Mordell–Weil lattices for higher genus fibration over a curve

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To Sōichi Kawai for his 60th birthday

1 Introduction, notation

Let $K = k(C)$ be the function field of an algebraic curve C over an algebraically closed ground field k. Let Γ/K be a smooth projective curve of genus $g > 0$ with a K-rational point $O \in \Gamma(K)$, and let J/K denote the Jacobian variety of Γ/K. Further let (τ, B) be the K/k-trace of J (see §2 below and [4]).

Then the Mordell–Weil theorem (in the function field case) states that the group of K-rational points $J(K)$ modulo the subgroup $\tau B(k)$ is a finitely generated Abelian group.

Now, given Γ/K, there is a smooth projective algebraic surface S with genus g fibration $f\colon S \to C$ which has Γ as its generic fibre and which is relatively minimal in the sense that no fibres contain an exceptional curve of the first kind (-1-curve). It is known that the correspondence $\Gamma/K \leftrightarrow (S, f)$ is bijective up to isomorphisms (cf. [7], [5]).

The main purpose of this paper is to give the Mordell–Weil group $M = J(K)/\tau B(k)$ (modulo torsion) the structure of Euclidean lattice via intersection theory on the algebraic surface S. The resulting lattice is the *Mordell–Weil lattice* (MWL) of the Jacobian variety J/K, which we call sometimes MWL of the curve Γ/K or of the fibration $f\colon S \to C$.

For this, we first establish the relationship between the Mordell–Weil group and the Néron–Severi group $\mathrm{NS}(S)$ of S (Theorem 1, stated in §2 and proved in §3). Then (in §4) we introduce the structure of lattice on the Mordell–Weil group by defining a natural pairing in terms of the intersection pairing on $\mathrm{NS}(S)$. Thus the basic idea is the same as the elliptic case $g = 1$ (cf. [8]).

We think that Theorem 1 must have been known to mathematicians such as Lang, Néron, Tate, Weil, ... who did the foundational work on the Mordell–Weil theorem and the theory of canonical height. But we could not locate the statement of Theorem 1 (as given below) in the literature; compare e.g., [16], [4, Chap. 6, §6], [14, §4].

For the sake of completeness, we give a proof of Theorem 1 in §3. In the proof, we use (a consequence of) Theorem 2 due to Raynaud which describes a precise relationship between the K/k-trace of J/K, the Picard variety of S and the Jacobian variety of C. As far as we know, Theorem 2 is new, in that it takes care of the infinitesimal part, which is a delicate point in dealing with the K/k-trace. Actually, to prove Theorem 1, we do not need the full strength of Theorem 2; its consequence on the k-valued points (i.e., ignoring the infinitesimal part) is sufficient. This was how we proved Theorem 1 in the original version of this paper. But it is much nicer to have a statement as definitive as Theorem 2.

The main results of this paper were announced in our short note [11] under the assumption that the K/k-trace in question is trivial. We apologize for the delay in providing a detailed, generalized version. During this delay, two important results have come to light; one is Raynaud's theorem mentioned above, and the other is a new viewpoint to use the canonical height for Abelian variety over over higher dimensional function fields (cf. [13]).

I am very grateful to M. Raynaud for informing me of his proof of Theorem 2, and for allowing me to include it here. Also I would like to thank many people for giving me the opportunity to reconsider the subject in a stimulating atmosphere, especially J. Coates (Cambridge Univ.), M. Reid (Warwick Univ.), S. Ramanan (Tata Inst.), F. Hirzebruch (Max-Planck Inst.) and others. The final version was completed during my stay at Johns Hopkins Univ. (JAMI).

List of Notation

- k: an algebraically closed field
  C or C/k: a smooth, projective curve defined over k
  $K = k(C)$: the function field of C/k
  $J_C = J_{C/k}$: the Jacobian variety of C

- Γ or Γ/K: a smooth, projective curve of genus $g > 0$ defined over K
  $J = J_{\Gamma/K}$: the Jacobian variety of Γ/K
  (τ, B): the K/k-trace of J

- $f\colon S \to C$: minimal genus g fibration with generic fibre Γ
  S or S/k: a smooth projective surface defined over k
  $\Gamma(K)$: the set of K-rational points of Γ, identified with the set of sections of $f\colon S \to C$
  (P) (for $P \in \Gamma(K)$): the image of $P\colon C \to S$ (a curve on S)
  $\mathrm{Pic}_{S/k}$: the Picard scheme of S/k

PicVar$_S$: the Picard variety of S/k
Pic$_{S/C}$: the relative Picard scheme of S/C (see [6])
$\mathrm{Pic}(S) := \mathrm{Pic}_{S/k}(k); \mathrm{PicVar}(S) := \mathrm{PicVar}_S(k)$
NS(S): the Néron–Severi group of S
$\rho(S)$: the Picard number of S ($= \operatorname{rank} \mathrm{NS}(S)$)

- $R := \{v \in C | f^{-1}(v) = \sum_{i=0}^{m_v-1} \mu_{v,i}\Theta_{v,i}$ is a reducible fibre$\}$
  $m_v (v \in R)$: the number of irreducible components of $f^{-1}(v)$
  $\Theta_{v,i}$ (for $0 \le i \le m_v - 1$): irreducible components
  $\Theta_{v,0}$: the identity components of $f^{-1}(v)$ (with $\mu_{v,0} = 1$)
  T_v: the subgroup of NS(S) generated by $\Theta_{v,i}$ (for $1 \le i \le m_v - 1$)
  F: the class of any fibre of f in NS(S)
  T: the subgroup of NS(S) generated by $(O), F$ and T_v (for $v \in R$)

2 K/k-trace, Mordell–Weil group and Néron–Severi group

We use the notation introduced above. Thus J is the Jacobian variety of a curve Γ defined over $K = k(C)$, C being a curve over an algebraically closed field k. Unless otherwise mentioned, we always assume:

$$\Gamma \text{ has a } K\text{-rational point } O. \tag{A_0}$$

According to the Mordell–Weil theorem for the function field case, the group of K-rational points $J(K)$ modulo the subgroup $\tau B(k)$, i.e., the quotient group $M = J(K)/\tau B(k)$, is a finitely generated Abelian group, where (τ, B) denotes the K/k-trace of J/K (cf. [4] for this and the following). We call M the *Mordell–Weil group* of J/K or of Γ/K.

For the convenience of the reader, let us recall here the definition of the K/k-trace of an Abelian variety A defined over K. It is a pair (τ, B) in which B is an Abelian variety defined over k and $\tau\colon B \to A$ is a homomorphism of Abelian varieties over K with the following universal mapping property: given any pair (τ', B') as above, there is a unique homomorphism $\psi\colon B' \to B$ such that $\tau' = \tau \circ \psi$. It is known that the K/k-trace exists (Chow) and that $\tau\colon B \to A$ is a radicial map (that is, it is injective on points, but it may have an infinitesimal kernel in case of characteristic $p > 0$).

Now, letting $A = J$ be a Jacobian variety as before, we state the following basic theorems.

Theorem 1 *Under* (A_0)*, the Mordell–Weil group* $M = J(K)/\tau B(k)$ *is naturally isomorphic to the quotient group of* NS(S) *by* T*:*

$$J(K)/\tau B(k) \simeq \mathrm{NS}(S)/T. \tag{1}$$

Theorem 2 (Raynaud) *There is a short exact sequence of Abelian varieties over k:*

$$0 \longrightarrow J_C \longrightarrow \mathrm{PicVar}_S \longrightarrow B \longrightarrow 0 \tag{2}$$

The maps in these theorems are induced from the natural maps: restricting line bundles on S to the generic fibre Γ and taking the pullback (via $f\colon S \to C$) of line bundles on the base curve C to S.

Postponing the proof of Theorems 1 and 2 to the next section, let us deduce some obvious consequences, which will be the most interesting case for application.

Theorem 3 *The following conditions are equivalent:*

(i) The K/k-trace of J/K is trivial, i.e., $B = 0$.

(ii) $f^*\colon J_C \to \mathrm{PicVar}_S$ is an isomorphism.

(iii) The irregularity of S is equal to the genus of C.

If any (hence all) of these conditions holds, then $J(K)$ is a finitely generated Abelian group such that

$$J(K) \simeq \mathrm{NS}(S)/T. \tag{3}$$

In particular, we have

$$\operatorname{rank} J(K) = \rho(S) - 2 - \sum_{v \in R} (m_v - 1) \tag{4}$$

(cf. the notation in §1 and §4, Proposition 5).

The irregularity of a surface is, by definition, the dimension of its Picard variety. A surface with irregularity zero is classically called a *regular* surface. Examples of regular surfaces include rational surfaces, K3 surfaces and smooth surfaces in $\mathbb{P}^3$.

Corollary 4 *Suppose that S is a regular surface with a given genus g fibration $f\colon S \to C = \mathbb{P}^1$. Then the K/k-trace of its generic fibre Γ is trivial, and the conclusion of Theorem 3 holds.*

3 Proof of Theorems 1 and 2

First we prove Theorem 1, assuming Theorem 2. Restricting a line bundle on S to the generic fibre Γ defines a group homomorphism

$$\mathrm{Pic}(S) := \mathrm{Pic}_{S/k}(k) \longrightarrow \mathrm{Pic}_{\Gamma/K}(K) \tag{5}$$

In terms of divisors, it associates with each divisor class $\mathrm{cl}(D)$ on S the class of the divisor $D|_\Gamma = D \cdot \Gamma$ on the generic fibre Γ. By adjusting the image to have degree zero using the given rational point $O \in \Gamma(K)$, we get a homomorphism

$$\varphi\colon\ \mathrm{Pic}(S) \longrightarrow J(K) := \mathrm{Pic}^0_{\Gamma/K}(K) \tag{6}$$

which sends $\mathrm{cl}(D)$ to $\mathrm{cl}(\delta)$ with $\delta = D \cdot \Gamma - (D \cdot \Gamma)O$.

Obviously (5) and (6) are surjective, because given any K-rational divisor, say δ', on Γ, we can take the k-loci (schematic closure) of the components of δ' to obtain a divisor D' on S such that $D' \cdot \Gamma = \delta'$.

Next we determine the kernel of φ. Let $\widetilde{T}$ denote the subgroup of $\mathrm{Pic}(S)$ generated by the divisor classes of irreducible components of fibres and the zero section (O). It is obvious that $\widetilde{T}$ is contained in $\mathrm{Ker}(\varphi)$. Let us show that $\widetilde{T} = \mathrm{Ker}(\varphi)$. Take any element $\xi = \mathrm{cl}(D) \in \mathrm{Ker}(\varphi)$. By definition, $D|_\Gamma$ is linearly equivalent to zero on Γ, and hence it is equal to a principal divisor (h) for some rational function $h \in K(\Gamma)$. Since the function field $K(\Gamma)$ can be identified with the function field $k(S)$, we can find some function $H \in k(S)$ such that $(H)|_\Gamma = (h)$ (note that H is unique up to multiplication by elements in $K^\times = k(C)^\times$). Then every component of the divisor $D' := D - (H)$ on S must be contained in some fibre; in other words, D' is a linear combination of irreducible components of fibres. Hence $\xi = \mathrm{cl}(D)$ belongs to $\widetilde{T}$.

Therefore we have an exact sequence (of Abelian groups):

$$0 \longrightarrow \widetilde{T} \longrightarrow \mathrm{Pic}(S) \longrightarrow J(K) \longrightarrow 0 \tag{7}$$

Furthermore, the restriction of the map φ to the Picard variety $\mathrm{PicVar}(S) \subset \mathrm{Pic}(S)$ factors through the K/k-trace B of J/K (see the proof of Theorem 2 below), which fits in the exact sequence of k-points of (2):

$$0 \longrightarrow J_C(k) \longrightarrow \mathrm{PicVar}(S) \longrightarrow B(k) \longrightarrow 0 \tag{8}$$

From the above two sequences, we obtain a surjective map:

$$\widetilde{\varphi}\colon\ \mathrm{NS}(S) := \mathrm{Pic}(S)/\,\mathrm{PicVar}(S) \longrightarrow J(K)/\tau B(k) \tag{9}$$

which induces an isomorphism (1) in Theorem 1, since T is nothing but the image of $\widetilde{T}$ in $\mathrm{NS}(S)$. This proves Theorem 1 (assuming Theorem 2).

Proof of Theorem 2 Following Raynaud, we prove Theorem 2 under the assumption:

$$\text{For every fibre } f^{-1}(v) \text{ of } f\colon S \to C, \text{ the g.c.d. of multiplicities of irreducible components is equal to 1.} \tag{A_1}$$

This condition is weaker than our assumption (A_0) that Γ has a K-rational point O, because the latter implies that every fibre has at least one component with multiplicity 1, namely, the (identity) component intersecting the zero section (O).

As proved in [6] (see also [1]), under (A_1), the sheaf $R^1 f_* \mathcal{O}_S$ is locally free of rank g. The Leray spectral sequence for $\mathcal{O}_S$ gives a short exact sequence

$$0 \longrightarrow H^1(C, \mathcal{O}_C) \xrightarrow{f^*} H^1(S, \mathcal{O}_S) \longrightarrow H^0(C, R^1 f_* \mathcal{O}_S) \tag{10}$$

(N.B. We can add the arrow $\to 0$ at the end under the stronger assumption (A_0), but this is not needed for the argument below.)

At the level of Picard schemes, we have an exact sequence

$$0 \longrightarrow \mathrm{Pic}_{C/k} \longrightarrow \mathrm{Pic}_{S/k} \longrightarrow \mathrm{Pic}_{S/C} \tag{11}$$

in which $\mathrm{Pic}_{S/C}$ is the relative Picard scheme representing $R^1 f_* \mathcal{O}_S$ (see [6]), and (10) corresponds to taking the Lie algebras of (11), or more precisely to the Lie algebras of the identity components of group schemes in (11):

$$0 \longrightarrow \mathrm{Pic}^0_{C/k} \longrightarrow \mathrm{Pic}^0_{S/k} \longrightarrow \mathrm{Pic}^0_{S/C} \tag{12}$$

Note that the image of the map $J_C = \mathrm{Pic}^0_{C/k} \to \mathrm{Pic}^0_{S/k}$ lies in $(\mathrm{Pic}^0_{S/k})_{\mathrm{red}}$, since smooth part is mapped into smooth part, and that we have $\mathrm{PicVar}_S = (\mathrm{Pic}^0_{S/k})_{\mathrm{red}}$ by definition.

On the other hand, the restriction to the generic fibre gives a map

$$\mathrm{Pic}^0_{S/C} \longrightarrow J_{\Gamma/K} \tag{13}$$

such that the corresponding map of Lie algebras

$$H^0(C, R^1 f_* \mathcal{O}_S) \longrightarrow \mathrm{Lie}(J_{\Gamma/K}) \tag{14}$$

is injective, since $R^1 f_* \mathcal{O}_S$ is locally free under (A_1) as remarked before.

By (12) and (13), we have

$$0 \longrightarrow J_C \xrightarrow{\alpha} \mathrm{PicVar}_S \xrightarrow{\gamma} J = J_{\Gamma/K}. \tag{15}$$

The map γ factors through the K/k-trace (τ, B) by the universal mapping property: $\gamma = \tau \circ \beta$ for a unique homomorphism $\beta\colon \mathrm{PicVar}_S \to B$.

Now we claim that

$$0 \longrightarrow J_C \xrightarrow{\alpha} \mathrm{PicVar}_S \xrightarrow{\beta} B \longrightarrow 0 \tag{16}$$

is an exact sequence of Abelian varieties over k, which will prove Theorem 2.

The most crucial point is the exactness at the middle. To take care of infinitesimal part, look at the corresponding Lie algebras:

$$\begin{array}{ccccc} \mathrm{Lie}(J_C) & \xrightarrow{\alpha'} & \mathrm{Lie}(\mathrm{PicVar}_S) & \xrightarrow{\beta'} & \mathrm{Lie}(B) \\ || & & \bigcap & & \downarrow \tau' \\ H^1(C, \mathcal{O}_C) & \xrightarrow{f^*} & H^1(S, \mathcal{O}_S) & \longrightarrow & \mathrm{Lie}(J) \end{array} \tag{17}$$

The bottom row is (10), combined with (14); hence it is exact.

Now we observe that τ' is an injective map (of a k-vector space into a K-vector space). Indeed, if $\mathrm{Ker}(\tau') \neq 0$, then there would be some radicial subgroup scheme $G \neq 0$ of B giving rise to a homomorphism $B' = B/G \to J$, which contradicts the universality of the trace (τ, B).

It follows that the top row in (17) is exact (at the middle).

Going back to (16), we show the exactness at the middle. First we note that $\beta \circ \alpha = 0$. For, otherwise, $\mathrm{Im}(\beta \circ \alpha) \neq 0$ will be an Abelian subvariety of B which is mapped to zero under $\tau \colon B \to J$; again this contradicts the universality of τ.

To prove $\mathrm{Ker}(\beta) = \mathrm{Im}(\alpha)$, it is now enough to test it at the k-valued points. Thus, take any $\xi \in \mathrm{Ker}(\beta)$. Then we have $\xi = \mathrm{cl}(D)$ for some divisor D with $D|_\Gamma = 0$ (cf. (7)); hence it is a linear combination of components of fibres. Thus we can write $D = \sum_v D_v$ (a finite sum) where D_v has support in $F_v = f^{-1}(v)$ (for $v \in C$) and $(D_v \cdot \Theta) = 0$ for every irreducible component Θ of F_v. We want to see that D_v is an integral multiple of F_v: $D_v = n_v F_v$ (with $n_v \in \mathbb{Z}$). There is no problem at good fibres. At a bad fibre, we know in general from [6] that dD_v is an integral multiple of F_v if d denotes the g.c.d. of multiplicities of irreducible components of F_v. Hence, under the assumption (A_1), we conclude that $D = f^{-1}(\sum_v n_v v)$, and hence ξ belongs to $\mathrm{Im}(\alpha)$.

To complete the proof of Theorem 2, we should also see that, in (16), α is injective and β is surjective. The injectivity of α follows from the fact that $f \colon S \to C$ is a flat morphism with $f_* \mathcal{O}_S = \mathcal{O}_C$.

For the surjectivity of β, it is enough to show that $\tau(B(k)) \subset \tau(\beta(A(k)))$. Let b be a generic point of B over K (in Weil's language [15] which will be used in this section). Because the homomorphism $\tau \colon B \to J$ is K-rational, $b' := \tau(b)$ is a $K(b)$-rational point of J. Taking a generic point t of C over k, we identify the function field $K = k(C)$ with $k(t)$. Then we have

$$K' := K(b) = k(t)(b) = k(b)(t). \tag{18}$$

Let δ be a K'-rational divisor on Γ of degree 0 representing b', and let D be a $k(b)$-rational divisor on S such that $D|_\Gamma = \delta$. We have (cf. (6))

$$\varphi(\mathrm{cl}(D)) = b' = \tau(b). \tag{19}$$

Now D defines an algebraic family of divisors $\{D_u | u \in W\}$ on S parametrized by some open subset W of B; D_u is a specialization of D over the specialization $b \to u$ over k. Thus we have $\varphi(\mathrm{cl}(D_u)) = \tau(u)$. For any $u_1, u_2 \in W$, $D_{u_1} - D_{u_2}$ is algebraically equivalent to zero, which defines a point, say a, of the Picard variety $A = \mathrm{PicVar}_S$. Then

$$\tau(\beta(a)) = \varphi(\mathrm{cl}(D_{u_1} - D_{u_2})) = \tau(u_1 - u_2). \tag{20}$$

Fixing $u_2 \in W(k)$, let W' be the translate of W by $-u_2$ in B. Then the above shows that $\tau(W'(k)) \subset \tau(\beta(A(k)))$. Since $B(k)$ is obviously generated by $W'(k)$ as a group, we conclude that $\tau(B(k)) \subset \tau(\beta(A(k)))$, i.e., that $\beta\colon A \to B$ is surjective. *q.e.d*

4 Mordell–Weil lattices

In Theorem 1, we established an isomorphism of the Mordell–Weil group $M = J(K)/\tau B(k)$ with $\mathrm{NS}(S)/T$. Once this is done, we can introduce a natural pairing $\langle\ ,\ \rangle$ on M by "splitting" this isomorphism and using the intersection pairing $(\ \cdot\ )$ on $\mathrm{NS}(S)$. This simple idea has proved to be fruitful in the elliptic case ([8]) and we follow it here.

Let $N := \mathrm{NS}(S)$ and $\overline{N} := N/N_{\mathrm{tor}}$. Then $\overline{N}$ is an integral lattice with respect to the intersection pairing, and it has signature $(1, \rho - 1)$ by the Hodge index theorem, where $\rho = \rho(S)$ is the Picard number of S (cf. [3]). We can regard T as a sublattice of $\overline{N}$, since it is torsion-free by the following well-known result (see e.g., [2, Chap. 9]).

Proposition 5 *The lattice T has an orthogonal decomposition:*

$$T = U \oplus \sum_{v \in R} T_v, \tag{21}$$

where U is the rank 2 unimodular lattice spanned by $(O), F$ and T_v is the negative-definite sublattice of rank $m_v - 1$ spanned by the irreducible components $\Theta_{v,i}$ (with $i > 0$), other than the identity component, of a reducible fibre $f^{-1}(v)$ (for $v \in R$).

Let us denote by

$$\overline{\varphi}\colon M \to N/T \tag{22}$$

the map giving the isomorphism (1) of Theorem 1.

Lemma 6 *There is a unique map*

$$\varphi\colon M \longrightarrow N \otimes \mathbb{Q} \tag{23}$$

such that (a) it splits $\overline{\varphi}$ in the sense that

$$\varphi(x) \bmod T \otimes \mathbb{Q} = \overline{\varphi}(x) \ \textit{in}\ (N/T) \otimes \mathbb{Q} \quad \textit{for any}\ x \in M \tag{24}$$

and (b) the image of φ is orthogonal to T w.r.t. $(\ \cdot\ )$:

$$\varphi(x) \perp T \quad \textit{for any}\ x \in M. \tag{25}$$

Moreover, φ is a group homomorphism such that

$$\mathrm{Ker}(\varphi) = M_{\mathrm{tor}}. \tag{26}$$

Proof This is essentially the same as in the elliptic case (cf. [8, §8]). First we assume the existence of the map φ, and prove (i) its uniqueness, (ii) that φ is a group homomorphism, and (iii) $\mathrm{Ker}(\varphi) = M_{\mathrm{tor}}$.

(i) Suppose that both φ_1 and φ_2 are maps satisfying the conditions (a) and (b). Take any $x \in M$. Then, by (a), $t = \varphi_1(x) - \varphi_2(x)$ belongs to $T \otimes \mathbb{Q}$, while (b) implies $t \perp T$. Using Proposition 5, it is easy to show that $t = 0$, that is, $\varphi_1(x) = \varphi_2(x)$. Hence $\varphi_1 = \varphi_2$.

(ii) Take any $x, y \in M$ and let $z = x + y$. We show that $\varphi(x) + \varphi(y) = \varphi(z)$. Indeed, noting that $\overline{\varphi}$ is a homomorphism, we have by (a)

$$(\varphi(x) + \varphi(y)) \bmod T \otimes \mathbb{Q} = \overline{\varphi}(x) + \overline{\varphi}(y) = \overline{\varphi}(z).$$

On the other hand, $\varphi(x) + \varphi(y)$ is obviously orthogonal to T by (b). Hence the uniqueness (i) implies $\varphi(x) + \varphi(y) = \varphi(z)$, proving (ii).

(iii) Suppose that $x \in M$ belongs to $\mathrm{Ker}(\varphi)$. Then $\overline{\varphi}(x)$ is an element of N/T which is killed in $(N/T) \otimes \mathbb{Q}$. This happens precisely when $\overline{\varphi}(x) \in (N/T)_{\mathrm{tor}}$. Since $M \simeq N/T$, we conclude that $\mathrm{Ker}(\varphi) = M_{\mathrm{tor}}$.

Now we prove the existence of the map φ satisfying (a), (b). This will be done elementwise. Thus fix any $x \in M$. Take an element $\xi \in N$ such that $\xi \bmod T = \overline{\varphi}(x)$. We can choose ξ so that $(\xi \cdot F) = 0$ by using $(O), F \in T$. To satisfy the condition (a), we write $\varphi(x) = \xi + t$ for some $t \in T \otimes \mathbb{Q}$, and express t as a $\mathbb{Q}$-linear combination of the free generators $(O), F, \Theta_{v,i}$ $(v \in R, i > 0)$ of T. Then by Proposition 5, this expression has a unique solution satisfying the condition (b). Explicitly, we have

$$\begin{aligned} \varphi(x) = \xi &- (\xi \cdot (O))F \\ &+ \sum_{v \in R} (\Theta_{v,1}, \cdots, \Theta_{v,m_v-1})(-I_v^{-1})^t((\xi \cdot \Theta_{v,1}), \cdots, (\xi \cdot \Theta_{v,m_v-1})), \end{aligned} \tag{27}$$

where I_v denotes the intersection matrix of $\Theta_{v,i}$ (for $1 \le i \le m_v - 1$) and ^{t} denotes transpose. *q.e.d*

Theorem 7 *For any $P \in J(K)$, denote by $\overline{P}$ its image in $M = J(K)/\tau B(k)$. Define a symmetric bilinear form on $J(K)$ by*

$$\langle P, Q\rangle = -(\varphi(\overline{P}) \cdot \varphi(\overline{Q})) \quad (P, Q \in J(K)). \tag{28}$$

Then it defines the structure of a positive definite lattice on

$$M/M_{\mathrm{tor}} = J(K)/(J(K)_{\mathrm{tor}} + \tau B(k)) \tag{29}$$

This will be called the Mordell–Weil lattice of J/K.

The pairing $\langle\ ,\ \rangle$ is called the height pairing. The formula (27) gives an explicit formula for the height pairing. If we take a representative divisor class $\mathrm{cl}(D_P)$ of $\overline{P} \in M \simeq \mathrm{NS}(S)/T$ for each $P \in J(K)$, then it reads as

$$\langle P, Q\rangle = -(D_P \cdot D_Q) - \sum_{v\in R} \mathrm{contr}_v(P, Q) \tag{30}$$

where the local contribution term is defined by

$$\begin{aligned}\mathrm{contr}_v(P, Q) = ((D_P \cdot \Theta_{v,1}), \cdots, (D_P \cdot \Theta_{v,m_v-1}))(-I_v^{-1})&\\ \times\, {}^t((D_Q \cdot \Theta_{v,1}), \cdots, (D_Q \cdot \Theta_{v,m_v-1}))&\end{aligned} \tag{31}$$

(cf. [8]).

In the following, we assume that the K/k-trace of J/K is trivial: $B = 0$, and hence $M = J(K)$ itself is finitely generated (cf. Theorem 3 in §2). Further, for the sake of simplicity, we also assume that $\mathrm{NS}(S)$ is torsion-free.

Then the above process defines the structure of lattice on $J(K)/J(K)_{\mathrm{tor}}$, the Mordell–Weil lattice of J/K. To define the narrow Mordell–Weil lattice of J/K, let $J(K)^0$ be the subgroup of $J(K)$ which is the image of $L = T^\perp \subset \mathrm{NS}(S)$ under (1). For $P \in J(K)^0$, we obviously have $\mathrm{contr}_v(P, Q) = 0$ for any $Q \in J(K)$. In particular, we have $\langle P, P\rangle = -(D_P^2)$ by (30). Since we can take $D_P \in L$ by definition, and L is negative definite under our assumption, we obtain:

Theorem 8 *Assume that (i) the K/k-trace of J/K is trivial and (ii) $\mathrm{NS}(S)$ is torsion-free. Then $J(K)^0$ is a positive definite integral lattice, isomorphic to the opposite lattice L^- of L. It will be called the narrow Mordell–Weil lattice of J/K.*

As in the elliptic case, the Mordell–Weil lattice embeds into the dual lattice of the narrow one, and we have the following:

Theorem 9 *In addition to the assumptions (i) and (ii) above, suppose that the Néron–Severi lattice of S is unimodular, that is, $\det \mathrm{NS}(S)/(\mathrm{tor}) = \pm 1$. Then the Mordell–Weil lattice of J/K is isomorphic to the dual lattice of the narrow Mordell–Weil lattice $J(K)^0$.*

The proof is the same as in [8, §9], and is omitted. This theorem applies, for example, when S is a rational surface (see §2, Corollary 4).

5 Examples

5.1 Milnor lattice of the A_n singularities

For any positive integer $n > 1$, consider the hyperelliptic curve $\Gamma = \Gamma_\lambda$ over $K = k(t)$, defined by the equation:

$$y^2 = x^{n+1} + p_2 x^{n-1} + \cdots + p_n x + p_{n+1} + t^2, \ \lambda = (p_2, \ldots, p_{n+1}) \in k^n. \quad (32)$$

Its genus g is given by $n = 2g$ or $n = 2g+1$ according to the parity of n. For simplicity, assume that k has characteristic 0.

This equation is a familiar one in the theory of rational singularities; it defines a "semi-universal deformation" of the A_n singularity $y^2 = x^{n+1} + t^2$, with parameter λ. We are interested in comparing the Mordell–Weil lattice arising from this situation with the Milnor lattice which, for the A_n singularity, is known to be the root lattice A_n. Such a viewpoint led to rather interesting results in the framework of MWL of elliptic surfaces, covering the exceptional case E_6, E_7, E_8 (cf. [9], [10]).

Let $u_1, \ldots, u_n, u_{n+1}$ be the roots of the algebraic equation

$$\Phi(x) = \Phi_\lambda(x) = x^{n+1} + p_2 x^{n-1} + \cdots + p_n x + p_{n+1} = 0. \quad (33)$$

Corresponding to them, there are K-rational points of Γ:

$$P_i : (x = u_i, y = t) \quad \text{and} \quad P'_i : (x = u_i, y = -t) \quad \text{for } i = 1, \ldots, n+1. \quad (34)$$

(I) Assume first that $n = 2g$ is even and that the u_i are all distinct. Let $O \in \Gamma(K)$ be the unique point at infinity, and we take an embedding Γ into J sending O to the origin of J. Then the 2(2g+1) points above satisfy the following relation in $J(K)$:

$$\sum_{i=1}^{2g+1} P_i = 0, \quad P'_i = -P_i. \quad (35)$$

(For the former, look at the divisor of the function $h = y - t$ on Γ.)

Theorem 10 *With the above notation, assume that $n = 2g$ and the u_i are mutually distinct. Then the Mordell–Weil group $J(K)$ is a torsion-free Abelian group of rank $r = 2g$. More precisely, the narrow MWL $J(K)^0$ is* *isomorphic to the root lattice A_{2g}, and the full MWL $J(K)$ is isomorphic to its dual lattice A_{2g}^*:*

$$\begin{array}{ccc} J(K) & \simeq & A_{2g}^* \\ \bigcup & & \bigcup \\ J(K)^0 & \simeq & A_{2g} \end{array} \qquad \textit{index } 2g+1 \tag{36}$$

Moreover $\{P_i, P_i'\}$ correspond to the minimal vectors of A_{2g}^* (with minimal norm $2g/2g+1$); in particular, $\{P_1, \ldots, P_{2g}\}$ forms a set of free generators of $J(K)$.

The algebraic surface S associated with (32) is a rational surface, and so we can use Theorem 9. It has a unique reducible fibre at $t = \infty$, and we have

$$\operatorname{rank} T_\infty = 2g+4, \quad \det\ T_\infty = 2g+1. \tag{37}$$

The case $g = 1$ reduces to the known result (Case (A_2) in [9]); in that case, we have $T_\infty^- \simeq E_6$. For $g > 1$, we need to resolve the singularity. Omitting the details, we find that T_∞^- is not a root lattice, but its dual graph can be described as follows: first take the Dynkin graph of type D_{2g+2} and then adjoin a new vertex with norm $g+1$ to each extreme vertex of the two short branches in D_{2g+2}. The sections (P_i) (or (P_i')) pass through the irreducible component corresponding to one (or the other) of the two new vertices.

(II) Still assuming that $n = 2g$ is even, we consider the general case where $\Phi_\lambda(x)$ has multiple roots. Let

$$\Phi_\lambda(x) = (x-u_1)^{n_1+1} \cdots (x-u_l)^{n_l+1} \tag{38}$$

with $u_1, \ldots, u_l$ distinct, $n_1, \ldots, n_l \geq 0$ and $\sum_j (n_j+1) = n+1$. In this case, we have

Theorem 11 *Assume $n = 2g$, and denote by L the orthogonal complement of $T_{\text{fin}} = A_{n_1} \oplus \cdots \oplus A_{n_l}$ in A_n; let L^* be its dual lattice. Then, the narrow MWL $J(K)^0$ is isomorphic to L, and $J(K) \simeq L^* \oplus (torsion)$. In particular, $J(K)$ has rank $r = n - \sum n_j = l - 1$.*

It is easy to see that the trivial lattice T is equal to $U \oplus T_{\text{fin}} \oplus T_\infty$ where T_{fin} is as above and T_∞ is the same as in (I). By the theory of simultaneous resolution of rational singularity (see the references in [10]), the surfaces S_λ form a smooth family when a suitable change of parameter is chosen. In particular, $N = \mathrm{NS}(S_\lambda)$ is constant over such a parameter space. Then the orthogonal complement of T in N is equal to that of $T_{\rm fin}$ in A_{2g} by (I), hence the assertion follows from Theorem 9.

As a consequence, we have $r \le 2g$ where equality holds only for the case treated in (I). On the other hand, we have $r = 0$ iff $l = 1$, which corresponds to the totally degenerate case $\lambda = 0$ with $\Phi_\lambda(x) = x^{2g+1}$. In this case, we have $J(K)$ is a finite cyclic group of order $2g+1$, i.e., $J(K) \simeq A_{2g}^*/A_{2g} \simeq \mathbb{Z}/(2g+1)\mathbb{Z}$.

(III) In case $n = 2g+1$ is odd, the structure of MWL depends more on each individual parameter λ. In any case, we choose one of the two rational points over $x = \infty$ as the origin O of J.

Theorem 12 *Assume $n = 2g+1$. In the general case where there are no reducible fibres other than $f^{-1}(\infty)$, $\{P_i\}$ generate a lattice isomorphic to the standard unimodular lattice $\mathbb{Z}^{2g+2}$. The narrow MWL $J(K)^0$ is a sublattice of index $g+1$, consisting of $(x_i) \in \mathbb{Z}^{2g+2}$ satisfying $\sum x_i \equiv 0(g+1)$, and $J(K)$ is isomorphic to its dual lattice.*

Even if $\Phi_\lambda(x)$ has no multiple roots, the structure of MWL can be different from the above general case. For instance, for $\Phi_\lambda(x) = x^{n+1} - 1 (n+1 = 2g+2)$, there are two more reducible fibres over $t = \pm 1$. In this case, we can prove the following:

$$\begin{array}{ccccc} J(K) & \simeq & A_g^* \oplus A_g^* \oplus \mathbb{Z}/(g+1)\mathbb{Z} & & \\ \cup & & \cup & \text{index } (g+1)^3 & \\ J(K)^0 & \simeq & A_g \oplus A_g & & \end{array} \tag{39}$$

Here the torsion part is generated by Q which is the other point at $x = \infty$ other than the chosen origin O. We have $P_i + P_i' = Q$ for all i. The $g+1$ points P_i corresponding to the $g+1$ roots of $x^{g+1} = 1$ (or $x^{g+1} = -1$) generate a sublattice of $J(K)$ isomorphic to A_g^*.

5.2 The Jacobian of a pencil of hyperplane sections of a regular surface

The following result was announced in [12]:

Theorem 13 *Let X be a regular surface such that* $\mathrm{NS}(X)$ *is torsion-free and suppose that it is embedded as a surface of degree d in a projective space $\mathbf{P}^N$. Let $\{\Gamma_t | t \in \mathbf{P}^1\}$ be a linear pencil of hyperplane sections of X such that every member is irreducible. Let Γ be the generic member of this pencil and let J be* *its Jacobian variety defined over the rational function field* $K = k(t)$. *Then* $J(K)$ *is a positive definite integral lattice of rank*

$$r = \rho(X) + d - 2$$

whose determinant is equal to $|\det \mathrm{NS}(X)|$.

Proof By Bezout's theorem, the linear pencil has d base points. By blowing up X at these points, we obtain a regular surface S with a fibration $f : S \to \mathbb{P}^1$ whose generic fibre is Γ and which has no reducible fibres. We have d sections arising from the base points; choose one of them as the zero section O. The Néron–Severi group $\mathrm{NS}(S)$ is torsion-free and has rank $\rho(S) = \rho(X) + d$, and the Néron–Severi lattice $\mathrm{NS}(S)$ has $|\det \mathrm{NS}(S)| = |\det \mathrm{NS}(X)|$. The trivial lattice T is equal to the rank 2 unimodular lattice U generated by (O) and F (a fibre), and we have $\mathrm{NS}(S) = U \oplus L$ with $L = T^\perp$. Therefore, by Theorem 3 (cf. Cor. 4) in §2 and Theorem 8 above, we have $J(K) = J(K)^0 \simeq L^-$. It has rank $\rho(S) - 2$ and determinant $\det L = \det \mathrm{NS}(S)$ up to sign. *q.e.d*

As noted in [12], an example is given by a linear pencil of degree m curves in a projective plane all of whose members are irreducible. More precisely, take X to be the isomorphic image of the projective plane $\mathbb{P}^2$ under the embedding given by the complete linear system $|mH|$ (H: a line). In this case, we have $d = m^2$ and $J(K)$ is a unimodular lattice of rank $r = m^2 - 1$. This gives a generalization of Manin–Shafarevich theorem for $m = 3$ saying that, if we choose one of the base points as the origin, the remaining $r = m^2 - 1$ points are linearly independent and generate a subgroup of finite index m in $J(K)$.

References

[1] Artin, M., Winters, G.: Degenerate fibres and stable reduction of curves, Topology 10, 373–383 (1971).

[2] Bosch, S., Lütkebohmert, W., Raynaud, M.: Néron models, Springer-Verlag (1990).

[3] Hartshorne, R.: Algebraic geometry, Springer-Verlag (1977).

[4] Lang, S.: Fundamentals of Diophantine geometry, Springer-Verlag (1983).

[5] Lichtenbaum, S.: Curves over discrete valuation rings, Am. J. Math. 90, 380–404 (1968).

[6] Raynaud, M.: Spécialisation du foncteur de Picard, Publ. Math. IHES 38, 27–76 (1970).

[7] Shafarevich, I.R.: Lectures on minimal models and birational transformations of two dimensional schemes, Tata Inst. Fund. Res. (1966).

[8] Shioda, T.: On the Mordell–Weil lattices, Comment. Math. Univ. St. Pauli 39, 211–240 (1990).

[9] — : Construction of elliptic curves with high rank via the invariants of the Weyl groups, J. Math. Soc. Japan 43, 673–719 (1991).

[10] — : Mordell–Weil lattices of type E_8 and deformation of singularities, in: Lecture Notes in Math. 1468, 177–202 (1991).

[11] — : Mordell–Weil lattices for higher genus fibration, Proc. Japan Acad. 68A, 247–250 (1992).

[12] — : Generalization of a theorem of Manin-Shafarevich, Proc. Japan Acad. 69A, 10–12 (1993).

[13] — : Constructing curves with high rank via symmetry (to appear).

[14] Tate, J. : On the conjectures of Birch and Swinnerton-Dyer and a geometric analog, Sém. Bourbaki 1965/66, $n^0$306; Dix exposés sur la cohomologie des schémas, Masson & CIE; North-Holland Pub. Co. (1968).

[15] Weil, A. : Foundation of Algebraic Geometry, 2nd ed., AMS (1962).

[16] — : Sur les critères d'équivalence en géométrie algébrique, Math. Ann. 128, 95–127 (1954); Collected Papers, II, Springer, 127–159 (1980).

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