,k}$.

If $(C, \mathbf{x})$ is just pre-stable we still have a pair $(\mathcal{C} \to S, \underline{\mathbf{x}})$. But now $\operatorname{Aut}(C, \mathbf{x})$ is higher dimensional and $\dim S = 3g - 3 + k + \dim \operatorname{Aut}(C, \mathbf{x})$. There is the germ of an action of $\operatorname{Aut}^0(C, \mathbf{x})$ on $\mathcal{C} \to S$, which is a map from a neighbourhood of $\{\mathrm{Id}\} \times C$ in $\operatorname{Aut}^0(C, \mathbf{x}) \times \mathcal{C}$ to $\mathcal{C}$ (respectively, from a neighbourhood of $(\mathrm{Id}, 0) \in \operatorname{Aut}^0(C, \mathbf{x}) \times S$ to S). Now $(\mathcal{C} \to S, \underline{\mathbf{x}})$ is no longer a universal deformation, but only semiuniversal, which means that uniqueness holds only on the level of tangent maps at $0 \in S$. The moduli "space" $\mathfrak{F}_{g,k}$ of pre-stable curves of genus $g = g(C)$ with $k = \#\underline{\mathbf{x}}$ marked points should locally around $(C, \mathbf{x})$ be thought of as the quotient of S by the analytic equivalence relation generated by this action. Now $\operatorname{Aut}^0(C, \mathbf{x})$ decomposes into a product with factors $\mathbb{C}^*$ for each bubbling component with only two special (i.e., marked or singular) points and $\mathbb{C} \rtimes \mathbb{C}^*$ for each bubbling component with one special point. In appropriate coordinates, the restriction of the action to one such $\mathbb{C}^*$ looks like the standard $\mathbb{C}^*$ action on $\mathbb{C}$ cross a trivial factor. So a quotient does not even exist as a Hausdorff topological space, let alone analytic space or scheme.

Despite this, $\mathfrak{F}_{g,k}$ behaves in many respects like a scheme. It has a structure of what is called an *Artin stack*. We do not go into details of this, but instead keep in mind the local description as a quotient of the base S of a semiuniversal deformation of $(C, \mathbf{x})$ by the analytic equivalence relation generated by $\mathrm{Aut}(C, \mathbf{x})$.

It is also useful to observe that the semiuniversal deformation $\mathcal{C} \to S$ of $(C, \mathbf{x})$ fibers over the universal deformation $\overline{\mathcal{C}} \to \overline{S}$ of its stabilization $(C, \mathbf{x})^{\mathrm{st}}$. The map $S \to \overline{S}$ is smooth (a linear projection in appropriate coordinates) unless $(C, \mathbf{x})$ has bubble chains (bubbles inserted at a double point of $(C, \mathbf{x})^{\mathrm{st}}$), in which case it is only of complete intersection type (with factors of the form $(x_1, \ldots, x_r) \mapsto x_1 \cdot \cdots \cdot x_r$ in appropriate coordinates). This is important in the proof of the isogeny axiom of GW invariants in Section 4.2.

There is a natural topology on the set of stable J-holomorphic curves, the *Gromov topology* [Gv], §1.5 cf. also [Pn], Def. 2.12. We do not give the definition here because it will become obvious once we introduce local charts for the ambient Banach manifold in Chapter 3. To state the compactness theorem, let $R \in H_2(M, \mathbb{Z})$ and $g, k \geq 0$.

Theorem 1.2 ([Gv], [Pn], [PrWo], [RuTi1], [Ye]) *Assume that the almost complex structure J is tamed by some symplectic form ω on M, i.e., $\omega(X, JX) > 0$ for all $X \in T_M \setminus \{0\}$. Then the space*

$$\mathcal{C}^{\mathrm{hol}}_{R,g,k}(M, J) \;:=\; \left\{ \begin{array}{c|l} (C, \mathbf{x}, \varphi) \text{ is a stable} & \varphi_*[C] = R, \#\mathbf{x} = k, \\ J\text{-holomorphic curve} & \text{and } g(C) = g \end{array} \right\} \Big/ \text{iso}$$

with the Gromov topology is compact and Hausdorff. □

The taming condition allows us to bound the volume of J-holomorphic curves in terms of its homology class R by the analogue of the Wirtinger theorem. In fact, Ye's method of proof uses only a bound on the volume. That such a bound is crucial in compactness results has been understood in complex analysis since [Bi]. The Hausdorff property is not proved in the given references but requires some additional arguments as given in any of [FkOn], [LiTi2], [Ru2], [Si1].

We also need to enlarge the ambient Banach manifold. We discuss this in a separate chapter since it involves a number of subtleties.

2 The ambient space

To carry out the program of Section 1.2 for the spaces $\mathcal{C}^{\mathrm{hol}}_{R,g,k}(M,J)$ of stable J-holomorphic curves in an almost complex manifold tamed by some symplectic form ω, we want to construct a Banach manifold into which $\mathcal{C}^{\mathrm{hol}}_{R,g,k}(M,J)$ embeds. Obvious choices are spaces of triples $(C,\mathbf{x},\varphi)$ with $(C,\mathbf{x})$ a k-marked pre-stable curve of genus g and $\varphi\colon C\to M$ just a continuous map such that $\varphi_*[C]=R$, together with some kind of regularity. We will see that requiring φ to be of Sobolev class L^p_1 with $2<p<\infty$, i.e., with one distributional derivative in L^p, is a very natural condition. The measure is with respect to a metric on C with certain weights at the singular points. For general facts on Sobolev spaces we refer to the books [Ad] (for Sobolev spaces on $\mathbb{R}^n$) and [Au] for the case of manifolds. Most standard textbooks on partial differential equations also contain ample information on Sobolev spaces. Note that since the domain is two dimensional, L^2_1 is a critical case of the Sobolev embedding theorem: there exist discontinuous L^2_1-functions on $\mathbb{R}^2$, but functions in L^p_1 with $p>2$ always have continuous representatives. Thus L^p_1 with $p>2$ is the minimal possible regularity for a sensible formulation of the $\overline{\partial}_J$-equation. Conversely, it is unreasonable to expect maps of higher regularity to give rise to a smooth total space, as will be clear from Section 2.4.

2.1 Charts

To produce charts, observe that, intuitively, a small deformation of $(C,\mathbf{x},\varphi)$ can be split into

(1) a deformation of the domain $(C,\mathbf{x})$ as pre-stable curve, arriving at a possibly less singular curve C', and

(2) a deformation of a pullback $\varphi\circ\kappa$, where $\kappa\colon C'\to C$ is some comparison map that is a diffeomorphism away from the singularities of C.

As in our discussion of the Artin stack $\mathfrak{F}_{g,k}$ in Section 1.3, let $(q\colon\mathcal{C}\to S,\underline{\mathbf{x}})$ be an analytically semiuniversal deformation of $(C,\mathbf{x})=(q^{-1}(0),\underline{\mathbf{x}}(0))$. Let us write C_s for $q^{-1}(s)$. If $\varphi\colon C\to M$ is L^p_1, we want $\varphi\circ\kappa$ also to be L^p_1. Since $\bigcap_{p>2}L^p_1=L^\infty_1$ (the Sobolev space of functions with essentially bounded first derivative) and $L^\infty_1=C^{0,1}$ is the space of Lipschitz maps, a good choice that works for all p should be Lipschitz. Using an analytic description of $q\colon\mathcal{C}\to S$, it is not hard to construct a retraction

$$\kappa\colon\mathcal{C}\longrightarrow C_0=C,$$

which, when restricted to C_s, is a diffeomorphism away from $C_{\rm sing}$, and which near the smoothing $zw = t$ of a node $zw = 0$ is given by a linear rescaling

$$(z = re^{i\varphi}) \;\longmapsto\; z = \frac{r - |t|^{1/2}}{1 - |t|^{1/2}}\, e^{i\varphi} \quad \text{if } |z| \geq |w|,$$

and similarly for w if $|w| \geq |z|$. In particular the circle $|z| = |w| = |t|^{1/2}$ is contracted to the node and κ is Lipschitz (note that $\mathcal{C}$ is also smooth).

Next, we define our weighted Sobolev spaces. The choice is distinguished by the fact that we want κ to induce isomorphisms of L^p-spaces. Since the ordinary Lebesgue measure on a nonsingular C_s corresponds to the finite cylindrical measure $\mathrm{d}r\mathrm{d}\varphi = r^{-1}\mathrm{d}x\mathrm{d}y$ on each branch of C near a singular point (where $z = re^{i\varphi} = x + iy$), our measure μ on C is required to be of this type near $C_{\rm sing}$ and locally equivalent to Lebesgue measure away from this set. We write

$$\check{L}^p(C, \mathbb{R}) \;:=\; L^p(C, \mathbb{R}; \mu)$$

and $\check{L}^p_1(C, \mathbb{R}) \subset L^p_1(C, \mathbb{R})$ for the functions possessing one weak derivative in $\check{L}^p(C, \mathbb{R})$ (on each irreducible component of C). Since $\check{L}^p(C, \mathbb{R}) \subset L^p(C, \mathbb{R})$, by the Sobolev embedding theorem there is an embedding of $\check{L}^p_1(C, \mathbb{R})$ into the space of continuous functions $C^0(C, \mathbb{R})$ for $p > 2$. We adopt the usual abuse of notation and identify $\check{L}^p_1(C, \mathbb{R})$ with its image in the space of continuous functions, i.e., we take the unique continuous representative of any class in $\check{L}^p_1(C, \mathbb{R})$. Note that in general there is no distinguished choice of metric on C, so these spaces are well defined only as topological vector spaces, not as normed spaces.

For vector bundles E over C one defines similarly $\check{L}^p(C, E)$ and $\check{L}^p_1(C, E)$. And as usual, spaces of maps $\check{L}^p_1(C, M)$ can be defined either by embedding M into some $\mathbb{R}^N$ and requiring the Sobolev property componentwise, or by taking local coordinates on M and require composition with the coordinate functions to be $\check{L}^p_1$. This is well defined for $p > 2$ by continuity.

Here is the definition of our ambient space. We fix once and for all some p with $2 < p < \infty$.

Definition 2.1 $(C, \mathbf{x}, \varphi)$ is a *stable complex curve* in M of Sobolev class L^p_1 if and only if

- $(C, \mathbf{x})$ is a pre-stable marked curve;
- $\varphi \in \check{L}^p_1(C, M)$;
- for any unstable component D of $(C, \mathbf{x})$, $\varphi|_D$ is homotopically non-trivial. $\square$

We use $\mathcal{C}(M;p)$ to denote the set of isomorphism classes of such curves and

$$\mathcal{C}_{R,g,k}(M;p) \ := \ \Big\{(C,\mathbf{x},\varphi)\in\mathcal{C}(M;p)\ \Big|\ \varphi_*[C]=R, \#\mathbf{x}=k, g(C)=g\Big\}\Big/\text{iso.}$$

By abuse of notation, we write $(C,\mathbf{x},\varphi)\in\mathcal{C}(M;p)$ to mean a representative of the isomorphism class.

By construction $\kappa_s^*\colon \check{L}^p(C,\mathbb{R}) \to \check{L}^p(C_s,\mathbb{R})$ is an isomorphism for any $s\in S$. On $\check{L}_1^p$-spaces pullback is also well defined, because κ_s is Lipschitz, but since $\varphi\circ\kappa_s$ is constant on the contracted circles, κ_s^* is certainly not surjective. What we are interested in for the construction of charts are identifications $\Pi_s\colon \check{L}_1^p(C,\varphi^*T_M)\to\check{L}_1^p(C_s;(\varphi\circ\kappa_s)^*T_M)$, i.e., a structure of Banach bundle on $\coprod_s \check{L}_1^p(C_s;(\varphi\circ\kappa_s)^*T_M)$. The latter space will be denoted $q_*^{1,p}(\kappa^*\varphi^*T_M)$, which captures the idea of being the direct image of a sheaf of sections of $(\varphi\circ\kappa)^*T_M$ that are fiberwise locally of class $\check{L}_1^p$. We will sketch a proof of the following result in Section 2.4:

Theorem 2.2 *Let $(C,\mathbf{x},\varphi: C\to M)$ be a stable map of class L_1^p and $(q:\mathcal{C}\to S,\underline{\mathbf{x}})$ an analytically semiuniversal deformation of $(C,\mathbf{x})$. Thus S is an open set in $\mathbb{C}^N$, where $N=3g-3+k+\dim\operatorname{Aut}(C,\mathbf{x})$. Let $\kappa:\mathcal{C}\to C_0=C$ be the Lipschitz retraction to the central fiber as above. Then there exists a family of isomorphisms*

$$\Pi_s\colon \check{L}_1^p(C,\varphi^*T_M)\to\check{L}_1^p(C_s;(\varphi\circ\kappa_s)^*T_M).$$

Π_s enjoys the following continuity property: The composition with pullback to the central fibre

$$(\kappa_s^*)^{-1}\circ\Pi_s\colon \check{L}_1^p(C,\varphi^*T_M)\to L^\infty(C,\varphi^*T_M)$$

is a norm continuous family of operators (between fixed Banach spaces!) parametrized by S.

Here the stated norm continuity implies that for $v\in\check{L}_1^p(C,\varphi^*T_M)$, small changes of s lead to small pointwise changes of $\Pi_s v$ in an intuitive sense. Thus the graphs of $\Pi_s v$ in the total space of $\kappa^*\varphi^*T_M$ fit together continuously. More regularity properties of Π_s will be discussed later.

Given Π_s we just need to write down the analogue of the charts for fixed domains (cf. 1.1) to get charts for $\mathcal{C}(M;p)$:

$$\Phi\colon S\times\check{L}_1^p(C,\varphi^*T_M)\supset S\times V\longrightarrow\mathcal{C}(M;p),\quad (s,v)\longmapsto\varphi(s,v)$$
$$\varphi(s,v)(z)\ :=\ \exp_{\varphi\circ\kappa_s(z)}\Big(\Pi_s v\Big)(z).$$

2.2 Automorphisms and differentiable structure

There are still two problems with the proposed charts Φ of 3.1. First, since S does not in general parametrize pre-stable curves near $(C, \mathbf{x})$ effectively, Φ need not be injective. Not every $\Psi \in \operatorname{Aut}(C, \mathbf{x})$ has $\operatorname{im}\Phi \cap \operatorname{im}(\Psi^* \circ \Phi) \neq \emptyset$, but this is certainly the case for Ψ close to $\operatorname{Aut}(C, \mathbf{x}, \varphi) \subset \operatorname{Aut}(C, \mathbf{x})$. Here we write Ψ both for the automorphism of the central fibre of the deformation and for the germ of the action on the total space $\mathcal{C}$. Thus the best we can hope for is the structure of a Banach *orbifold* on $\mathcal{C}(M; p)$.

Orbifolds are a generalization of the notion of manifolds where we replace open subsets of vector spaces as local models by quotients of such open subsets by linear actions of finite groups. More precisely, one defines

Definition 2.3 A *local uniformizing system* (of Banach orbifolds) is a triple $(q\colon \widehat{U} \to U, G, \alpha)$ where:

(1) α is a continuous linear action of the finite group G on some Banach space T;

(2) $\widehat{U}$ is a G-invariant open subset of T;

(3) q induces a homeomorphism $\widehat{U}/G \to U$.

For a more intuitive notation we often write $U = \widehat{U}/G$ instead of (q, G, α). □

Compatibility of local uniformizing systems is defined through the notion of *open embeddings*: let $V = \widehat{V}/G'$, $U = \widehat{U}/G$ be two local uniformizing systems. An open embedding $V = \widehat{V}/G' \hookrightarrow U = \widehat{U}/G$ is a γ-equivariant open embedding $\widehat{f}\colon \widehat{V} \to \widehat{U}$ for a monomorphism $\gamma\colon G' \to G$. Thus this induces an open embedding of the quotient spaces $f\colon V \hookrightarrow U$. If the actions of the groups are not effective one should also require a maximality condition for γ, namely

$$\operatorname{im}\gamma \;=\; \{g \in G \mid \widehat{f}(\widehat{V}) \cap g \cdot \widehat{f}(\widehat{V}) \neq \emptyset\}.$$

This ensures that for any $\widehat{X} \in \widehat{V}$, γ induces an isomorphism of stabilizers $G'_{\widehat{X}} \simeq G_{\widehat{f}(\widehat{X})}$.[3]

Recall that a covering $\{U_i\}_{i \in I}$ of a set X is called *fine* if for any $i, j \in I$ with $U_i \cap U_j \neq \emptyset$ there exists $k \in I$ with $U_k \subset U_i \cap U_j$. An *atlas* for the structure of Banach orbifold on a Hausdorff space X is now a fine covering of X by local

[3]The group actions are traditionally required to be effective. This is too restrictive for our purposes: because any curve of genus 2 has a hyperelliptic involution, $\mathcal{M}_2$ is an instance of an orbifold with $\mathbb{Z}_2$-kernel of the action everywhere, cf. below.

uniformizing systems $\{U_i = \widehat{U}_i/G_i\}_{i\in I}$ (i.e., $\{U_i\}$ is an open covering of X) such that for any $i, j \in I$ there is a $k \in I$ and open embeddings

$$U_k = \widehat{U}_k/G_k \hookrightarrow U_i = \widehat{U}_i/G_i \quad \text{and} \quad U_k = \widehat{U}_k/G_k \hookrightarrow U_j = \widehat{U}_j/G_j.$$

It is in general not possible to find an open embedding of the restriction of $U_i = \widehat{U}_i/G_i$ to $U_i \cap U_j$ into $U_j = \widehat{U}_j/G_j$. Consider, for example, the orbifold structure on S^2 with cyclic quotient singularities of orders 2 and 3 at the poles. This orbifold can be covered by two local uniformizing systems $\mathbb{R}^2 = \mathbb{C}/\mathbb{Z}^2$, $\mathbb{R}^2 = \mathbb{C}/\mathbb{Z}^3$ via stereographic projection from the poles. So unlike the case of manifolds, one has to restrict to sufficiently small open sets to compare two local uniformizing systems.

As usual a *(topological) Banach orbifold* is defined as an equivalence class of atlases (or as a maximal atlas). If one requires all open embeddings $\widehat{f}$ to be differentiable or holomorphic immersions, one arrives at *differentiable* and *complex* Banach orbifolds. In the latter case, the representations $G \to \mathrm{GL}(T)$ should of course also be complex.

Quotients of manifolds by finite groups are examples of orbifolds, but the whole point of the concept of orbifold is that not every orbifold is of this form. An easy example is S^2 with a $\mathbb{Z}_m$-quotient singularity at one point P: if $q\colon X \to S^2$ were a global cover (connected, w.l.o.g.), $X \setminus q^{-1}(P) \to S^2 \setminus \{P\}$ is an unbranched cover, hence trivial by simply connectivity of the base, hence bijective. But any local uniformizer at P is m to one. Contradiction, such a global cover X does not exist! According to Thurston, orbifolds are called *good* or *bad* depending on whether or not they are globally covered by manifolds.

Note that to build up an orbifold from a set of local uniformizing systems $\{U_i = \widehat{U}_i/G_i\}$ through glueing by open embeddings, we need only require the cocycle condition to hold on the level of the underlying sets U_i; on the level of uniformizers $\widehat{U}_i$, it needs to hold only up to the group actions.

Note also that any $x \in X$ has an associated group G_x, the isomorphism class of the stabilizer $G_{\widehat{X}}$ of a lift $\widehat{X}$ of x to any local uniformizing system containing x. It is the smallest group of a local uniformizing system containing x. Thus the concept of orbifold incorporates groups intrinsically associated to the points of X.

Natural examples of orbifolds are thus moduli spaces of objects in complex analysis with finite automorphism groups and unobstructed deformation theory (the latter to ensure smoothness of the local covers). As we saw in the last section, moduli spaces $\mathcal{M}_g$ (or $\mathcal{M}_{g,k}$) of Deligne–Mumford stable curves of fixed genus are examples of this: local uniformizers at C are of the form $S \to S/\operatorname{Aut}(C)$, where $S \subset \operatorname{Ext}^1(\Omega_C, \mathcal{O}_C)$ is an open set. What is nice about viewing $\mathcal{M}_g$ as orbifold is that, unlike the underlying scheme or complex space, the orbifold is a fine moduli space, i.e., it wears a universal family of stable curves. The latter is the orbifold $\mathcal{M}_{g,1}$ of stable 1-pointed curves $(C,\mathbf{x})$ fibered over $\mathcal{M}_g$ via the forgetful and stabilization map $(C,x) \to C^{\mathrm{st}}$: any family $X \to T$ of stable curves of genus g over a complex manifold, say, is isomorphic to the pullback family $T \times_{\mathcal{M}_g} \mathcal{M}_{g,1} \to T$ for some morphism of orbifolds $T \to \mathcal{M}_g$ (where T is viewed as orbifold with trivial groups). Similarly for $\mathcal{M}_{g,k}$.

Such a *morphism of orbifolds* is just a continuous map of the underlying topological spaces with compatible lifts to local uniformizers. We also need the notion of *(vector) orbibundle*. This is a morphism of orbifolds $\pi\colon E \to X$ that is locally uniformized by projections $E_0 \times \widehat{U} \to \widehat{U}$, with E_0 a Banach space. If the local groups are G^E and G for $E|_U$ and U then the action of G^E on $E_0 \times \widehat{U}$ is required to be diagonal via a linear representation of G^E on E_0 and an epimorphism $G^E \to G$. Open embeddings of E have to be linear on the fibers E_0. Note that the topological fiber $\pi^{-1}(x)$ is isomorphic to E_0/G^E_x and thus does not have an additive structure in general. Tangent bundles of differentiable orbifolds are examples of orbibundles (with $G^E = G$ everywhere).

We now proceed with our discussion of charts for $\mathcal{C}(M;p)$. We will see that if the domain $(C,\mathbf{x})$ of $(C,\mathbf{x},\varphi)$ is stable as an abstract curve then the map $\Phi\colon S \times V \to \mathcal{C}(M;p)$ from the end of Section 2.1 will indeed provide a local uniformizing system at $(C,\mathbf{x},\varphi)$. But if $(C,\mathbf{x})$ is not stable as an abstract curve, i.e., if $(C,\mathbf{x},\varphi)$ has bubbles, then $\dim \mathrm{Aut}(C,\mathbf{x}) > 0$ and any $\eta \in \mathrm{Lie}\,\mathrm{Aut}(C,\mathbf{x})$ induces a vector field v_η on S in such a way that the prestable curves are mutually isomorphic along any integral curve of v_η. We will see in the next section how to deal with this problem by taking a slice of the induced equivalence relation on $S \times V$. The slice will again be a family of Banach manifolds over S but with tangent space at $(0,0) \in S \times V$ of the form $S \times \overline{V}$, where $\overline{V} \subset \check{L}^p_1(C,\varphi^* T_M)$ is a linear subspace of codimension equal to the dimension of $\mathrm{Aut}(C,\mathbf{x})$. Moreover, the slice can be taken to be $\mathrm{Aut}(C,\mathbf{x},\varphi)$-invariant.

The existence of the slice is not merely a simple application of the implicit function theorem; this is related to the second problem that we face with our charts: the action of the group of self-diffeomorphisms of C (and even of $\mathrm{Aut}^0(C,\mathbf{x})$) on $\check{L}^p_1(C,M)$ is only continuous, not differentiable. In fact, the differential with respect to a one parameter group of diffeomorphisms would mean applying the corresponding vector field to the maps $\varphi\colon C \to M$, and this costs one derivative. So looking at the simple case of nonsingular C, two choices of retraction $\kappa,\kappa'\colon \mathcal{C} \to C$ will lead to two different structures of differentiable Banach orbifold near $(C,\mathbf{x},\varphi)$: the coordinate change need not be differentiable. The solution to the problem is that, locally, the differentiable structure *relative* to S is well defined. Since S is finite dimensional, this is enough to make the implicit function theorem work, albeit locally in a version relative to S.

From a categorical point of view, we are thus led to a category of topological Banach orbifolds that locally have a well-defined differentiable structure relative to some finite dimensional spaces. From a point of view closer to algebraic geometry, one might alternatively view our Banach orbifold as "fibered in differentiable Banach orbifolds over the Artin stack $\mathfrak{F}_{g,k}$".

2.3 Slices

According to the discussion in the last section, when bubbles are present in $(C, \mathbf{x}, \varphi)$, we need to find slices of the equivalence relation generated by the germ of the action of $\mathrm{Aut}^0(C, \mathbf{x})$ on $S \times V \subset S \times \check{L}^p_1(C, \varphi^* T_M)$. The usual method, applied both in algebraic geometric and analytic approaches, is *rigidification.* This means adding enough points $\mathbf{y} = (y_1, \ldots, y_l)$ to $(C, \mathbf{x})$ to make $(C, \mathbf{x} \vee \mathbf{y})$ stable as an abstract curve; here $\mathbf{x} \vee \mathbf{y} = (x_1, \ldots, x_k, y_1, \ldots, y_l)$ is the concatenation of $\mathbf{x}$ and $\mathbf{y}$. Explicitly, this means that we add at least $3 - i$ points to each rational component having only i special points, $i \in \{1, 2\}$. By stability (!) the y_i can be chosen in such a way that φ is locally injective there. Choose locally closed submanifolds $H_1, \ldots, H_l \subset M$ of real codimension two and transversal to φ through $\varphi(y_1), \ldots, \varphi(y_l)$. The slice is

$$\gamma \;=\; \Big\{ (s, v) \in S \times V \;\Big|\; \varphi(s, v)(y_i) \in H_i \Big\}.$$

This will be a submanifold at $(0, 0) \in S \times V$ if transversality to H_i is an open condition in the function spaces we employ. This is indeed the case in function spaces with at least one continuous derivative. So the idea of rigidification is to let the map rule the deformation of the points we add.

Unfortunately, this method does not work in our case, because local injectivity is not an open condition in L^p_1. The way out is an integral version of rigidification: let $z\colon U \to C$ restrict to holomorphic coordinates on $U_s = U \cap C_s$, where $U \subset \mathcal{C}$ is an open set with U_0 contained in a bubble we want to rigidify. By stability, if U_0 is sufficiently large, there are differentiable bump functions ρ on M with $\varphi^* \rho|_{U_0}$ nontrivial and having compact support. Consider

$$\lambda(s, v) \;=\; \frac{\int_{U_s} z \cdot \varphi(s, v)^* \rho \, \mathrm{d}\mu(z)}{\int_{U_s} \varphi(s, v)^* \rho \, \mathrm{d}\mu(z)}\,,$$

which computes the center of gravity of $\varphi^* \rho$ in the coordinate $z(s)$ on U_s. Assembling one (respectively, two different) such λ for each unstable component of $(C, \mathbf{x})$ with two (respectively, one) special points into a vector valued function $\Lambda\colon S \times V \to \mathbb{C}^b$, where $b = \dim \mathrm{Aut}^0(C, \mathbf{x})$, our candidate for a slice is

$$\gamma \;=\; \Lambda^{-1}(\lambda_0),\ \lambda_0 = \Lambda(0,0).$$

Due to the lack of differentiability it seems hard to prove that this is in fact a slice, i.e., induces a local homeomorphism $\mathrm{Aut}^0(C,\mathbf{x}) \times \gamma \to S \times V$ on appropriate domains of definition. However, this is easy if we choose z to be a *linear* coordinate, i.e., such that the action of $\mathrm{Aut}^0(C,\mathbf{x})$ is affine linear. Such coordinates can be constructed explicitly, cf. [Si1]. Then the implicit function theorem allows us to change coordinates on $S \times V$ relative to S in such a way that $\gamma = S \times \overline{V}$, with $\overline{V} \subset V$ a linear subspace of codimension equal to $\dim \mathrm{Aut}(C,\mathbf{x})$. Moreover, the slice can be chosen $\mathrm{Aut}(C,\mathbf{x},\varphi)$-invariant.

An alternative in the differentiable setting is to take directly linear slices of the form $S \times \overline{V}$, with $\overline{V} \subset V$ complementary to the finite dimensional subspace spanned by the action of $\mathrm{Lie}\,\mathrm{Aut}(C,\mathbf{x})$. This is the approach of [FkOn]. Since in the final analysis we are only interested in the germ of $\mathcal{C}(M;p)$ along $\mathcal{C}^{\mathrm{hol}}(M,J)$, we may assume the centers $(C,\mathbf{x},\varphi)$ of our charts are J-holomorphic. The map φ is then smooth by elliptic regularity. So $D\varphi$ maps $\mathrm{Lie}\,\mathrm{Aut}(C,\mathbf{x})$ to a finite dimensional subspace in $\check{L}^p_1(C,\varphi^*T_M)$, to which we may choose a complementary subspace $\overline{V}$. However, in our setting, it still seems hard to prove that $S \times \overline{V}$ is in fact a slice.

2.4 Trivializing the relative tangent bundle

What is still missing is a structure of Banach bundle on

$$q^{1,p}_*(\kappa^*\varphi^*T_M) \;=\; \coprod_{s\in S} \check{L}^p_1(C_s;(\varphi\circ\kappa_s)^*T_M),$$

which should be viewed as the *tangent bundle* of $\coprod_s \check{L}^p_1(C_s;M)$ *relative* to S, restricted to $\{(C_s,\mathbf{x}_s,\varphi\circ\kappa_s)\,|\,s\in S\}$. This problem is at the heart of our approach to symplectic GW invariants. Our solution has three ingredients:

(1) For any $\varphi \in \check{L}^p_1(C,M)$ there is a natural structure of *holomorphic* vector bundle on the complex vector bundle $(\varphi^*T_M,\varphi^*J)$, even though φ is only L^p_1. In particular, we get a φ^*J-linear first order linear differential operator

$$\overline{\partial}^{\varphi}_J\colon\ \check{L}^p_1(C,\varphi^*T_M) \longrightarrow \check{L}^p(C,\varphi^*T_M\otimes_{\mathbb{C}}\overline{\Omega}).$$

Here $\overline{\Omega} = \Lambda^{0,1}$ is a bundle only away from C_{sing}, and the right-hand side is defined using frames of the form $\mathrm{d}\overline{z}$ on a branch of C near $P \in C_{\mathrm{sing}}$, where z is a holomorphic coordinate of this branch at P.[4]

[4]Equivalently, one may use the relative dualizing bundle $\omega_{C/S}$, which is more natural from the point of view of algebraic geometry. Local frames near a singularity of C are now of the form $\mathrm{d}z/z$, which requires insertion of the p-dependent weight $|z|^p$ in the definition of the measure μ near C_{sing}.

(2) Prove the Poincaré Lemma for the above $\overline{\partial}$ operator. Then the sequence of coherent sheaves on C

$$0 \to \mathcal{O}(\varphi^* T_M) \longrightarrow \check{\mathcal{L}}_1^p(\varphi^* T_M) \xrightarrow{\overline{\partial}_J^\varphi} \check{\mathcal{L}}^p(\varphi^* T_M \otimes \overline{\Omega}) \to 0 \qquad (*)$$

is exact (this is well known for smooth C).

(3) Use (2), plus the trivialization of $\check{L}^p$-spaces via pullback by κ, plus a Čech construction for the holomorphic part to exhibit the Banach bundle structure on $q_*^{1,p}(\kappa^* \varphi^* T_M)$.

Informally speaking, the $\overline{\partial}$ operator is used to reduce the nonholomorphic part to the simple case of $\check{L}^p$-spaces, while the holomorphic part is taken care of by a Čech construction. We should remark that this kind of argument does not work for $\check{L}_k^p$ with $k > 1$ because only a subspace of $\check{L}_k^p$ is mapped to $\check{L}_{k-1}^p$ by $\overline{\partial}$. This is due to the continuity at the node imposed on sections of the sheaf $\check{\mathcal{L}}_{k-1}^p(\varphi^* T_M \otimes \overline{\Omega})$. Restricting to this subspace would mean a higher tangency condition of the two branches at the nodes which is not what we want in the application to J-holomorphic curves. This dictates the choice of L_1^p as modelling space in our approach.

The rest of this section is devoted to filling in the details of the above steps.

Holomorphic structure on $\varphi^* T_M$

The logic here is actually the opposite of that presented above; namely, the operator $\overline{\partial}_J^\varphi$ is constructed first. There are various ways to do this, but at J-holomorphic φ the choice should reduce to the $\varphi^* J$-linear part of the linearization of the $\overline{\partial}_J$ operator (which is independent of choices of local trivialization). Letting ∇ be the Levi-Civita connection on M with respect to some fixed Riemannian metric, $\nabla^\varphi = \varphi^* \nabla$ the induced connection on $\varphi^* T_M$, our choice for $(\overline{\partial}_J^\varphi)_\xi v$ is the $\varphi^* J$-linear part of

$$\frac{1}{2}\Big(\nabla_\xi^\varphi v + J \nabla_{j(\xi)}^\varphi v + (\nabla_v J) \partial_J \varphi(j(\xi))\Big),$$

for $\xi \in \Gamma(T_C)$ and $v \in \check{L}_1^p(C, \varphi^* T_M)$; here $\partial_J \varphi := \frac{1}{2}(D\varphi - J \circ D\varphi \circ j)$ and j is the complex structure on C. Note that since we assumed φ to be only of class $\check{L}_1^p$, this expression does not make sense pointwise, but only as $\check{L}^p$-section of $\varphi^* T_M$, itself only a complex vector bundle of class $\check{L}_1^p$. Nevertheless, an application of the implicit function theorem shows that local solutions of $\overline{\partial}_J^\varphi$ define a locally free coherent sheaf on C, i.e., induce the structure of a holomorphic vector bundle on $\varphi^* T_M$, cf. [HLiSk] and [IvSh], Lemma 6.1.1.

Poincaré Lemma for weighted Sobolev spaces

Away from the singularities of C, exactness of the stated sequence of sheaves is well known. What is left at a node is to prove surjectivity of the restriction of the ordinary $\overline{\partial}$ operator to each branch in nonstandard Sobolev spaces:

$$\overline{\partial}\colon \check{L}^p_1(\Delta) \to \check{L}^p(\Delta).$$

These spaces can be identified with Sobolev spaces on a half-infinite cylinder with exponential weights $e^{-\mu s}$ by the identification

$$\Delta^* \longrightarrow \mathbb{R}_{>0} \times S^1, \quad re^{i\varphi} \longmapsto (s, \psi) = (-\log r, \varphi),$$

under which the $\overline{\partial}$ operator transforms to an operator of the form $e^{-s} \cdot (\partial_s + i\partial_\psi)$. For such linear elliptic differential operators on manifolds with cylindrical ends ($\mathbb{R}_{>0} \times N$ with compact N) there does exist a general theory, which implies the result we need [LcMc].

Alternatively, and maybe even more enlightening than invoking general theory, one may employ the explicit right inverse to the $\overline{\partial}$ operator on the disk given by the Cauchy integral operator

$$T(g\mathrm{d}\overline{z})(z) \;=\; \frac{1}{2\pi i}\int_\Delta \frac{g(w)}{w-z}\mathrm{d}w \wedge \mathrm{d}\overline{w}.$$

To show that T does indeed map $\check{L}^p$ to $\check{L}^p_1$, one just needs to estimate $\partial \circ T$. The latter equals a singular integral operator

$$S(g\mathrm{d}\overline{z})(z) \;=\; \frac{1}{2\pi i}\left(\lim_{\varepsilon\to 0}\int_{\Delta\setminus B_\varepsilon(z)} \frac{g(w)}{(w-z)^2}\mathrm{d}w \wedge \mathrm{d}\overline{w}\right)\mathrm{d}z.$$

The classical Calderon–Zygmund inequality says that S is a continuous endomorphism of $L^p(\Delta)$. The corresponding statement for the relevant case with weights can be found in [CfFe].

The Čech construction

Because $(*)$ is a soft resolution of $\mathcal{O}(\varphi^*T_M)$, the long exact cohomology sequence reads

$$0 \to H^0(\varphi^*T_M) \longrightarrow \check{L}^p_1(\varphi^*T_M) \xrightarrow{\overline{\partial}^\varphi_J} \check{L}^p(\varphi^*T_M \otimes \overline{\Omega}) \longrightarrow H^1(\varphi^*T_M) \to 0,$$

where all sections are understood over the domain C of φ. Now let $\mathcal{U} = \{U_i\}_{i=0,\dots,d}$ be a finite open covering of C with the properties:

(1) U_0 has components conformally equivalent to the unit disk minus a number of pairwise disjoint closed disks in its interior (so is an open Riemann surface of genus 0);

(2) each U_i for $i > 0$ is conformally equivalent to (possibly degenerate) cylinders $Z_t = \{(z,w) \in \Delta^2 \mid zw = t\}$, and such that $U_i \cap U_j \cap U_k = \emptyset$ for any three pairwise different indices i, j, k.

Write $C^i(\mathcal{U}; \mathcal{O}(\varphi^* T_M))$ for the ith (alternating) Čech cochains of holomorphic sections extending continuously to the boundary. These are Banach spaces under the sup norm. Since $\mathcal{U}$ is a Stein (hence acyclic) cover, there is a similar sequence

$$0 \to H^0(\varphi^* T_M) \to C^0(\mathcal{U}; \mathcal{O}(\varphi^* T_M)) \xrightarrow{\check{d}} C^1(\mathcal{U}; \mathcal{O}(\varphi^* T_M)) \to H^1(\varphi^* T_M) \to 0,$$

where $\check{d}$ is the Čech coboundary operator. Note that $C^1(\mathcal{U}; \mathcal{O}(\varphi^* T_M))$ consists of cochains rather than cocycles, because by our choice of $\mathcal{U}$, triple intersections are empty. To find an explicit quasi-isomorphism between the two middle arrows $\overline{\partial}_J^\varphi$ and $\check{d}$, we just need to go through standard constructions of cohomology theory: define maps

$$\Theta \colon \check{L}_1^p(\varphi^* T_M) \to C^0(\mathcal{U}; \mathcal{O}(\varphi^* T_M)) \quad \text{by} \quad f \mapsto \Big(f|_{U_i} - T^i(\overline{\partial} f|_{U_i}) \Big)_i$$

and

$$\Lambda \colon \check{L}^p(\varphi^* T_M \otimes \overline{\Omega}) \to C^1(\mathcal{U}; \mathcal{O}(\varphi^* T_M)) \quad \text{by} \quad \alpha \mapsto \Big(T^j(\alpha|_{U_j}) - T^i(\alpha|_{U_i}) \Big)_{ij}.$$

where $T^i \colon \check{L}^p(U_i; \varphi^* T_M \otimes \overline{\Omega}) \to \check{L}_1^p(U_i; \varphi^* T_M)$ is a right inverse of the above $\overline{\partial}_J^\varphi$. One can show that Θ and Λ induce isomorphisms on kernels and cokernels of $\overline{\partial}_J^\varphi$ and $\check{d}$. This is equivalent to exactness of the associated mapping cone

$$0 \to \check{L}_1^p(\varphi^* T_M) \xrightarrow{\binom{\overline{\partial}_J^\varphi}{\Theta}} \begin{matrix} \check{L}^p(\varphi^* T_M \otimes \overline{\Omega}) \\ \oplus \\ C^0(\mathcal{U}; \mathcal{O}(\varphi^* T_M)) \end{matrix} \xrightarrow{(\Lambda, -\check{d})} C^1(\mathcal{U}; \mathcal{O}(\varphi^* T_M)) \to 0.$$

We have thus exhibited the $\check{L}_1^p$-spaces as kernels of epimorphisms of Banach spaces that we have good hope to trivialize in families. Alternatively, since all maps have right inverses, one may use a similar sequence with the arrows reversed, cf. [Si1].

Introducing parameters

So far we have discussed the situation at a fixed curve $(C, \mathbf{x}, \varphi)$. For the purpose of producing charts we wanted to identify $\check{L}_1^p(C_s; (\varphi \circ \kappa_s)^* T_M)$ with $\check{L}_1^p(C, \varphi^* T_M)$, where $q \colon \mathcal{C} \to S$ (together with $\underline{\mathbf{x}} \colon S \to \mathcal{C} \times_S \cdots \times_S \mathcal{C}$, which is not of interest here) is a semiuniversal deformation of $(C, \mathbf{x})$, and $\kappa \colon \mathcal{C} \to C$ is a Lipschitz retraction to the central fiber as in 2.1, $C_s = q^{-1}(s)$, $\kappa_s = \kappa|_{C_s}$.

Using κ_s we may identify $\check{L}^p(C, \varphi^* T_M \otimes \overline{\Omega})$ with $\check{L}^p(C_s; (\varphi \circ \kappa_s)^* T_M \otimes \overline{\Omega})$. In view of (a parametrized version of) the exact sequence of Banach spaces in the last paragraph, it remains to trivialize spaces of Čech cochains. We choose an open covering $\mathcal{U} = \{U_i\}_{i=0,\ldots,d}$ of the total space $\mathcal{C}$ in such a way that on $\overline{U_i}$ there are holomorphic functions z, w identifying $U_i(s) := U_i \cap C_s$, $i > 0$, with possibly degenerate cylinders $Z_{t_i(s)}$, $t_i \in \mathcal{O}(S)$, while on $\overline{U_0}$ there is just one holomorphic function z identifying $U_0(s)$ with a union of plane open sets as above. (Holomorphic *relative* to S would suffice for what follows.) Together with continuously varying holomorphic trivializations of $(\varphi \circ \kappa)^* T_M|_{U_i(s)}$ we are left to find isomorphisms between $\mathcal{O}(Z_t) \cap C^0(\overline{Z_t})$ for different t. This can be done by observing that these spaces are given (up to constants) by positive Fourier series $\sum_{n>0} a_n e^{in\varphi}$ on the two boundary circles $|z| = 1$ or $|w| = 1$ via

$$(a_n, b_n) \longmapsto \sum_{n>0} a_n z^n + \sum_{n>0} b_n w^n.$$

A similar method works for U_0.

2.5 Summary

We summarize our discussion in the following

Theorem 2.4 *Let (M, J) be an almost complex manifold. Then the space $\mathcal{C}(M; p)$ of stable complex curves in M of Sobolev class $\check{L}^p_1$ has the structure of a weakly differentiable Banach orbifold with local group* $\mathrm{Aut}(C, \mathbf{x}, \varphi)$ *at* $(C, \mathbf{x}, \varphi)$.

Moreover, $\mathcal{C}(M; p)$ has a weakly differentiable Banach orbibundle $\mathcal{E}$ with fibers $\mathcal{E}_{(C,\mathbf{x},\varphi)}$ uniformized by $\widehat{\mathcal{E}}_{(C,\mathbf{x},\varphi)} = \check{L}^p(C, \varphi^* T_M \otimes \overline{\Omega})$. $\mathcal{E}$ has a weakly differentiable orbibundle section $s_{\bar\partial,J}$ sending $(C, \mathbf{x}, \varphi)$ to $\bar\partial_J \varphi$. Its zero locus $Z(s_{\bar\partial,J})$ is the space $\mathcal{C}^{\mathrm{hol}}(M, J)$ of stable J-holomorphic curves with the Gromov topology. $\square$

Here "weak differentiability" means that the differentiable structure is well defined locally only relative to some finite dimensional space. The differentiability properties of the section $s_{\bar\partial,J}$ will be discussed further in 3.1. The construction of $\mathcal{E}$ is straightforward.

3 Construction of the virtual fundamental class

In this chapter we outline our construction of the virtual fundamental class along the lines of Chapter 1 inside the ambient space of Chapter 2.

3.1 Local transversality

We first show how to solve the problem locally, i.e., how to construct a manifold $\widetilde{Z}$ containing a neighbourhood of $(C, \mathbf{x}, \varphi)$ in $\mathcal{C}^{\mathrm{hol}}(M, J)$ as the zero set of a map to a finite dimensional space. In view of the analogy with the construction of germs of moduli spaces of complex manifolds by Kuranishi [Ku], this data is often called a *Kuranishi model* (here for $\mathcal{C}^{\mathrm{hol}}(M, J)$ at $(C, \mathbf{x}, \varphi)$). Since the construction relies on the implicit function theorem we must now discuss the regularity properties of $s_{\overline{\partial},J}$.

Recall that charts at $(C, \mathbf{x}, \varphi)$ are of the form $S \times \overline{V} \hookrightarrow S \times \check{L}_1^p(C, \varphi^* T_M)$, where $\overline{V}$ is of finite codimension in $\check{L}_1^p(C, \varphi^* T_M)$, of positive codimension whenever $(C, \mathbf{x}, \varphi)$ has bubbles. Fixing s means fixing the domain $(C, \mathbf{x})$, which implies differentiability of all objects involved, including the local uniformizers $\widehat{\mathcal{E}}$ of $\mathcal{E}$ and $\widehat{s}_{\overline{\partial},J}$ of $s_{\overline{\partial},J}$. However, due to the phenomenon discussed in Section 2.2, one cannot even expect the differential σ of $\widehat{s}_{\overline{\partial},J}$ *relative* to S to be uniformly continuous. But since we used the $\varphi^* J$-linear part of σ as $\overline{\partial}$ operator to trivialize $q_*^{1,p}(\kappa^* \varphi^* T_M)$, σ turns out to be uniformly continuous at the center of our charts. This is just enough to apply the implicit function theorem in a version relative to S.

Now let $(C, \mathbf{x}, \varphi)$ be J-holomorphic and let σ_0 be the differential of $\widehat{s}_{\overline{\partial},J}$ relative to S at $(0, 0) \in S \times \overline{V}$. Then

$$\sigma_0(w) \;=\; \overline{\partial}_J^{\varphi} w + \varphi^* N_J(w, D\varphi) \quad \text{for } w \in T_0 \overline{V} \subset \check{L}_1^p(C, \varphi^* T_M).$$

So possibly up to a zeroth order term and restriction to a subspace of finite codimension in $\check{L}_1^p(C, \varphi^* T_M)$, σ_0 is just the $\overline{\partial}$ operator on $\varphi^* T_M$. By the results of Section 2.4, the latter is a Fredholm operator to $\check{L}^p(C, \varphi^* T_M \otimes \overline{\Omega})$, and so is σ_0. Moreover, it is not hard to see that when restricted to sufficiently small neighbourhoods of C_{sing}, the corresponding operators are surjective. Therefore we may choose $\alpha_1, \ldots, \alpha_c \in \check{L}^p(C, \varphi^* T_M \otimes \overline{\Omega})$ supported away from C_{sing}, $c = \dim \operatorname{coker} \sigma_0$, such that $\operatorname{im} \sigma_0 + \sum_i \mathbb{C}\alpha_i \;=\; \check{L}^p(C, \varphi^* T_M)$. We say that the α_i *span* $\operatorname{coker} \sigma_0$. Define a morphism τ from a trivial bundle $F = \underline{\mathbb{R}}^c$ over $S \times \overline{V}$ to $\widehat{\mathcal{E}}$ by sending the ith standard section to the parallel transport of α_i. Then an application of the implicit function theorem relative to S to the section $\widetilde{s} := q^* s + \tau$ of $q^* \mathcal{E}$ ($q \colon F \to S \times \overline{V}$ the bundle projection), viewed as map from $S \times \overline{V} \times \mathbb{R}^c$ to $\widehat{\mathcal{E}}_{(C,\mathbf{x},\varphi)} = \check{L}^p(C, \varphi^* T_M \otimes \overline{\Omega})$, shows that $\widetilde{Z} = Z(\widetilde{s})$ is a topological submanifold of F of expected dimension plus $\operatorname{rank} F$. The restriction of $q^* F$ to $\widetilde{Z}$ has a tautological section s_{can} (mapping $f \in F$ to f). A germ of $\mathcal{C}^{\mathrm{hol}}(M, J)$ at $(C, \mathbf{x}, \varphi)$ is given by the zero locus of s_{can}.

If $(C, \mathbf{x}, \varphi)$ has nontrivial automorphisms we would like to make the Kuranishi model $\operatorname{Aut}(C, \mathbf{x}, \varphi)$-equivariant. Since it is not always possible to span $\operatorname{coker} \sigma_0$ by $\operatorname{Aut}(C, \mathbf{x}, \varphi)$-invariant sections (this is the notorious obstruction to transversality under the presence of multiply covered components) this inevitably forces a nontrivial action of $G = \mathrm{Aut}(C, \mathbf{x}, \varphi)$ on the fibers of F. The easiest way to make τ equivariant is then to replace F by F^G ($\#G$ copies of F) and define $\tau^G \colon F^G \to \widehat{\mathcal{E}}$ on the Ψth copy of F, $\Psi \in \mathrm{Aut}(C, \mathbf{x}, \varphi)$, by $(\Psi^{-1})^* \circ \tau$.

Choosing α to have support away from C_{sing} will be convenient in going over to other charts that we will need below.

3.2 Globalization

To globalize we would like:

- to extend F^G to an orbibundle over $\mathcal{C}(M;p)$;
- to extend τ by multiplication with a bump function that is differentiable relative to S in *any* chart $S \times \overline{V}$.

Neither of these problems is immediate. On Banach orbifolds the existence of finite rank orbibundles with effective actions of the local groups on the fibers, say on a neighbourhood of a compact set, seems to be a nontrivial condition. The general solution to this question given in a previous version of [Si1] is insufficient, because the cocycle condition cannot be verified. Fortunately, such orbibundles do exist on $\mathcal{C}(M;p)$ by a method similar to that given in [Be1], Prop. 5.

To this end, we now assume J tamed by some symplectic form ω. By slightly deforming ω and taking a large multiple, we may assume ω to represent an integral de Rham class. Then there exists a $\mathrm{U}(1)$-bundle L over M with $[\omega] = c_1(L)$. L is a substitute for an ample line bundle in the algebraic setting. Let ∇ be a $\mathrm{U}(1)$-connection on L. Let $\pi \colon \Gamma \to \mathcal{C}(M;p)$ be the universal curve and $\mathrm{ev} \colon \Gamma \to M$ the evaluation map sending $p \in C$ over $(C, \mathbf{x}, \varphi) \in \mathcal{C}(M;p)$ to $\varphi(p)$. Thus π is a morphism of topological orbifolds with fiber over $(C, \mathbf{x}, \varphi)$ equal to the complex analytic orbispace $C/\mathrm{Aut}(C, \mathbf{x}, \varphi)$. As in Section 2.4 one shows that via ∇, ev^*L has naturally the structure of a continuously varying family of holomorphic line bundles over the fibers of π. And since $[\omega]$ evaluates positively on any nonconstant J-holomorphic curve, φ^*L is ample on any bubbling component. To achieve ampleness on the other components we just need to tensor with $\omega_C(x_1 + \cdots + x_k)$, which is the sheaf of meromorphic 1-forms on C with at most simple poles at C_{sing} and the marked points x_i. These sheaves again fit into a continuously varying family of holomorphic line bundles $\omega_\pi(\underline{\mathbf{x}})$ over the fibers of π. Then $\mathrm{ev}^*L \otimes \omega_\pi(\underline{\mathbf{x}})$ is π-ample (i.e., ample on each fiber), hence a sufficiently large multiple (power $\geq N_0$, say) has vanishing H^1 on any fiber of π. Let $N > N_0$ be an even bigger natural number such that for any $\#\,\mathrm{Aut}(C, \mathbf{x}, \varphi)$ points on

C there exist a section of $(\varphi^L \otimes \omega_C(\mathbf{x}))^{\otimes N}$ vanishing at all but one point. We consider

$$\pi_*(\mathrm{ev}^* L \otimes \omega_\pi(\underline{\mathbf{x}}))^{\otimes N} \;:=\; \coprod_{(C,\mathbf{x},\varphi)\in\mathcal{C}(M;p)} \Gamma(C, \mathrm{ev}^* L^{\otimes N} \otimes \omega_\pi^{\otimes N}(\underline{\mathbf{x}})).$$

Using a Čech construction as in Section 2.4, one shows that this is locally uniformized by the kernel of a Fredholm epimorphism of Banach bundles (of Čech cocycles) over $\mathcal{C}(M;p)$ and hence glues to an orbibundle F of finite rank. Moreover, by the choice of N, for any section α of $\check{L}^p(C, \varphi^* T_M \otimes \overline{\Omega})$ with sufficiently small support there exists a vector v of the fiber of F at $(C, \mathbf{x}, \varphi)$ such that the dimensions of the linear subspace spanned by the $\mathrm{Aut}(C, \mathbf{x}, \varphi)$-orbits of v in F and of α in $\check{L}^p(C, \varphi^* T_M \otimes \overline{\Omega})$ coincide. Direct sums of bundles of this type allow us to extend F^G on a neighbourhood of $\mathcal{C}^{\mathrm{hol}}(M, J) \subset \mathcal{C}(M;p)$.

As for extending the morphism τ, one might try to use parallel transports of differentiable bump functions on $\check{L}^p_1(C, \varphi^* T_M)$, which do in fact exist provided p is even. This will be insufficient for our purposes though. The problem is that if we look at such bump functions constructed at a curve with bubbles (with deformation space S say) from a chart centered at a curve without bubbles (with deformation space $\overline{S}$ say) then, locally, S fibers over $\overline{S}$. Differentiability holds relative to S but will fail relative to $\overline{S}$. The way out is to take a "bump function" χ which only takes into account the behaviour on an open set $U \subset C \setminus C_{\mathrm{sing}}$. U has to be chosen in such a way that the coordinates of S ruling the deformations of nodes belonging to the bubbles do not influence the trivialization of $q_*^{1,p}(\kappa^* \varphi^* T_M)$ over U. χ will not have bounded support on $\mathcal{C}(M;p)$, but its restriction to $\mathcal{C}^{\mathrm{hol}}(M, J)$ does. This is enough for extending τ along a neighbourhood of $\mathcal{C}^{\mathrm{hol}}(M, J)$ in $\mathcal{C}(M;p)$. Note that by choosing $\operatorname{supp} \alpha_i$ inside U, τ will be differentiable even relative to $\mathcal{M}_{g,k}$ in any appropriately chosen coordinate chart.

3.3 The Main Theorem

Since we need compactness (and for the construction of F) we further assume J tamed by some symplectic form ω. Fix $R \in H_2(M, \mathbb{Z})$, g, k. Then $\mathcal{C}^{\mathrm{hol}}_{R,g,k}(M, J)$ is compact. The direct sum of finitely many morphisms to $\mathcal{E}$ as in 3.2 yields a morphism $\tau\colon F \to \mathcal{E}$ spanning the cokernels of the differentials of $s_{\overline{\partial},J}$ relative to S in any chart $S \times \overline{V}$ centered at J-holomorphic $(C, \mathbf{x}, \varphi)$. Thus $\widetilde{Z} = Z(\widetilde{s})$, where $\widetilde{s} = q^* s + \tau$, $q\colon F \to \mathcal{C}(M;p)$, is a finite dimensional (topological) suborbifold of the total space of F. It is also not hard to see that $\widetilde{Z}$ can be naturally oriented by complex linearity of $\overline{\partial}^{\varphi}_J$, provided that F is also oriented. The latter can be achieved by taking $F \oplus F$ if necessary (this is just a matter of convenience; what one needs is a relative orientation of q^*F over $\widetilde{Z}$). Let Θ_F be the Thom class of F. For the virtual fundamental class of $\mathcal{C}^{\mathrm{hol}}_{R,g,k}(M,J)$, we set

$$\mathcal{GW}^{M,J}_{R,g,k} \;:=\; [\widetilde{Z}] \cap \Theta_F \;\in\; H_{2d(M,R,g,k)}(\mathcal{C}^{\mathrm{hol}}_{R,g,k}(M,J)),$$

where $d(M,R,g,k) = \dim_{\mathbb{C}} \mathcal{M}_{g,k} + c_1(M,J)\cdot R + (1-g)\dim_{\mathbb{C}} M$ is computed by the Riemann–Roch theorem to be the index of $\overline{\partial}^{\varphi}_J$ plus $\dim \mathcal{M}_{g,k}$ (this needs to be corrected if $2g+k<3$).

Theorem 3.1 *The class $\mathcal{GW}^{M,J}_{R,g,k}$ is independent of the choices made. Its image in $H_*(\mathcal{C}(M;p))$ depends only on the symplectic deformation class of ω.* $\square$

Independence of choices (of τ, and of the Sobolev index p) is easy to establish. The second claim asserts independence under deformations of J inside the space of almost complex structures tamed by some symplectic form. To this end, one sets up a family version of the approach with fixed J, from which independence of the image in $\mathcal{C}(M;p)$ follows immediately. For details we refer to [Si1].

3.4 Alternative approaches

The purpose of this section is to discuss, in a rather sketchy way, various other approaches to the construction of virtual fundamental classes, as given by Fukaya and Ono [FkOn], Li and Tian [LiTi2] and Ruan [Ru2]. Still another definition can be extracted from a paper of Liu and Tian [LiuTi] on a solution to the closely related Arnol'd conjecture on nondegenerate exact symplectomorphisms (the latter is also covered in [FkOn] and [Ru2]).

Recall that in formulating our problem as that of constructing a localized Euler class of a section of a Banach orbibundle over a Banach orbifold we had to pay the following price:

(1) working in spaces of maps with very weak differentiability (this caused problems in the slice theorem, cf. 2.3),

(2) the loss of differentiability in a finite dimensional direction (which made the construction of τ more subtle, cf. 3.1), and

(3) having to construct a finite dimensional orbibundle F with effective actions of the local groups on the fibers, cf. 3.2.

But what we are finally interested in is the zero locus $\widetilde{Z} \subset F$ of a perturbed section $\widetilde{s} = q^*s + \tau$. As a set, $\widetilde{Z}$ consists of (isomorphism classes of) tuples

$(C, \mathbf{x}, \varphi, f)$ with $(C, \mathbf{x}, \varphi) \in \mathcal{C}(M; p)$ and $f \in F_{(C,\mathbf{x},\varphi)}$ such that $\overline{\partial}_J \varphi = \tau(f)$. Thus if the sections $\alpha \in \check{L}^p(C'; (\varphi')^* T_M \otimes \overline{\Omega})$ spanning the cokernel at various $(C', \mathbf{x}', \varphi') \in \mathcal{C}^{\mathrm{hol}}(M, J)$ used to construct τ, are chosen to be smooth, solutions φ of $\overline{\partial}_J \varphi = \tau(f)$ will also be smooth by elliptic regularity. That is, in constructing $\widetilde{Z}$ we may safely restrict to spaces of smooth maps. A common feature of the other approaches to GW theory is that they work in ambient spaces of C^∞ maps, and that $\widetilde{Z}$ is first constructed locally for any local, finite dimensional perturbation. The problem is then to find a global object that matches up the local perturbations.

The local construction of $\widetilde{Z}$ can be done by more or less straightforward modifications of the known glueing constructions for J-holomorphic curves in generic situations (i.e., when the linearization of the relevant Fredholm operator is already surjective), as given in [RuTi1], [Liu], [McSa]. "Glueing" means the following: given a nodal J-holomorphic curve $\varphi\colon C \to M$ and a family $\{C_s\}_{s\in S}$ of deformations of C as a pre-stable curve, one wants to deform φ to a family of J-holomorphic curves $\varphi_s\colon C_s \to M$. This is achieved by first constructing φ_s approximately by some kind of differentiable glueing construction involving bump functions. The $\overline{\partial}_J$ operators on the C_s set up a family of elliptic problems, albeit with varying Banach spaces (here: versions of L^p_1 and L^p). The basic analytic problem is to establish a uniform estimate on the norm of the inverse of the linearized problem. Here one has to assume that the linearization is invertible at $s = 0$, which is true for generic situations as in op. cit. The inverse of the linearized problem enters into effective versions of the implicit function theorem, which can then be applied to identify the solution set as a manifold. In the nongeneric case one can consider a perturbed problem by introducing abstract perturbation terms spanning the cokernel. A solution to the latter problem will yield an ambient smooth space into which the original solution set is embedded as the zero set of finitely many functions, i.e., a Kuranishi model for $\mathcal{C}^{\mathrm{hol}}(M, J)$ at $(C, \mathbf{x}, \varphi)$. Several choices of spaces, differentiable glueing and deformation of abstract perturbation terms are possible, cf. op. cit. Notice that if the perturbations are chosen to be smooth, then the solutions of the perturbed equation will also be smooth by elliptic regularity.

The problem of globalization of local transversality in this setting (in particular in the presence of local automorphisms) is new. This is where the approaches differ most.

1. Fukaya and Ono let the dimension of the perturbation space ($\operatorname{rank} F$ in our setting) and hence also the dimensions of the manifolds containing $Z = \mathcal{C}^{\mathrm{hol}}_{R,g,k}(M, J)$ locally ($\dim \widetilde{Z}$ in our setting) vary along a finite open cover of Z. The result is that Z is the zero locus of a section s' of a strange fiber space $F \to \widetilde{Z}$. Locally, the fiber space is a finite union of orbibundles of finite ranks over finite dimensional orbifolds fitting together nicely, but of jumping ranks and dimensions. The basic observation of [FkOn] is that while it is usually impossible to make orbifold sections transverse by perturbation, one may do so by going over to sufficiently high multivalued sections ("multisections"). These are sections of a symmetric product $S^l F$ that locally lift to $F^{\oplus l}$. And transversality means transversality of *each* component of a lift, i.e., of each branch of the multisection. The zero locus of a multisection is defined as the union of the zero loci of its branches. A generic perturbation of $\widetilde{s}$ will thus have a zero locus which locally is the finite union of (oriented) orbifolds of the expected dimension. The sums of the fundamental classes of these orbifolds, appropriately normalized, glue to a homology class on the base. The same works for sections of the strange fiber space $F \to \widetilde{Z}$. The homology class on $\widetilde{Z}$ thus obtained is the virtual fundamental class of Z. Note that if one insists on a class localized on Z one might take a limit of these classes as the perturbations tend to zero. But since the maps $\mathrm{ev}\colon Z \to M^k$ and $q\colon Z \to \mathcal{M}_{g,k}$ extend to $\widetilde{Z}$, this is not important for GW theory.

2. Li and Tian also describe Z as the zero locus of a section of a fiber space $F \to \widetilde{Z}$ with jumping dimensions as in (1). But instead of trying to perturb the section, they show how to glue cycles representing the Euler class and supported on $\widetilde{Z}$ directly.

3. Ruan works inside the stratified Fréchet orbifold of C^∞-stable complex curves in M. This is a topological space, but locally stratified into finitely many Fréchet orbifolds, depending on the combinatorial type of the curve. Nevertheless, by the glueing construction, it suffices to work within this space. The argument proceeds in an analogous way to Section 1.2, i.e., one constructs the perturbation as a morphism from a stratified orbibundle F of finite rank over the ambient space to a Banach bundle. Ruan claimed that one may take a trivial orbibundle of the form (base space) $\times$ $(\mathbb{R}^N/G)$, where G is the product of the local groups of finitely many Kuranishi models covering $Z = \mathcal{C}^{\mathrm{hol}}_{R,g,k}(M, J)$. This is not in general possible. The argument is however right if one takes a nontrivial orbibundle, e.g., as in Section 3.2.

4. Another method, due to Liu and Tian, uses a compatible system of perturbation terms in the following sense: Z can be covered by finitely many local uniformizers $\{V_I = \widehat{V}_I/\Gamma_I\}_I$, $I = \{i_1, \ldots, i_k\}$, $i_\nu \in \{1, \ldots, n\}$, $k \le m$, with

- $V_I \cap V_J = \emptyset$ if $\#I = \#J$ and $I \neq J$
- whenever $I \subset J$ there are morphisms $\pi_{IJ} : V_J = \widehat{V}_J/\Gamma_J \to V_I = \widehat{V}_I/\Gamma_I$

uniformizing open embeddings $V_J \subset V_I$, and these are compatible in the obvious way.

The $V_I = \widehat{V}_I/\Gamma_I$ are open sets in fibered products $\widehat{U}_{i_1} \times_{\mathcal{B}} \cdots \times_{\mathcal{B}} \widehat{U}_{i_k}$, and so are not smooth for $\#I > 1$, but finite unions of manifolds. Using the π_{IJ} one may compare perturbation terms over different V_I and thus define compatible systems $\{v_I\}_I$ of perturbation terms. For a generic choice of $\{v_I\}$ the zero loci of the perturbed section $\{(\widehat{s}_{\overline{\partial},J})_I - \nu_I\}$ form a compatible system of finite dimensional oriented orbifolds $\widehat{Z}_I^\nu \subset \widehat{V}_I$. This is enough to produce a homology class on the underlying space of the expected dimension of $Z^\nu \subset \mathcal{B}$.

While it might be somewhat tedious to do this in detail, it is rather obvious that all these definitions lead to the same homology class in an appropriate common ambient space, say $\mathcal{C}(M;p)$. In fact, in all these approaches one might take as Kuranishi model the restriction of our embedding $Z = Z(s_{\mathrm{can}}) \subset \widetilde{Z}$ to open sets, at least if we choose our perturbations α sufficiently smooth. The problem is then essentially reduced to comparing various constructions of Euler classes for orbibundles in finite dimensions.

4 Axioms for GW invariants

4.1 GW invariants

There are several ways of extracting symplectic invariants from the virtual fundamental classes $\mathcal{GW}^{M,J}_{R,g,k} \in H_*(\mathcal{C}^{\mathrm{hol}}_{R,g,k}(M,J))$. Assume that $2g+k \geq 3$. Then $\mathcal{M}_{g,k}$ exists as orbifold and, in particular, it satisfies rational Poincaré duality. There are diagrams

$$\begin{array}{ccc} \mathcal{C}^{\mathrm{hol}}_{R,g,k}(M,J) & \xrightarrow{\mathrm{ev}} & M^k \\ {\scriptstyle p}\downarrow & & \\ \mathcal{M}_{g,k} & & \end{array}$$

with ev and p the evaluation map and the forgetful map, sending $(C,\mathbf{x},\varphi)$ respectively to $(\varphi(x_1),\dots,\varphi(x_k))$ and to the stabilization of $(C,\mathbf{x})$. Note that both maps extend to $\mathcal{C}_{R,g,k}(M;p)$. By Theorem 3.1, we conclude the following:

Proposition 4.1 *The associated* GW correspondence

$$\begin{aligned} \mathrm{GW}^M_{R,g,k}\colon H^*(M)^{\otimes k} &\longrightarrow H_*(\mathcal{M}_{g,k}), \\ \alpha_1 \otimes \cdots \otimes \alpha_k &\longmapsto p_*\Big(\mathcal{GW}^{M,J}_{R,g,k} \cap ev^*(\alpha_1 \times \cdots \times \alpha_k)\Big) \end{aligned}$$

is invariant under deformations of J inside the space of almost complex structures tamed by some symplectic form. In particular, $\mathrm{GW}^M_{R,g,k}$ *is an invariant of the symplectic deformation type of (M,ω).* $\square$

The following equivalent objects also occur commonly:

- composition of $\mathrm{GW}^M_{R,g,k}$ with Poincaré duality $H_*(\mathcal{M}_{g,k}) \to H^*(\mathcal{M}_{g,k})$;
- the associated homomorphism $H_*(\mathcal{M}_{g,k}) \otimes H_*(M)^k \to \mathbb{Q}$;
- the cycle $(p \times \mathrm{ev})_* \mathcal{GW}^M_{R,g,k} \in H_*(\mathcal{M}_{g,k} \times M^k)$.

Of these the second one is perhaps the most intuitive. For cycles $K \subset \mathcal{M}_{g,k}$, $A_1, \ldots, A_k \subset M$, it counts the "ideal" number of k-marked stable J-holomorphic curves $(C, \mathbf{x}, \varphi)$ in M of genus g with $(C, \mathbf{x})^{\mathrm{st}} \in K$ and the ith point mapping to A_i. "Ideal" means that this agrees with the actual (signed) number only in nice situations, say when $\mathcal{C}^{\mathrm{hol}}_{R,g,k}(M, J)$ is indeed an orbifold of the expected dimension which is transversal to $K \times A_1 \times \cdots \times A_k$ under $p \times \mathrm{ev}$. I prefer to reserve the name *GW invariant* for such numbers, i.e., by applying the second map to a product of cycles.

As already pointed out in [RuTi1], Remark 7.1 one can also define invariants by restricting the domain to certain singular curves and requiring homological conditions for the restriction of the maps to subcurves. The full perspective of this point of view has been given in [BeMa], where *marked modular graphs* τ are introduced as a bookkeeping device for the combinatorial data, cf. [Be2], Definition 1.2 (we adopt the abuse of notation and use τ both for the marked modular graph and the associated stable modular graph, i.e., with the marking omitted). If τ has n vertices and l edges, the corresponding moduli spaces $\mathcal{C}_\tau(M) = \mathcal{C}^{\mathrm{hol}}_\tau(M, J)$ (which corresponds to $\overline{M}(M, \tau)$ in [Be2]) are constructed as the fiber over a product of diagonals $\Delta^l \subset M^{2l}$ of a partial evaluation map

$$\mathrm{pev}\colon \prod_{i=1}^n \mathcal{C}^{\mathrm{hol}}_{R_i, g_i, k_i}(M, J) \longrightarrow M^{\Sigma k_i} \longrightarrow M^{2l}.$$

The meaning of this is that any edge of τ implements the requirement that the two marked points of the subcurves (= vertices of τ) bounding the edge, map to the same point in M. We refer to [Be2] for details of this concept.

To define virtual fundamental classes on these more general moduli spaces, let $\delta_{\Delta^l} \in H^*_{\Delta^l}(M^{2l})$ be the Poincaré dual class of Δ^l. We may then set

$$\mathcal{GW}^M_\tau := \Big(\prod_i \mathcal{GW}^M_{R_i, g_i, k_i} \Big) \cap \mathrm{pev}^* \delta_{\Delta^l} \in H_*(\mathcal{C}_\tau(M))$$

as the virtual fundamental class of $\mathcal{C}_\tau(M)$ (this corresponds to $J(M, \tau)$ in [Be2]). As above, we get an associated GW correspondence

$$\mathrm{GW}^M_\tau\colon H^*(M)^{\otimes \# S_\tau} \longrightarrow H_*\Big(\mathcal{M}_\tau := \prod_i \mathcal{M}_{g_i, k_i}\Big),$$

where S_τ is the set of tails of τ (which encode the positions of marked points).

4.2 Properties

From the intuitive geometric meaning one expects GW invariants to have a number of properties. These turned up as proved identities for a restricted class of varieties [RuTi1], Theorem A and Prop. 2.5, and in [RuTi3], or as axioms for $GW^M_{R,g,k}$ in [KoMa]. The corresponding axioms for the system of GW correspondences parametrized by marked modular graphs are given in [BeMa], cf. also [Be2]. As the presentation in the last reference is quite appropriate we just indicate what is to be added to establish Axioms I–V in op. cit. in the symplectic context. For statement and geometric explanation of the axioms we mostly refer to op. cit.

One should probably add to the axioms the important property of invariance under deformations of the (tamed) almost complex structure (respectively, under smooth projective deformations in the algebraic setting) that we have already commented on.

I. Mapping to point This is the case $R = 0$. Since by the Wirtinger inequality, $\omega(\varphi_*[C]) = 0$ for a connected J-holomorphic curve $\varphi\colon C \to M$ implies $\varphi \equiv \text{const}$, we get

$$\mathcal{C}^{\text{hol}}_{0,g,k} \;=\; \mathcal{M}_{g,k} \times M$$

with universal curve $\pi = p \times \text{Id}\colon \mathcal{M}_{g,k+1} \times M \to \mathcal{M}_{g,k} \times M$. This is an orbifold, but possibly of the wrong dimension. In fact, the cokernels of the linearization of $s_{\overline{\partial},J}$ glue to $R^1\pi_*\text{ev}^*T_M = R^1p_*\mathcal{O} \boxtimes T_M$ (that we view as orbibundle, rather than its orbisheaf of sections). We set $F = R^1\widetilde{\pi}_*\widetilde{\text{ev}}^*T_M$ and define $\tau\colon F \to \mathcal{E} = \widetilde{\pi}^p_*(\widetilde{\text{ev}}^*T_M \otimes \overline{\Omega})$ in such a way that it restricts to a lift of this identification. For clarity, this time we write $\widetilde{\pi}$, $\widetilde{\text{ev}}$ for the extensions of π, ev to $\mathcal{C}_{0,g,k}(M;p)$. Then $\widetilde{Z} = Z(\widetilde{s} = q^*s + \tau) \subset F$ is nothing other than $\mathcal{C}^{\text{hol}}_{0,g,k}(M, J)$, and $[\widetilde{Z}] \cap \Theta_F$ computes the Euler class of $R^1\pi_*\text{ev}^*T_M = R^1p_*\mathcal{O} \boxtimes T_M$ as claimed in the mapping to point axiom.

II. Products This axiom forced our definition of the virtual fundamental class for nonconnected τ.

III. Glueing tails/cutting edges Again, this axiom follows directly from our definition of virtual fundamental classes, now for connected components of τ with more than one vertex.

IV. Forgetting tails Forgetting tails in a marked modular graph means omitting marked points from a stable complex curve and stabilizing (as a complex curve in M). Let us restrict to $\tau = (R, g, k)$, from which the general case follows easily. In view of the analogous fact for $\mathcal{M}_{g,k}$ and our construction of charts for $\mathcal{C}_{R,g,k}(M;p)$, it is easy to check that the corresponding map

$$\Phi\colon \mathcal{C}_{R,g,k+1}(M;p) \longrightarrow \mathcal{C}_{R,g,k}(M;p)$$

is the universal curve. If $\mathcal{E}_k$ and $s_k = s_{\overline{\partial},J}$ are the Banach bundle and section over $\mathcal{C}_{R,g,k}(M;p)$, then $\Phi^*\mathcal{E}_k$, $\Phi^* s_k$ can be identified with the bundle and section $\mathcal{E}_{k+1}$, s_{k+1} over $\mathcal{C}_{R,g,k+1}(M;p)$. Let $\tau\colon F \to \mathcal{E}_k$ span the cokernel of the (relative) linearization of s_k. Then $\Phi^*\tau$ will span the cokernel of the (relative) linearization of s_{k+1}. We obtain

$$\widetilde{Z}_{k+1} \;=\; Z(\widetilde{s}_{k+1} = \Phi^*\widetilde{s}_k) \;=\; \Phi^{-1}\Big(Z(\widetilde{s}_k = q^* s_k + \tau)\Big) \;=\; \Phi^{-1}(\widetilde{Z}_k),$$

and hence

$$\mathcal{GW}^M_{R,g,k+1} \;=\; [\widetilde{Z}_{k+1}] \cap \Theta_{\Phi^*F} \;=\; \Phi^!\Big([\widetilde{Z}_k] \cap \Theta_F\Big) \;=\; \Phi^! \mathcal{GW}^M_{R,g,k}.$$

V. Isogenies Among the axioms this is the most interesting, having as consequence for instance the associativity of quantum products. The axiom comprises those modifications of marked modular graphs that do not change its genus. There are four basic cases:

(1) Contraction of a loop: omitting a loop, i.e., an edge connecting a vertex with itself, from a modular graph corresponds to dropping the requirement that a certain subcurve has a nondisconnecting double point. In a sense this case says something about potential smoothings of such double points of the domain.

(2) Contraction of a nonlooping edge: nonlooping edges correspond to disconnecting double points of the curve. Contraction of such an edge means that we consider two adjacent subcurves of genera g_1, g_2 as one subcurve of genus $g_1 + g_2$. So here we deal with potential smoothings of disconnecting double points.

(3) Forgetting a tail: as in axiom IV, but the conclusion is different.

(4) Relabelling: this treats isomorphisms of marked modular graphs, which in particular covers renumberings of the set of marked points.

Let τ be the marked modular graph obtained from σ by any of the operations (1–4). There is an embedding $\mathcal{C}_\sigma(M) \hookrightarrow \mathcal{C}_\tau(M)$ over the closed embedding of moduli spaces of curves $\mathcal{M}_\sigma \hookrightarrow \mathcal{M}_\tau$. The latter is divisorial in the first three cases and an isomorphism in the last case. Except possibly in (2), the choice of τ and the underlying modular graph of σ determine the marking of σ and the diagram

$$\begin{array}{ccc} \mathcal{C}_\sigma(M) & \longrightarrow & \mathcal{C}_\tau(M) \\ {\scriptstyle q_\sigma}\downarrow & & \downarrow{\scriptstyle q_\tau} \\ \mathcal{M}_\sigma & \longrightarrow & \mathcal{M}_\tau \end{array}$$

is Cartesian. Let $\delta_{\mathcal{M}_\sigma} \in H^*_{\mathcal{M}_\sigma}(\mathcal{M}_\tau)$ be Poincaré dual to $\mathcal{M}_\sigma$. The axiom can then be stated by requiring

$$\mathcal{GW}^M_\sigma \;=\; \mathcal{GW}^M_\tau \cap q_\tau^* \delta_{\mathcal{M}_\sigma}.$$

In case (2) the homology class R of the joined subcurve of τ can be arbitrarily distributed to the two adjacent subcurves of σ. We get a proper surjection

$$h\colon \coprod_{R=R_1+R_2} \mathcal{C}_{\sigma=\sigma(R_1,R_2)}(M) \;\longrightarrow\; q_\tau^{-1}(\mathcal{M}_\sigma) \;=\; \mathcal{M}_\sigma \times_{\mathcal{M}_\tau} \mathcal{C}_\tau(M).$$

Note that h is not injective if there are J-holomorphic curves with bubbles inserted at the double points. The claim is

$$\sum_{R=R_1+R_2} h_*\Big(\mathcal{GW}^M_\sigma\Big) \;=\; \mathcal{GW}^M_\tau \cap q_\tau^* \delta_{\mathcal{M}_\sigma}.$$

Except in the obvious case (4), the proof runs as follows. We again restrict to the basic case $\tau = (R, g, k)$. The embedding $\mathcal{M}_\sigma \hookrightarrow \mathcal{M}_\tau$ identifies $\mathcal{M}_\sigma$ with a divisor parametrizing singular curves or curves with two infinitely near marked points. By the form of our charts it is not hard to see that the Kuranishi space $\widetilde{Z}_\tau \subset F$ for $\mathcal{C}_\tau(M)$ intersected with $q_\tau^{-1}(\mathcal{M}_\sigma)$ is a union of suborbifolds $\widetilde{Z}_\sigma$ that can be identified with Kuranishi spaces for the components of $q_\tau^{-1}(\mathcal{M}_\sigma)$. Capping with the Thom class of F yields the result.

5 Comparison with algebraic GW invariants

For a smooth complex projective variety $M \subset \mathbb{P}^N$ we have now two definitions of virtual fundamental classes fulfilling the list of axioms plus deformation invariance: the symplectic ones $\mathcal{GW}^M_\sigma$ discussed so far (where $J = I$ is the integrable complex structure tamed by the Fubini–Study form), and the algebraic ones $J(M, \sigma)$ discussed in [Be2]. The latter are taken here in $H_*(\mathcal{C}_\sigma(M))$ by sending the analogous Chow class in the Deligne–Mumford stack $\overline{M}(M, \tau)$ to its homology class on the underlying complex space. It is natural to expect

Theorem 5.1 [Si4] *For any marked modular graph* σ

$$\mathcal{GW}^M_\sigma = J(M,\sigma). \quad \square$$

It is of course enough to treat the case $\sigma = (R, g, k)$. To compare the two definitions it is most convenient to work in the category of complex orbispaces, which are defined analogously to complex orbifolds, but with local models taken as finite group quotients of complex spaces (the underlying space will also be a complex space, but we want to keep in mind the group actions).

We first present an argument that does not work as stated, but where the basic reason for this equivalence is clear, and then outline the actual proof.

5.1 A model argument

Let us pretend that we can find $\tau\colon F \to \mathcal{E}$ such that

- $\widetilde{Z} = Z(\widetilde{s} = q^*s + \tau) \subset F$ is a *complex* suborbifold and $\widetilde{F} = q^*F|_{\widetilde{Z}}$ is a *holomorphic* vector bundle with *holomorphic* tautological section $s_{\mathrm{can}}\colon \widetilde{Z} \to \widetilde{F}$.
- The induced structure of complex orbispace on $\mathcal{C}_\sigma(M) = Z = Z(s_{\mathrm{can}})$ is the right one (coming from the notion of holomorphic families of stable holomorphic curves in M).

According to [Fu], §14.1, the Euler class of $\widetilde{F}$ can be expressed in terms of the normal cone $C_{Z|\tilde{Z}}$ of Z in $\widetilde{Z}$ and the total Chern class of $\widetilde{F}|_Z$ by

$$\left\{c(\widetilde{F}) \cap s(C_{Z|\tilde{Z}})\right\}_d.$$

Here $d = \dim \widetilde{Z} - \operatorname{rank} \widetilde{F} = d(M, R, g, k)$ is the expected dimension of Z and $s(C_{Z|\tilde{Z}})$ is the Segre class of $C_{Z|\tilde{Z}}$. By construction, $\widetilde{Z}$ is smooth over the Artin stack $\mathfrak{F}_{g,k}$ of pre-stable curves (we should work with the analytic analogue here). Let $T_{\tilde{Z}|\mathfrak{F}_{g,k}}$ be the relative tangent bundle, which is in fact an ordinary vector bundle over $\widetilde{Z}$, cf. below for an explicit construction. Next observe that $c_F(Z/\mathfrak{F}_{g,k}) := c(T_{\tilde{Z}|\mathfrak{F}_{g,k}}) \cap s(C_{Z|\tilde{Z}})$ is a class intrinsically associated to $Z \to \mathfrak{F}_{g,k}$, i.e., does not depend on the choice of embedding into a space smooth over $\mathfrak{F}_{g,k}$. This is a relative version of Fulton's *canonical class* [Fu], Expl. 4.2.6. We may thus write

$$\mathcal{GW}^M_\sigma = \left\{c(\widetilde{F} - T_{\tilde{Z}/\mathfrak{F}_{g,k}}) \cap c_F(Z/\mathfrak{F}_{g,k})\right\}_d.$$

There is a quasi-isomorphism

$$\begin{array}{ccccc}
T_{\tilde{Z}/\mathfrak{F}_{g,k}} & \xrightarrow{q_*} & T_{\mathcal{C}_\sigma(M)/\mathfrak{F}_{g,k}} & = & \pi_*^{1,p}\mathrm{ev}^*T_M \\
{\scriptstyle Ds_{\mathrm{can}}}\downarrow & & \downarrow{\scriptstyle \bar\partial = D_{\mathfrak{F}_{g,k}} s_{\bar\partial,I}} & & \\
\widetilde{F} & \xrightarrow{\tau} & \mathcal{E} & = & \pi_*^p(\mathrm{ev}^*T_M \otimes \overline{\Omega})
\end{array}$$

where q_* and the tangent bundle and differential relative to $\mathfrak{F}_{g,k}$ are only meant formally, but are defined directly. The right-hand vertical arrow in turn is quasi-isomorphic to $R\pi_*\mathrm{ev}^*T_M$ (to make this precise one should represent $R\pi_*\mathrm{ev}^*T_M$ by a morphism of vector bundles, cf. below). We obtain

$$\mathcal{GW}_\sigma^M \;=\; \Big\{c(R\pi_*\mathrm{ev}^*T_M)\cap c_F(Z/\mathfrak{F}_{g,k})\Big\}_d.$$

This is exactly the formula that one can derive for $J(M,\sigma)$ [Si3], for Behrend's relative obstruction theory is of the form $(R\pi_*\mathrm{ev}^*T_M)^\vee \to L^\bullet_{\mathcal{C}_\sigma(M)/\mathfrak{F}_{g,k}}$.

This argument is of course somewhat ad hoc and does not work as stated, because a τ with the required properties does not in general exist. But it shows already the basic reason behind the equivalence of the two theories: $R\pi_*\mathrm{ev}^*T_M$ can be interpreted as the *virtual tangent bundle* of $\mathcal{C}_\sigma(M)$ relative to $\mathfrak{F}_{g,k}$ when viewed either as zero locus of $s_{\overline{\partial},I}$ or as equipped with an obstruction theory relative to $\mathfrak{F}_{g,k}$.

5.2 Sketch proof by comparing cones

Recall that the construction of $J(M,\sigma)$ worked by writing $R^1\pi_*\mathrm{ev}^*T_M$ as homomorphism of vector bundles $[G\to H]$, constructing a cone $C^H\subset H$ invariant under the additive action of G and intersecting $[C^H]$ with the zero section of H. Our proof that this class coincides with the symplectic virtual fundamental class is divided into three steps:

(1) The local construction of complex analytic Kuranishi models;

(2) a limit construction to obtain a (bundle of) cone(s) $C(\tau)\subset F|_Z$ of dimension $d+\operatorname{rank}F$ and supporting a homology class $[C(\tau)]$ of the same dimension;

(3) finding an inclusion of vector bundles $\mu\colon H\hookrightarrow F|_Z$ with $\mu^![C(\tau)] = [C^H]$. Here $\mu^!$ is defined as the cap product with the pullback of the Thom class of $(F|_Z)/H$.

The theorem then follows from

$$\mathcal{GW}_\sigma(M) \;=\; [\widetilde{Z}]\cap\Theta_F \;=\; [C(\tau)]\cap\Theta_F \;=\; [C^H]\cap\Theta_H \;=\; J(M,\sigma).$$

Analytic Kuranishi models

Finding Kuranishi models in an integrable situation is actually easier than generally, because we may restrict to holomorphic maps near the double points. Let $(C,\mathbf{x},\varphi)$ be a stable holomorphic curve, i.e., the map $\varphi\colon C\to M$ is holomorphic. As in Section 2.4, let $(q\colon\mathcal{C}\to S,\underline{\mathbf{x}})$ be a semiuniversal deformation of $(C,\mathbf{x})$. If C_i are the irreducible components of C, we choose this time an open covering $\mathcal{U}=\{\mathcal{U}_i\}_{i=0,\dots,d}$ of $\mathcal{C}$ with the following properties:

- for $i > 0$ there are holomorphic maps
$$z_i \colon U_i \longrightarrow \Delta$$
extending holomorphically to $\overline{U_i}$ and inducing isomorphisms $U_i(s) := U_i \cap q^{-1}(s) \to \Delta$ for any $s \in S$;
- $U_i(0) \subset C_i$ and $U_i \cap U_j = \emptyset$ for $i, j > 0$;
- $U_0 = \mathcal{C} \setminus \bigcup_{i>0} z_i^{-1}(\overline{\Delta_{1/2}})$, $\Delta_{1/2} = \{z \in \mathbb{C} \mid |z| < 1/2\}$;
- for $i > 0$ there are holomorphic charts
$$M \supset W_i \xrightarrow{\gamma_i} \mathbb{C}^n$$
with $\varphi(\overline{U_i(0)}) \subset W_i$.

The part over U_0 is dealt with by the space $\overline{\mathrm{Hom}}_S(U_0; M)$, that as a set consists of holomorphic maps $U_0(s) \to M$ extending continuously to $\overline{U_0(s)}$. Using a Čech construction together with the fact that open Riemann surfaces have vanishing higher coherent cohomology (they are Stein), one can show

Proposition 5.2 $\overline{\mathrm{Hom}}_S(U_0; M)$ *is a complex Banach manifold mapping submersively onto* S. $\square$

By this we mean of course that this complex Banach manifold represents a certain functor. The functor sends a morphism $\varphi \colon T \to S$ to the set of holomorphic maps from $T \times_S U_0$ to M that extend continuously to $T \times_S \overline{U_0}$.

For $i > 0$ we may identify (an open set in) $L_1^p(U_i(s); M)$ with $L_1^p(\Delta; W_i)$ via z_i and γ_i, and $L_1^p(U_0(s) \cap U_i(s); M)$ with $L_1^p(A_{1/2}; W_i)$, $A_{1/2} = \Delta \setminus \overline{\Delta_{1/2}}$. Consider the differentiable map of complex Banach manifolds

$$\begin{aligned} H \colon \overline{\mathrm{Hom}}_S(U_0; M) \times \prod_{i>0} L_1^p(\Delta; \gamma_i(W_i)) &\longrightarrow \prod_{i>0} L_1^p(A_{1/2}; \mathbb{C}^n) \\ \Big(\psi_0 \colon U_0(s) \to M; \psi_i\Big) &\longmapsto \Big(\psi_i - \gamma_i \circ \psi_0 \circ z_i^{-1}\Big). \end{aligned}$$

$H^{-1}(0)$ can be identified with an open neighbourhood of φ in the space of L_1^p maps $\psi \colon C_s \to M$, some $s \in S$, that are holomorphic on $U_0(s)$. H is a split submersion along $H^{-1}(0)$. Hence

Proposition 5.3 $\mathcal{B} := H^{-1}(0)$ *is a complex Banach manifold.* $\square$

The $\overline{\partial}$ operator can now be viewed as a holomorphic map

$$G \colon \mathcal{B} \longrightarrow \prod_{i>0} L^p(A_{1/2}; \mathbb{C}^n),$$

and this induces the complex analytic structure on $\mathrm{Hom}_S(\mathcal{C}; M) = G^{-1}(0)$. An embedding of $\mathrm{Hom}_S(\mathcal{C}; M)$ into a finite dimensional complex manifold submerging onto S can be found as follows: let $Q \subset \prod_{i>0} L^p(A_{1/2}; \mathbb{C}^n)$ be a finite dimensional linear subspace spanning the cokernel of the linearization of G at some holomorphic φ. Q exists by the Stein property of $U_0(0)$. Then $G^{-1}(Q)$ is the desired finite dimensional complex manifold containing (an open part of) $\mathrm{Hom}_S(\mathcal{C}; M)$ as a closed complex subspace.

Note that by taking a basis of Q as perturbation terms α and a trivialization of $\mathcal{E}$ compatible with the complex analytic structure over W_i in the construction of $\tau\colon F \to \mathcal{E}$ (Section 3.2), we can achieve:

> Let $(C, \mathbf{x}, \varphi) \in \mathcal{C}_\sigma(M)$. Then, locally, there is a complex subbundle $F^h \subset F$ such that $\tau^h := \tau|_{F^h}$ spans the cokernel of the linearization of $s_{\overline{\partial}}$ relative to $\mathfrak{F}_{g,k}$ and $\widetilde{Z}^h := \widetilde{Z} \cap F^h$ is a *complex* orbifold.

For this purpose let $\overline{\mathcal{B}}$ be the image of $\mathcal{B}$ in $\mathcal{C}(M; p)$. Then by the choice of τ, $\widetilde{Z}^h$ must be a subset of $F^h|_{\overline{\mathcal{B}}}$, while over $\overline{\mathcal{B}}$ a uniformizer of $\widetilde{s}^h$ factorizes over G. Note that $\widetilde{Z}^h$ is $Z(\widetilde{s}^h)$ with $\widetilde{s}^h = (q^h)^* s + \tau^h$, $q^h = q|_{F^h}$.

The limit cone

The next step concerns the construction of the cone $C(\tau) \subset F|_Z$ that we get as the limit of $t \cdot \widetilde{Z} \subset F$ as t tends to infinity. This has nothing to do with holomorphicity.

We start with any $\tau\colon F \to \mathcal{E}$ over our Banach orbifold $\mathcal{C}_\sigma(M; p)$ spanning the cokernel of σ and write as usual $q\colon F \to \mathcal{C}_\sigma(M; p)$ for the bundle projection. For any $l > 0$,

$$F \times \mathbb{R}^l \ni (f, v) \longmapsto s_{\overline{\partial}}(q(f)) + |v|^2 \cdot \tau(f) \in \mathcal{E}$$

defines a section $\widetilde{s}_l$ of $q_l^* \mathcal{E}$, where $q_l = q \circ \mathrm{pr}_1$ is the projection from $F \times \mathbb{R}^l$ to $\mathcal{C}_\sigma(M; p)$. $\widetilde{s}_l$ is constant on spheres $\{f\} \times S_t^{l-1}(0)$. For $t \neq 0$ the zero locus $\widetilde{Z}_l$ of $\widetilde{s}_l$ restricted to $F \times S_t^{l-1}(0)$ is just $(t \cdot \widetilde{Z}) \times S_t^{l-1}(0)$, while $\widetilde{Z}_l \cap (F \times \{0\}) = F|_{\mathcal{C}_\sigma(M)}$.

Definition 5.4 Set $A = \widetilde{Z}_l \cap (F \times (\mathbb{R}^l \setminus \{0\}))$ and write $\overline{A}$ for its closure in $F \times \mathbb{R}^l$. The *limit cone* $C(\tau) \subset F$ of $s_{\overline{\partial}}$ with respect to τ is defined to be $\overline{A} \cap (F \times \{0\})$. $\square$

$C(\tau)$ is the set theoretic limit of $t \cdot \widetilde{Z}$ as t tends to infinity. As such:

(1) it does not depend on l; and

(2) it lies over the zero locus $\mathcal{C}_\sigma(M)$ of $s_{\overline{\partial}}$.

The reason for introducing l is the exact sequence of homology groups (of the second kind, see the discussion in 1.2)

$$H_{l+d+r}(C(\tau)) \longrightarrow H_{l+d+r}(\overline{A}) \longrightarrow H_{l+d+r}(A) \longrightarrow H_{l+d+r-1}(C(\tau)).$$

Here $r = \operatorname{rank} F$ and $d = d(M, R, g, k)$. The fundamental class $[A]$ of the oriented manifold A extends uniquely to a $(l+d+r)$-homology class (conveniently denoted $[\overline{A}]$ by abuse of notation) on $\overline{A}$, provided that $l+d+r-1 > \dim C(\tau)$. This uses the general vanishing theorem for homology, cf. [Iv], IX.1, Prop. 1.6. But from $C(\tau) \subset F|_{\mathcal{C}_\sigma(M)}$, it follows that $\dim C(\tau) < r + \dim \mathcal{C}_\sigma(M)$ is always finite, so the inequality can be satisfied by choosing l large enough. We can now define a homology class on $C(\tau)$ that is the limit of $[t \cdot \widetilde{Z}]$.

Proposition 5.5 *Let $\delta_0 \in H^l_{\{0\}}(\mathbb{R}^l)$ be the Poincaré dual of $\{0\} \subset \mathbb{R}^l$. Then*

$$[C(\tau)] \;:=\; [\overline{A}] \cap \delta_0 \;\in H_{d+r}(C(\tau))$$

is independent of l and homologous to $[\widetilde{Z}]$ as class on F. $\square$

Note that the construction of $C(\tau)$ and $[C(\tau)]$ actually happens in finite dimensions. This is more apparent if we work in q^*F over the fixed finite dimensional orbifold $\widetilde{Z}$. Let s_{can} be the tautological section of q^*F. The natural map $q^*F \to F$ identifies the graph $\Gamma_{t \cdot s_{\mathrm{can}}}$ of $t \cdot s_{\mathrm{can}}$ with $t \cdot \widetilde{Z}$, and we may as well work with these graphs.

In a holomorphic situation we retrieve the following familiar picture [Fu], §14.1 (the following construction also works for singular spaces): let E be a holomorphic vector bundle over a complex manifold N, and let Z be the zero locus of a holomorphic section s of E. The differential of s induces a closed embedding of the *normal bundle* $N_{Z|N}$ of Z in N into E. $N_{Z|N}$ is the linear fiber space over Z associated to the conormal sheaf $\mathcal{I}/\mathcal{I}^2$ (the analytic analogue of $\operatorname{Spec}_Z S^\bullet \mathcal{I}/\mathcal{I}^2$ in the algebraic situation), $\mathcal{I}$ the ideal sheaf of Z in N. The *normal cone* $C_{Z|N}$ (the analytic analogue of $\operatorname{Spec}_Z \oplus_{d \geq 0} \mathcal{I}^d/\mathcal{I}^{d+1}$) is a closed subspace of $N_{Z|N}$. Let $\iota\colon C_{Z|N} \hookrightarrow E$ be the induced closed embedding. Take the identity morphism $E \to E$ for τ. Then $t \cdot \widetilde{Z}$ is the graph of $t \cdot s$. One can show ([Fu], Remark 5.1.1) that

$$C(\tau) \;=\; \iota(C_{Z|N}) \quad \text{(as spaces) and} \quad [C(\tau)] \;=\; \iota_*[C_{Z|N}] \;=\; [\iota(C_{Z|N})].$$

This is used below to identify $C(\tau^h)$ with the image in F^h of the normal cone of $Z = \mathcal{C}_\sigma(M)$ in $\widetilde{Z}^h$. The use of s^h_{can} is equivalent to the present use of the identity morphism.

The other ingredient is the following method to get rid of a nonholomorphic part of τ locally.

Proposition 5.6 *Let $F = F^h \oplus \overline{F}$ be a decomposition such that*

- *$\tau^h := \tau|_{F^h}$ spans the cokernel of the linearization σ along $Z = Z(s)$ and has the regularity properties of τ;*
- *$\overline{\tau} := \tau|_{\overline{F}}$ maps to $\mathrm{im}\,\sigma$ along Z.*

Then $C(\tau) = C(\tau^h) \oplus \overline{F}$ and $[C(\tau)] = [C(\tau^h)] \oplus [\overline{F}]$. $\square$

The *proof* runs by considering the two parameter family $\widetilde{s}_{t,u} := q^*s + t\cdot\tau^h + u\cdot\overline{\tau}$ of perturbed sections with $|u| \le |t|$. This interpolates between the original family $\widetilde{s}_t = \widetilde{s}_{t,t}$ and the family $\widetilde{s}_{t,0}$ having $\overline{F}$ added as trivial factor. As long as $t \neq 0$, $\widetilde{Z}_{t,u} = Z(\widetilde{s}_{t,u})$ is a suborbifold of F. The essential point is this:

Lemma 5.7 *The set theoretic limit of $\widetilde{Z}_{t,u}$ as $t, u \to 0$, $|u| \le |t|$, equals $C(\tau^h) \oplus \overline{F}$. More precisely,*

$$\mathrm{cl}\Big(\bigcup_{\substack{t\neq 0\\ |u|\le|t|}} \widetilde{Z}_{t,u} \times (t,u)\Big) \cap (F \times (0,0)) \;=\; \Big\{(f,g) \in F^h \oplus \overline{F} \,\Big|\, f \in C(\tau^h)\Big\}.$$

Intuitively if slightly imprecisely, we write $[C(\tau)] = \lim_{t\to 0}[\widetilde{Z}_t]$, to indicate both set theoretical and homological convergence. The proposition follows from

$$\begin{aligned}
[C(\tau)] &= \lim_{t\to 0}[\widetilde{Z}_{t,t}] = \lim_{\substack{t,u\to 0\\ |u|\le|t|}}[\widetilde{Z}_{t,u}]\\
&= \lim_{t\to 0}[\widetilde{Z}_{t,0}] = \lim_{t\to 0}[\widetilde{Z}^h_t] \oplus [\overline{F}] = [C(\tau^h)] \oplus [\overline{F}].
\end{aligned}$$

As for the lemma we may restrict attention to a fixed fiber F_z over $z \in \mathcal{C}_\sigma(M)$. One may then use uniform continuity of the relative differential σ at centers of local uniformizing systems in connection with the implicit function theorem to modify sequences $(f_\nu, g_\nu) \in \widetilde{Z}_{t_\nu,u_\nu}$ with limit $(f,g) \in C(\tau)$ to $(f'_\nu, g'_\nu) \in \widetilde{Z}_{t_\nu} \oplus \overline{F}$ with the same limit. This shows $C(\tau) \subset C(\tau^h) \oplus \overline{F}$. The converse inclusion is obvious. $\square$

The following will also be used.

Lemma 5.8 *Let χ be a continuous function on $\mathcal{C}_\sigma(M;p)$ without zeros on an open set U. Then*

$$C(\tau)|_U \;=\; C(\chi\cdot\tau|_U), \quad [C(\tau)]|_U \;=\; [C(\chi\cdot\tau|_U)].$$

This is because multiplication by χ on the fibers of F induces an isomorphism from $Z(q^*s + t\cdot\chi\cdot\tau)$ to $Z(q^*s + t\cdot\tau)$. $\square$

Global comparison

We begin by recalling the global free resolution of $R\pi_*\mathrm{ev}^*T_M$ used by Behrend [Be1], Prop. 5. Let $\pi\colon \Gamma \to \mathcal{C}_\sigma(M)$ be the universal family, $\mathrm{ev}\colon \Gamma \to M$ the universal morphism. By a twisting procedure with a relatively ample line bundle one obtains a sequence of holomorphic vector bundles

$$0 \to K \longrightarrow N \longrightarrow \mathrm{ev}^*T_M \to 0$$

with $\pi_*K = \pi_*N = 0$. Then $R\pi_*\mathrm{ev}^*T_M$ is (up to unique isomorphism) given by the homomorphism of vector bundles $G := R^1\pi_*K \to H := R^1\pi_*N$, viewed as a complex in degrees 0 and 1, as element of the derived category. The latter vector bundles can be described as cokernels of $\overline{\partial}$ operators obtained by resolving the above sequence by sheaves of fiberwise Sobolev sections and pushing forward. We get a diagram of complex (rather than holomorphic) Banach bundles

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & \pi_*^{1,p}K & \longrightarrow & \pi_*^{1,p}N & \longrightarrow & \pi_*^{1,p}\mathrm{ev}^*T_M & =: \mathcal{T} & \to 0 \\
 & & \overline{\partial}_K\downarrow & & \downarrow\overline{\partial}_N & & \downarrow\overline{\partial} & & \\
0 & \to & \pi_*^{p}(K\otimes\overline{\Omega}) & \longrightarrow & \pi_*^{p}(N\otimes\overline{\Omega}) & \longrightarrow & \pi_*^{p}(\mathrm{ev}^*T_M\otimes\overline{\Omega}) & = \mathcal{E} & \to 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & G & \stackrel{d}{\longrightarrow} & H & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

As we occasionally do in the sequel, we omit to indicate some restrictions to $\mathcal{C}_\sigma(M)$. Similarly to the above construction of τ^h, we may now construct local homomorphisms $\tau_i\colon H \to \mathcal{E}$ that come from lifts to $\pi_*^p(N\otimes\overline{\Omega})$ of local holomorphic sections of H with support away from the singular locus of π. τ_i is easily seen to span the cokernel of $\sigma = \overline{\partial}$. In fact, locally, we even obtain a Cartesian diagram of vector bundles

$$\begin{array}{ccc}
G & \longrightarrow & \mathcal{T} \\
d\downarrow & & \downarrow\overline{\partial} \\
H & \stackrel{\tau_i}{\longrightarrow} & \mathcal{E}
\end{array} .$$

K and N extend naturally to $\mathcal{C}_\sigma(M;p)$, as do H, G and τ_i. Keeping the notations H, G and τ_i for the extended objects we may set

$$F := H^{\oplus l} \quad \text{and} \quad \tau = \sum_i \chi_i\tau^i\colon F \longrightarrow \mathcal{E},$$

where we now insist that the bump functions χ_i form a partition of unity along $\mathcal{C}_\sigma(M)$ (this can be done by going over to $\chi_i/\sum_j \chi_j$). Then τ composed with the diagonal embedding $H \hookrightarrow F$ spans the cokernel of σ along all of $\mathcal{C}_\sigma(M)$. To compare Behrend's cone $C^H \subset H$ and $C(\tau)$ we embed H in F diagonally:

$$\mu \colon H \longrightarrow F, \quad h \longmapsto (h, \ldots, h).$$

Over an open set where τ_i spans the cokernel, put $F^h := F_i$ and $\iota_i \colon F_i \hookrightarrow F$ the embedding. Then, up to a harmless scaling factor, $\tau|_{F_i} = \chi_i \tau_i$ is of the form as given in the construction of analytic Kuranishi models. Put $\tau^h := \tau_i$. To find the complementary subbundle $\overline{F}$, let $\widetilde{T} := F \oplus_\tau \mathcal{T} = \{(f, v) \mid \tau(f) = \sigma(v)\}$. $\widetilde{T}$ should be viewed as the tangent bundle of $\widetilde{Z}$ relative to $\mathfrak{F}_{g,k}$. Both F_i and $\operatorname{im}\mu$ span the cokernel of the projection $\widetilde{T} \to F$. A linear algebra argument gives:

Lemma 5.9 *Over the open set under consideration there exists a (continuous) suborbibundle $P \subset \widetilde{T}$ with $\overline{F} := \sigma(P)$ complementary to both $\mu(H)$ and $\iota_i(F_i)$.* $\square$

Proposition 5.6 applied to $F = F^h \oplus \overline{F}$ now shows that

$$C(\tau) \;=\; C(\chi_i\tau_i) \oplus \overline{F} \quad \text{and} \quad [C(\tau)] \;=\; [C(\chi_i\tau_i)] \oplus [\overline{F}].$$

By Lemma 5.8 we may also replace $\chi_i\tau_i$ by $\tau^h = \tau_i$. Let $\rho \colon F \to Q$ be the cokernel of μ. By transversality of $\overline{F}$ to $\operatorname{im}\mu$ we may identify $\overline{F}$ with Q via ρ. Let Θ_Q be the Thom class of Q. Then

$$\mu^![C(\tau)] \;=\; [C(\tau)] \cap \rho^*\Theta_Q \;=\; [C(\tau^h)].$$

It thus remains to show that $C(\tau^h)$ coincides with $C^H \subset H$. To this end note that the morphism

$$\varphi^\bullet \colon\; [\mathcal{F}_i \to \Omega_{\widetilde{Z}^h/\mathfrak{F}}|_Z] \;\longrightarrow\; [\mathcal{I}/\mathcal{I}^2 \to \Omega_{\widetilde{Z}^h/\mathfrak{F}}|_Z]\,,$$

that we obtain from the description of Z as zero locus of s^h_{can} in $\widetilde{Z}^h$ (with ideal sheaf $\mathcal{I}$), is a *(global resolution of a perfect) obstruction theory* as defined in [BeFa], or a *free global normal space* in the language of [Si2]. Note that the right-hand side of $\varphi^\bullet$ is isomorphic to the truncated cotangent complex $\tau_{\geq -1}L^\bullet_Z$ of Z. And $C(\tau^h)$ is exactly the closed subcone of $F^h|_Z$ obtained from this obstruction theory.

Let $\mathcal{G} = \mathcal{O}(G^\vee)$, $\mathcal{H} = \mathcal{O}(H^\vee)$ be the sheaves corresponding to G and H. Then $\mathcal{H} = \mathcal{F}_i$ and $\mathcal{G}$ can be identified with $\Omega_{\widetilde{Z}^h/\mathfrak{F}}|_Z$. Let

$$\psi^\bullet \colon\; [\mathcal{H} \to \mathcal{G}] \;\longrightarrow\; [\mathcal{I}/\mathcal{I}^2 \to \Omega_{\widetilde{Z}^h/\mathfrak{F}}|_Z]$$

be the obstruction theory used for algebraic GW invariants in [Be1], cf. [BeFa] before Prop. 6.2. The central result is

Proposition 5.10 *With the identifications $\mathcal{H} = \mathcal{F}_i$ and $\mathcal{G} = \Omega_{\widetilde{Z}^h/\mathfrak{F}}|_Z$, the morphisms $\varphi^\bullet$ and $\psi^\bullet$ are locally homotopic, i.e., equal as morphisms in the derived category.* $\square$

Since the cone belonging to an obstruction theory depends only on the morphism in the derived category, this shows that $C^H = C(\tau^h)$ as complex subspaces of H. So the proposition will finish the proof of Theorem 5.1.

To prove the proposition it suffices to check equality of the maps in cohomology, because we are dealing here with locally split two-term complexes [Si2], Lemma 2.4. $\psi^\bullet$ is constructed from the morphisms of the universal curve over $\mathcal{C}_\sigma^{\mathrm{hol}}(M)$ to M (evaluation map) and to $\mathcal{C}_\sigma^{\mathrm{hol}}(M)$ (projection) by constructions in the derived category. The difficulty in proving the proposition is to make the abstract constructions in derived categories explicit in a way suitable for comparison with the $\overline{\partial}$ operator. Let us just briefly indicate here how the $\overline{\partial}$ operator shows up, which is the key part.

First note that it suffices to work with truncated cotangent complexes $\tau_{\geq -1}L^\bullet$. By embedding into smooth spaces these can always be expressed in the form "conormal sheaf maps to restriction of cotangent sheaf of ambient smooth space". The smooth spaces we take are of course $\widetilde{Z}^h$ and the universal curve $\widetilde{\Gamma}$ over $\widetilde{Z}^h$. The holomorphic evaluation map from the universal curve Γ over Z does not in general extend holomorphically to $\widetilde{\Gamma}$. The point is that $\widetilde{\mathrm{ev}}$ provides a *differentiable* extension. The defect in holomorphicity leads to the $\overline{\partial}$ operator in the following explicit description of the map

$$\ker(\mathcal{H} \to \mathcal{G}) \;\simeq\; \pi_*(\mathrm{ev}^*\Omega_M \otimes \omega) \longrightarrow \mathcal{I}/\mathcal{I}^2.$$

Namely, for $U \subset \widetilde{Z}$ an open set, we send $\alpha \in (\mathrm{ev}^*\Omega_M \otimes \omega)(\pi^{-1}U)$ to $f_\alpha \in \mathcal{I}(U)$ by

$$U \ni z \longmapsto \int_{\widetilde{\Gamma}_z} \alpha(\overline{\partial}\widetilde{\varphi}_z),$$

where $\widetilde{\varphi}_z \colon \widetilde{\Gamma}_z \to M$ is the curve parametrized by z, and where we apply the dual pairing $\Omega_M \otimes T_M \to \underline{\mathbb{C}}$ to make $\alpha(\overline{\partial}\widetilde{\varphi}_z)$ a $(1,1)$-form on $\widetilde{\Gamma}_z$. Note that $\widetilde{\varphi}_z$ is holomorphic near the singularities, so this form is smooth (in contrast to α, which may have poles at the singularities of $\widetilde{\Gamma}_z$). It should be more or less clear, and can be checked easily that this is exactly $H^{-1}(\varphi i^\bullet)$, the map induced by s^h_{can}. Similarly for the cokernels of $\varphi^\bullet$ and $\psi^\bullet$.

One final remark concerning rigidification: the limit cones that one obtains over an unrigidified chart $S \times V$ is invariant under the automorphism group of $(C, \mathbf{x})$ and hence restricts to the limit cone uniformizing $C(\tau)$ on the actual, rigidified chart $S \times \overline{V}$. A similar statement holds for the algebraic cones.

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