The Bogomolov–Pantev resolution, an expository account

Kapil H. Paranjape*

Statement of the result

Bogomolov and Pantev [3] have recently discovered a rather elegant geometric proof of the weak Hironaka theorem on resolution of singularities:

Theorem 0.1 *Let X be a projective variety and Z a proper Zariski closed subset of X. There is a projective birational map $\varepsilon\colon \widetilde{X} \to X$ such that $\widetilde{X}$ is smooth and the set theoretic inverse image $\varepsilon^{-1}(Z)$ is a divisor with simple normal crossings.*

Before their work, and that of Abramovich and de Jong [1] (appearing at roughly the same time) the only proof of this theorem was as a corollary of the famous result of Hironaka [5]. These new proofs were inspired by the recent work of de Jong [6], which Bogomolov and Pantev combine with a beautiful idea of Belyi [2] "simplifying" the ramification locus of a covering of $\mathbb{P}^1$ by successively folding up the $\mathbb{P}^1$ onto itself, over a fixed base. This latter step unfortunately only works in characteristic zero, limiting the scope of the argument (Abramovich and de Jong's paper gives some results even in characteristic p). Hence, we work over the field of complex numbers; the argument also works (with suitable modifications about rationality) over any field of characteristic zero.

The outline of the argument we follow is the same as that of the paper of Bogomolov and Pantev; however we offer different (and we hope simpler) proofs of the corresponding lemmas. To begin with, their argument using Grassmannians is replaced by an application of Noether normalisation in Section 1. Belyi's argument to reduce the degree of individual components of the ramification locus is presented in purely algebraic form in Section 1, Lemmas 1.3 and 1.4. The presentation of Bogomolov and Pantev refers to

*The author thanks the University of Warwick for hospitality during the period when this work was done.

"semi-stable families of pointed curves of genus 0"; the precise result required from that theory is proved here by means of blowups in Section 2. Finally we also give a summary (in Section 3) of the desingularisation of toroidal embeddings which is used by both the papers [3, 1]. To summarise, the aim of this account is to give all the details of the argument, so that it should be accessible to anyone with a basic knowledge of algebraic geometry (as contained for example in Mumford's book [8]).

Acknowledgments

Discussions with Lawrence Ein and D. S. Nagaraj have greatly helped me in understanding this proof better. V. S. Sunder patiently heard out my ramblings on "convex" barycentric subdivisions. M. V. Nori pointed out a flaw in an earlier argument in the section on genus 0 fibrations. I thank the referee for reading the paper very carefully and providing a number of suggestions. Of course, many thanks to Fedor Bogomolov and Tony Pantev for their beautiful proof!

Trivial reductions

We begin the proof with some trivial reductions.

0.1 We may assume that X is normal.

0.2 Any normal curve is smooth and any proper closed subset of a smooth curve is a simple normal crossing divisor (!). Hence the result is true in dimension 1.

0.3 We prove the theorem by induction on the dimension of X. Thus we may assume that $n = \dim X > 1$ and that the result holds for all varieties of smaller dimension than X.

0.4 We can blow up X along Z if necessary, and so we may assume that Z is of codimension 1 in X.

1 $\mathbb{P}^1$-bundles

We will prove the following:

Claim 1.1 *After replacing X by X', where the latter is the blow up of X at a finite set of smooth points we have:*

1. *There is a finite surjective map $f\colon X \to \mathbb{P}_Y(F)$, where Y is smooth and F is a rank 2 vector bundle on Y.*

2. *The branch locus of f is contained in the union $B = \bigcup_{i=1}^{n} s_i(Y)$ of a finite number of sections $s_i\colon Y \to \mathbb{P}_Y(F)$ of the $\mathbb{P}^1$-bundle $\mathbb{P}_Y(F)$.*

3. *The image $f(Z)$ of Z under f is a divisor contained in B.*

Lemma 1.2 *After replacing X by its blowup at a suitable finite set of smooth points, we have the following situation. There is a smooth variety Y, a line bundle L and a finite surjective map $g\colon X \to Q = \mathbb{P}_Y(\mathcal{O} \oplus L)$ such that if E denotes the section corresponding to the quotient homomorphism $\mathcal{O} \oplus L \to \mathcal{O}$, then there is a divisor $B \subset Q$ disjoint from E and containing $g(Z)$ and the branch locus of g.*

Proof This is essentially just Noether normalisation. Let $X \hookrightarrow \mathbb{P}^N$ be a projective embedding of X and $L = \mathbb{P}^{N-n-1}$ a linear subspace in $\mathbb{P}^N$ that does not meet X, where $n = \dim X$. The projection from L gives a finite map $X \to \mathbb{P}^n$. Let B be the hypersurface in $\mathbb{P}^n$ that contains the image of Z and the branch locus of the map. Let p be a point of $\mathbb{P}^n$ not on B. Replace X by its blowup at the points lying over p (which are all smooth) and let $Y = \mathbb{P}^{n-1}$ with $L = \mathcal{O}(1)$. The blowup of $\mathbb{P}^n$ at p is naturally isomorphic to $\mathbb{P}_Y(\mathcal{O} \oplus L)$ and the resulting morphism from X to this $\mathbb{P}^1$-bundle has the required form. $\square$

Now B is a divisor in the $\mathbb{P}^1$-bundle $Q \to Y$ that does not meet a section E. Thus the projection $B \to Y$ is finite and flat. In fact we have:

Lemma 1.3 *In the above situation, if $\mathcal{O}_Q(1)$ denotes the universal quotient bundle, then $\mathcal{O}_Q(B) = \mathcal{O}_Q(b)$ where b is the degree of the map $B \to Y$.*

Proof Consider the line bundle $\mathcal{O}_Q(B) \otimes \mathcal{O}_Q(-b)$. Since this line bundle is trivial on the fibres of $Q \to Y$, it is the pullback of a line bundle from Y. But it restricts to the trivial bundle on E, which is a section of $Q \to Y$. Hence it is the trivial line bundle. $\square$

Set $Q = \mathbb{P}_Y(\mathcal{O} \oplus L)$, and let B be a divisor in Q which is finite over Y; assume moreover that $B = B_1 \cup \bigcup_{i=1}^{r} s_i(Y)$, where s_i are sections, and B_1 does not meet E. Now let $d(B)$ denote the maximum degree over Y of any irreducible component of B_1 and $m(B)$ the number of components of this degree. We wish to construct a new map $X \to Q' = \mathbb{P}_Y(\mathcal{O} \oplus L^N)$ for which B_1 is empty. Thus we may assume that $d(B) > 1$ and $m(B) > 0$. The following lemma shows that we can arrange for at least one of these numbers to drop, and that completes the required inductive step.

Lemma 1.4 *There is a map $h\colon Q \to Q' = \mathbb{P}_Y(\mathcal{O} \oplus L^d)$ such that if B' is the union of $h(B)$ and the branch locus of h then $(d(B), m(B)) > (d(B'), m(B'))$ in the lexicographic ordering.*

Proof Let A be an irreducible component of B_1 of degree $d = d(B)$ and consider the two maps $\mathcal{O}_Q \to \mathcal{O}_Q(d)$ and $L_Q^d \to \mathcal{O}_Q(d)$ given by A and $d \cdot E$. Since these two divisors do not meet, the direct sum $\mathcal{O}_Q \oplus L_Q^d \to \mathcal{O}_Q(d)$ is surjective. Thus, by the universal property of Q' we obtain a morphism $Q \to Q'$ which on every fibre is a map $\mathbb{P}^1 \to \mathbb{P}^1$ of degree d. Its ramification divisor R has the form $R = (d-1)E + R'$ for some divisor R' that does not meet E. Moreover, the Hurwitz formula on the general fibre $\mathbb{P}^1$ (equivalently, computing the canonical divisors) gives $\mathcal{O}_Q(R') = \mathcal{O}_Q(d-1)$. Thus we have the required result. $\square$

We compose the morphism obtained from Lemma 1.2 with a succession of morphisms obtained from Lemma 1.4. By the latter lemma the pair $(d(B), m(B))$ can be reduced until eventually B_1 becomes empty and thus the composite morphism f is as stated in Claim 1.1.

2 Genus 0 fibrations

We will prove the following:

Claim 2.1 *We can replace X by a blow up X' so that:*

1. *There is a finite map $f\colon X \to W$ with W smooth.*
2. *The union of the image $f(Z)$ of Z and the branch locus of f is contained in a divisor D with simple normal crossings (or strict normal crossings).*

For completeness, we recall the definition of a simple normal crossing divisor D in a smooth projective variety. If $D = \bigcup_{i=1}^n D_i$ is the decomposition of the divisor into irreducible components, then for each $I \subset \{1, \dots, n\}$ the scheme theoretic intersection $D_I = \bigcap_{i \in I} D_i$ is reduced and smooth of codimension equal to the cardinality $\#I$ of the set I.

Write $g\colon X \to P = \mathbb{P}_Y(F)$ for the map constructed in Section 1 and let $\{s_i\}_{i=1}^m$ be sections of $P \to Y$ such that the union $\bigcup_{i=1}^m s_i(Y)$ contains $g(Z)$ and the branch locus of g. Let

$$Z_1 = \bigcup_{1 \le i < j \le m} p_Y(s_i(Y) \cap s_j(Y))$$

be the divisor in Y obtained as the image of the pairwise intersections of the sections. We apply the induction hypothesis 0.3 to obtain a map $\varepsilon_Y\colon \widetilde{Y} \to Y$ such that the inverse image $Z_1 = \varepsilon_Y^{-1}(Z)$ is a divisor with simple normal crossings in $\widetilde{Y}$. Replacing Y by $\widetilde{Y}$ and P by its pullback, we obtain a configuration with the following properties (A):

(1) $p\colon P \to Y$ is a flat morphism of smooth varieties whose reduced fibres are trees of projective lines $\mathbb{P}^1$ (in other words, p is a *genus* 0 *fibration*);

(2) we have a finite collection $\{s_i\}_{i=1}^m$ of sections of p such that

$$\bigcup_{1\leq i<j\leq m} p_Y(s_i(Y)\cap s_j(Y)) \subset Z_1$$

where Z_1 is a divisor with simple normal crossings in Y;

(3) the morphism p is smooth outside $p^{-1}(Z_1)$ and the latter is a divisor with simple normal crossings.

We wish to perform a succession of blowups resulting in a configuration still having the same properties, but with the sections disjoint.

For each component C of Z_1 and each section s_i, the image $C' = s_i(C) \subset P$ is a codimension 2 subvariety. Write $n(C')$ for the number of j such that $s_j(C) = C'$, and n_P for the maximum of such $n(C')$.

Lemma 2.2 *Let C' be so chosen that $n(C') = n_P$. Then for any j, either $s_j(C) = C'$ or $s_j(Y) \cap C' = \emptyset$.*

Proof Suppose j is such that $s_j(Y)\cap C'$ is a proper nonempty subset of C', and let C'' be an irreducible component of this intersection. Choose i so that $s_i(C) = C'$, let D' be a component of $s_i(Y)\cap s_j(Y)$ containing C'', and denote by D the image $p(D')$. On the one hand we have

$$s_j(C)\cap C' \subset s_j(Y)\cap C' \subsetneq C' = s_i(C),$$

while $D' = s_i(D) = s_j(D)$. Thus we see that D and C are different components of Z_1 containing $p(C'')$. But $C''\subset C'$ has codimension 3 in P, and p restricts to an isomorphism from C' to C. Thus $p(C'')$ is of codimension 2 in Y; hence it is an irreducible component of $D\cap C$. Since Z_1 is a simple normal crossing divisor we see that D and C are the *only* components of Z_1 that contain $p(C'')$.

Take any k such that $s_k(C) = C'$; then $s_k(Y)\cap s_j(Y)$ contains a component E' which contains C''. Then by the above reasoning we must have $D = p(E')$, and thus

$$D' = s_j(D) = s_j(p(E')) = E' = s_k(p(E')) = s_k(D)$$

But then we get $s_k(D) = D'$ for all k such that $s_k(C) = C'$, and in addition, $s_j(D) = D'$ so that $n(D') = n(C') + 1$. This contradicts the maximality of $n(C')$. $\square$

Together with the preceding lemma, the following result shows that blowing up C' with $n(C') = n_P$ again leads to a configuration of type (A).

Lemma 2.3 *Let $p\colon P \to Y$ be a genus 0 fibration which is smooth outside Z_1 so that the inverse image $p^{-1}(Z_1)$ of Z_1 is a simple normal crossing divisor in P. For $s\colon Y \to P$ a section and C a component of Z_1, consider the blowup $P' \to P$ along $s(C)$.*

Then $p'\colon P' \to Y$ is again a genus 0 fibration which is smooth outside Z_1 so that the inverse image ${p'}^{-1}(Z_1)$ of Z_1 is a simple normal crossing divisor, and the birational transform (or strict transform) of $s(Y)$ gives a section of P' over Y. Moreover, if t is a section of p which is disjoint from $s(C)$ then the birational transform of $t(Y)$ continues to be a section of p'.

Proof The first two statements and the final statement about P' are obvious. The union of $s(Y)$ with $p^{-1}(Z_1)$ is a simple normal crossing divisor since s is a section. Moreover, $s(C)$ is the locus of intersection of $s(Y)$ and one of the components of $p^{-1}(C)$. Thus the blowup preserves the property of being a simple normal crossing divisor.

Suppose that $n_P > 1$ and let N_P be the number of C' attaining this maximum. For each such C' and each pair i, j such that $s_i(C) = s_j(C) = C'$ let $m(i, j, C')$ denote the multiplicity of intersection of $s_i(Y)$ and $s_j(Y)$ along C'. We are in the situation of the following lemma

Lemma 2.4 *Let P be a smooth variety, D_1 and D_2 smooth divisors meeting along a smooth codimension 2 locus B with multiplicity $m > 0$. Let P' be the blowup of P along C', and let D_1' and D_2' be the birational transforms of D_1 and D_2 respectively. Then the multiplicity of intersection of D_1' and D_2' is $m-1$ along a codimension two locus B' lying over B.*

Proof In a neighbourhood of the generic point of B, the given condition can be written as follows

$$\mathcal{O}_P(D_1) \otimes \mathcal{O}_{D_2} = \mathcal{O}_{D_2}(m \cdot B)$$

Let E denote the exceptional divisor of the blowup $\varepsilon\colon P' \to P$ and let B' be the intersection $D_2' \cap E$; this is a section of the $\mathbb{P}^1$-bundle $E \to B$. Now the birational transform of D_1 represents the divisor class $\varepsilon^*(D_1) - E$ and thus we see that

$$\mathcal{O}_{P'}(D_1') \otimes \mathcal{O}_{D_2'} = (\mathcal{O}_P(D_1) \otimes \mathcal{O}_{D_2'}) \otimes \mathcal{O}_{D_2'}(-B') = \mathcal{O}_{D_2'}((m-1) \cdot B'),$$

which proves the result.

From this lemma, we see that the multiplicity of intersection of the birational transforms of $s_i(Y)$ and $s_j(Y)$ is $m(i, j, C') - 1$. If this becomes zero then $n(C')$ drops, hence either N_P decreases or n_P does so. This completes the argument by induction since all these numbers are positive and we wish to obtain the situation where $n_P = 1$.

Now we replace X by the normalisation of the fibre product $X \times_{\mathbb{P}_Y(F)} P$, where $P \to Y$ is the configuration of type (A) with $n_P = 1$ obtained above. We then have a finite morphism $h\colon X \to P$. The image $h(Z)$ of Z and the branch locus of h are contained in the simple normal crossing divisor consisting of the finitely many disjoint sections of the configuration (A) and $p^{-1}(Z_1)$. This proves Claim 2.1.

3 Toric singularities

As a last step we must prove the result in the following situation. There is a finite map $f\colon X \to W$ with W smooth and X normal and $D = \bigcup_{i=1}^n D_i$ a divisor with simple normal crossings so that f is étale outside D; moreover Z is a union of (some of) the components of $f^{-1}(D)$. So we need to construct a birational morphism $\varepsilon\colon \widetilde{X} \to X$ so that $\widetilde{X}$ is smooth and $\varepsilon^{-1}f^{-1}(D)$ is a divisor with simple normal crossings in $\widetilde{X}$.

We will show (see Lemma 3.1 below) that the inclusion of $X \setminus f^{-1}(D)$ in X is a strict toroidal embedding in the sense of [7], where the desingularisation problem for such embeddings has been studied and solved. For the sake of completeness we also give a brief summary of their method. Many statements below are given without detailed proofs—readers are invited to complete the arguments on their own or look for proofs in one of the books on toric geometry such as Oda [9] or Fulton [4].

3.1 Affine toric singularities

Let us first consider the simple situation where $W = \mathbb{A}^n$ and D is a union of coordinate hyperplanes. In this case $W \setminus D$ is isomorphic to $\mathbb{G}_m^r \times \mathbb{A}^{n-r}$ where r is the number of components of D. Hence (because we are working over the complex numbers $\mathbb{C}$) its fundamental group is a product of infinite cyclic groups and the finite cover $X \setminus f^{-1}(D)$ is also isomorphic to $\mathbb{G}_m^r \times \mathbb{A}^{n-r}$. In fact, the homomorphism f^* on rings has the form

$$\begin{aligned} \mathbb{C}[z_1, z_1^{-1}, \ldots, z_r, z_r^{-1}, z_{r+1}, \ldots, z_n] &\to \mathbb{C}[t_1, t_1^{-1}, \ldots, t_r, t_r^{-1}, t_{r+1}, \ldots, t_n] \\ z_i &\mapsto \begin{cases} m_i & \text{for } i \leq r \\ t_i & \text{for } i > r \end{cases} \end{aligned} \tag{1}$$

where the m_i are certain monomials in $t_1, \ldots, t_r$ (with negative powers allowed). The natural action of the torus $\mathbb{G}_m^n$ on $X \setminus f^{-1}(D)$ then descends to an action via f making this map equivariant. Moreover, since X is the normalisation of W in $X \setminus f^{-1}(D)$, the action extends to X. Since the map f is equivariant there are only finitely many orbits for the torus action on X; in other words, X is an (affine) *toric variety.*

An explicit description of X can be given as follows. Let M be the free Abelian group of all monomials in the variables $t_1, \dots, t_r$. Let M^+ be the *saturated* submonoid of M generated by the m_i. Then $X = \operatorname{Spec} \mathbb{C}[M^+] \times \mathbb{A}^{n-r}$. For future reference, we note that there is a unique closed orbit in X which maps isomorphically to the closed orbit $0 \times \mathbb{A}^{n-r}$ in W. Moreover, M is the group of Cartier divisors supported on $f^{-1}(D)$ and M^+ is the submonoid of effective Cartier divisors.

Let N be the dual Abelian group to M and

$$N^+ = \{n \mid n(m) \geq 0 \text{ for all } m \in M^+\}$$

the "dual" monoid to M^+. Then N^+ is a finitely generated saturated monoid in N (like M^+ in M). Let σ be any finitely generated submonoid of N^+ and

$$M^\sigma = \{m \mid n(m) \geq 0 \text{ for all } n \in N^+\}$$

the dual monoid in M (which contains M^+). Then $X(\sigma) = \operatorname{Spec} \mathbb{C}[M^\sigma] \times \mathbb{A}^{n-r}$ is an affine toric variety and the natural morphism $X(\sigma) \to X$ is birational and equivariant for the torus action. Moreover, we see that $X(\sigma)$ is nonsingular and the pullback to $X(\sigma)$ of D is a simple normal crossing divisor if and only if the monoid σ is generated over nonnegative integers by a (sub-)basis of N; such a monoid is called *simplicial.*

Let $\Sigma = \{\sigma_i\}_{i=1}^n$ be a collection of finitely generated saturated submonoids of N^+ which give a subdivision of N^+; i.e., N^+ is the union of all the σ_i and for any pair σ_i, σ_j their intersection is a σ_k for some k. We then obtain a collection of equivariant birational morphisms $X_i = X(\sigma_i) \to X$ so that if $\sigma_j \subset \sigma_i$ then $X_j \subset X_i$ in a natural way. Thus we can patch together the X_i to obtain $X_\Sigma \to X$ which is birational and equivariant (but X_Σ need not be affine any more). Moreover, the condition that the σ_i cover N^+ implies that $X_\Sigma \to X$ is proper.

Thus to obtain a desingularisation of X it is enough to find a subdivision of N^+ consisting entirely of simplicial monoids; an easy enough combinatorial problem solved by barycentric subdivision. The intrepid reader is warned that proving that the resulting morphism is projective is a little intricate since an arbitrary simplicial subdivision *need not* result in a projective morphism; however, the barycentric subdivision does yield a projective morphism.

3.2 Local toric singularities

Now we examine the general case locally. Let $x \in X$ be any point such that $w = f(x)$ lies in D. There is an analytic neighbourhood U of w in W

and coordinates on U so that $D \cap U$ is given by the vanishing of a product of coordinate functions; by further shrinking U we can assume that U is a polydisk in these coordinates. Let V be the component of $f^{-1}(U)$ which contains x. The normality of X implies the normality of the analytic space V. Hence, the open subset $V \setminus f^{-1}(D)$ is connected. So it is a topological cover of $U \setminus D$. The coordinate functions give an inclusion $U \hookrightarrow W' = \mathbb{A}^n$ so that $D \cap U$ is the restriction to U of a union of coordinate hyperplanes in W'. The resulting inclusion $U \setminus D \hookrightarrow \mathbb{G}_m^r \times \mathbb{A}^{n-r}$ induces an isomorphism of fundamental groups. Thus there is a Cartesian square in which the horizontal maps are inclusions:

$$\begin{array}{ccc} V \setminus f^{-1}(D) & \longrightarrow & \mathbb{G}_m^r \times \mathbb{A}^{n-r} \\ f \downarrow & & \downarrow f' \\ U \setminus D & \longrightarrow & \mathbb{G}_m^r \times \mathbb{A}^{n-r} \end{array}$$

and f' is a covering of the form (1) above. Let X' denote the normalisation of W' in the cover f'. By the normality of X (and hence the normality of V as an analytic space) it follows that we obtain a commutative diagram

$$\begin{array}{ccc} V & \longrightarrow & X' \\ f \downarrow & & \downarrow f' \\ U & \longrightarrow & W' \end{array}$$

so that V is isomorphic to an analytic open neighbourhood of a point in the toric variety X'. By means of the two diagrams above we can carry over the desingularisation of 3.1 to the local analytic space V.

3.3 General toroidal embeddings

The local description 3.2 can be repeated in a suitable neighbourhood of any point $x \in X$. This shows that the inclusion of $X \setminus f^{-1}(D)$ in X is a *toroidal embedding*. The desingularisations obtained locally need to be constructed in a coherent manner so that they "patch up".

Let us stratify W by connected components of intersections of the form

$$(D_{i_1} \cap \cdots \cap D_{i_r}) \setminus \bigcup_{j \neq i_s} D_j. \tag{2}$$

If U and V are as in 3.2, then there is a unique stratum S of W so that $S \cap U$ is closed. Under the inclusions of U in W' and V in X' of 3.2 the strata correspond to the orbits of the torus action. As we have noted in

3.1 the unique closed strata in U and V then become isomorphic. Thus, if $T = f^{-1}(S)$ then $T \cap V$ is closed in V and the morphism $T \cap V \to S \cap U$ is an isomorphism; indeed $T \cap V$ and $S \cap U$ are the restrictions of the closed orbits in X' and W' respectively. Thus we see that if S is *any* stratum in W and T a connected component of $f^{-1}(S)$ then $T \to S$ is étale and proper. We thus stratify X by connected components of the inverse images of strata in W under f. Let $\{T_a\}_{a\in A}$ denote this stratification; by abuse of notation we define $S_a = f(T_a)$.

Let $E = \bigcup_{j=1}^{m} E_j$ be the decomposition into irreducible components of the inverse image $f^{-1}(D)$ of D. Then there is a function $i\colon \{1,\dots,m\} \to \{1,\dots,n\}$ so that $f(E_j) = D_{i(j)}$ for all j. For any stratum T_a let X^a denote the complement in X of all those E_j that do not meet T_a. Similarly, we denote by W^a the complement in W of all D_i that do not meet S_a. The morphism f clearly maps X^a to W^a.

Lemma 3.1 *Let M_a be the Abelian group of all Cartier divisors in X^a with support in $E \cap X^a$; let M_a^+ be the submonoid consisting of effective Cartier divisors. Then M_a has rank $r_a = \operatorname{codim}_X T_a$. The distinct analytic components of E in a neighbourhood of any point x of X are precisely the algebraic components; i.e., $X \setminus E \hookrightarrow X$ is a* strict *toroidal embedding.*

Proof Let M'_a be the group of Cartier divisors on W^a with support on $D \cap W^a$. Clearly, this is the free Abelian group on those D_i which contain S_a. Since S_a is a connected component of an intersection of the type (2), there are exactly $r_a = \operatorname{codim}_W S_a$ of such D_i. The pullback homomorphism makes M'_a a subgroup of M_a. On the other hand, consider an analytic neighbourhood $V = f^{-1}(U)$ of some point $x \in T_a$ as in 3.2. If M''_x denotes the group of all Cartier divisors on V supported on $E \cap V$; then we have noted in 3.1 that M''_x is a free Abelian group of rank r_a. Moreover, M_a is included as a subgroup of M''_x under the restriction from X_a to V.

It follows that the homomorphism $M_a \to M''_x$ has finite cokernel; but then the normality of X means that it is surjective. In particular, we see that the distinct analytic components of $E \cap V$ correspond to distinct algebraic components of E. This concludes the proof.

Let M_a^+ denote the monoid of effective divisors in M_a; under the isomorphism $M_a \to M''_x = M$ this maps isomorphically onto the submonoid M^+ considered in 3.1. Thus X_a is smooth (and the divisor E is a simple normal crossing divisor) if and only if M_a^+ is simplicial. If T_b lies in the closure of T_a then X_a is an open subset of X_b. This induces by restriction a (surjective) homomorphism $M_b \to M_a$ which further restricts to a surjection $M_b^+ \to M_a^+$. Thus we have a (finite) projective system of monoids.

Let N_a and N_a^+ be the dual objects as defined in 3.1. These form a finite injective system of monoids. By a compatible family $\mathcal{S}$ of subdivisions we mean a subdivision Σ_a of N_a^+ for each a so that the subdivision Σ_b restricts to the subdivision Σ_a on the submonoid N_a^+ of N_b^+. We then obtain a proper birational morphism $X_{\Sigma_b} \to X_b$ for each b which restricts to $X_{\Sigma_a} \to X_a$ on the open subset X_a of X_b. Thus, we see that any such compatible family leads to a proper birational morphism $X_{\mathcal{S}} \to X$.

Thus, in order to desingularise, we have to find a compatible family of subdivisions so that each of the new monoids is simplicial. This is achieved by the barycentric subdivision. As seen earlier, this ensures that the morphism $X_{\mathcal{S}} \to X$ is locally projective. Since X is projective, this morphism is indeed projective.

References

[1] D. Abramovich and A. J. de Jong, *Smoothness, semistability and toroidal geometry*, `alg-geom/9603018`, 1996.

[2] G. V. Belyi, *Galois extensions of a maximal cyclotomic field*, Izv. Akad. Nauk SSSR Ser. Math. **43** (1979), no. 2, 267–276.

[3] F. Bogomolov and T. Pantev, *Weak Hironaka theorem*, `alg-geom/9603019`, 1996.

[4] W. Fulton, *Introduction to Toric varieties*, Annals of Math Studies, Vol. 151, Princeton University Press, Princeton, USA, 1993.

[5] H. Hironaka, *Resolution of singularities of an algebraic variety over a field of characteristic zero: I, II*, Ann. of Math.(2) **79** (1964), 109–326.

[6] A. J. de Jong, *Smoothness, semistability and alterations*, Publ. Math. IHES **83** (1996), 51–93.

[7] G. Kempf, F. Knudsen, D. Mumford, and B. Saint-Donat, Lecture Notes in Mathematics, no. 339, Springer-Verlag, Berlin-Heidelberg-New York, 1973.

[8] D. Mumford, *The Red book of varieties and schemes*, Narosa Publishers, New Delhi, India, 1996, Indian edition.

[9] T. Oda, *Lectures on Torus Embeddings and applications*, TIFR Lecture Notes, TIFR, Bombay, India, 1982.

Kapil H. Paranjape,
Institute of Mathematical Sciences,
CIT Campus, Tharamani, Chennai
600 113, INDIA.
e-mail: kapil@imsc.ernet.in