On symplectic invariants of algebraic varieties coming from crepant contractions

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1 Introduction

Let X be a smooth projective variety over the complex numbers; in standard notation, $N_1(X)$ denotes the finite dimensional vector space of 1-cycles on X modulo numerical equivalence. $N_1(X)$ contains the cone $\mathrm{NE}(X)$ of effective 1-cycles, generated by the classes of curves in X. In some geometric situations, the effective cone of X provides important deformation theoretic invariants of the manifold. For example, Wisniewski [Ws 2] observed that the halfcone $\mathrm{NE}(X) \cap (K_X z < 0)$ is invariant in smooth families of projective varieties. Recently, however, Ruan has used the theory of Gromov–Witten invariants to show that, at least in dimension two and three, Mori extremal rays provide invariants of algebraic manifolds in a stronger sense [Ru]. To describe his results, let Y denote a complete C^∞ manifold of real dimension $2n$, and let J_0 and J_1 be integrable complex structures on Y such that $Y_0 = (Y, J_0)$ and $Y_1 = (Y, J_1)$ are projective varieties. Then we say that Y_0 and Y_1 are *symplectically deformation equivalent* if there exists a family ω_t of symplectic forms on Y such that ω_0 is a Hodge form on Y_0 (that is, $i/2\pi$ times the curvature form of an ample line bundle) and ω_1 a Hodge form on Y_1. There is a family of almost complex structures J_t joining J_0 and J_1, such that J_t is compatible with ω_t for every t. If $n \leq 3$, Ruan proved that every Mori extremal ray has a nonzero Gromov–Witten invariant, and so is common to Y_0 and Y_1. Thus Mori extremal rays are symplectic deformation invariants of low dimensional projective manifolds.

In a related direction, Wilson [Wl 1] proved that the nef cone of a Calabi–Yau 3-fold is locally constant in its Kuranishi family if and only if X contains no quasi-ruled surface over an elliptic curve (see [Wl 2] for definitions). In [P 1], we proved a similar result for quasi-Fano 3-folds, that is, 3-dimensional projective manifolds for which $-K_X$ is nef and big: if X is a quasi-Fano 3-fold, then the nef cone of X is locally constant in the Kuranishi family if and only if X contains no quasi-ruled surface over a smooth rational curve, such that the canonical divisor K_X is trivial along the fibres, and descends to the canonical divisor of the base curve. The results of Ruan described above then lead to the question of the symplectic deformation invariance of the effective cone of these varieties. This problem was dealt with for Calabi–Yau 3-folds by Wilson in [Wl 2], where it was shown that two Calabi–Yau 3-folds that are symplectically deformation equivalent have the same nef cone unless one of them contains a quasi-ruled surface over an elliptic curve. Similarly, for quasi-Fano 3-folds, the effective cone is a symplectic invariant unless X contains a surface as above. More precisely, in this case, the effective cone of X is finite rational polyhedral, and each edge (extremal ray) has an associated nonzero Gromov–Witten invariant, except when the associated contraction φ is crepant, that is, the canonical divisor is φ-trivial, and the exceptional locus is a quasi-ruled surface over a smooth rational curve C, such that K_X descends to the canonical divisor on C. This statement follows from the arguments of [P 2].

In [P 2], we applied these methods to associate symplectic invariants to crepant extremal rays of 3-folds with K_X nef and big. For such 3-folds, the face $\mathrm{NE}(X) \cap (K_X z = 0)$ of the effective cone is finite rational polyhedral, and again, each edge of this part of the cone (crepant extremal ray) has an associated nonzero Gromov–Witten invariant, unless something special happens. Namely, the associated crepant contraction (see [Rd] for terminology) should have as its exceptional locus a quasi-ruled surface over a smooth curve C, with the property that the canonical divisor K_X descends to a divisor of degree $2g-2$ on C, where g is the genus of C. In this case, the integer $d = \deg(\omega_C \otimes \omega_X^{-1})$ is the number of rational curves in the fibres of φ that deform with the almost complex structure. If the surface itself does not generically deform in the Kuranishi family, then (morally) this is the number of smooth rational curves in the exceptional locus of φ in a generic holomorphic deformation of X. This is illustrated by the examples of [P 1] and by the following.

Example 1.1 An element $t \in H^1(\Omega^1_{\mathbb{P}^1}) \cong \mathbb{C}$ corresponds to a short exact sequence $0 \to \mathcal{O}_{\mathbb{P}^1} \to \mathcal{F}_t \to \mathcal{O}_{\mathbb{P}^1}(2) \to 0$, and

$$\mathcal{F}_0 \cong \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2), \quad \text{but} \quad \mathcal{F}_t \cong \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus 2} \quad \text{for } t \neq 0.$$

Set $\mathcal{E}_t = \mathcal{O}_{\mathbb{P}^1}^{r-1} \oplus \mathcal{F}_t$ and write $X_t = \mathbb{P}(\mathcal{E}_t^*)$ for the corresponding $\mathbb{P}^1$-bundle. Then $\mathcal{O}_{X_t}(1)$ is nef and big for all t; indeed, it is globally generated because $\mathcal{E}_t$ is, and has top selfintersection number $\mathcal{O}_{X_t}(1)^{r+1} = 2$ by the Leray–Hirsch relation. Moreover, $K_{X_t} = \mathcal{O}_{X_t}(-(r+1))$, so that X_t is quasi-Fano for all t. Let $\Psi_t \colon X_t \to \overline{X}_t$ be the crepant birational contraction associated to $|-mK_{X_t}|$ for $m \gg 0$. Then Ψ_0 is a divisorial contraction, contracting $\mathbb{P}^1 \times \mathbb{P}^{r-1}$ to a subspace $\mathbb{P}^{r-1} \subset \overline{X}_0$ of canonical singularities, while Ψ_t is small, with exceptional locus $\mathbb{P}^1 \times \mathbb{P}^{r-2}$ contracted to a subspace $\mathbb{P}^{r-2} \subset \overline{X}_t$. For $m \gg 0$

and $i = 1, \dots, r-2$, let $V_{i,t} \in |-mK_{X_t}|$ be general, and set $V_t = \bigcap_{i=1}^{r-2} V_{i,t}$; the V_t form a smooth family of projective 3-folds with K_X nef and big, and $\varphi_t = \Psi_{t|V_t}$ is the pluricanonical contraction of V_t. For each t write $F_t \subset X_t$ for the exceptional locus of Ψ_t, and $E_t = F_t \cap V_t$ for that of φ_t. Then $E_0 \cong \mathbb{P}^1 \times C$, where C is a smooth complete intersection curve of type $(m, \dots, m)$ in $\mathbb{P}^{r-1}$, and φ_0 contracts E_0 to C. However, $K_C = \mathcal{O}_C(m(r-2)-r)$ and $K_{V_0|C} = K_C \otimes \mathcal{O}_C(-1)$. For $t \neq 0$, the contraction φ_t is small, with exceptional locus $d = \deg C$ copies of $\mathbb{P}^1$.

There may in principle be an exception to this, namely crepant contractions with exceptional locus a conic bundle surface whose generic fibre is a line pair, that degenerates somewhere to a double line. This is because in the case where K_X is nef and big one does not have such a good hold of the embedded deformation of subvarieties as when X is, say, a Calabi–Yau. I do not know of any example of this type associated with a discontinuity of the effective cone.

The aim of this paper is to make a few scattered remarks about this problem in the higher dimensional case. As above, we deal with varieties for which either K_X or $-K_X$ is nef and big, or K_X is trivial. Except in the case $K_X = 0$, there is a built-in crepant birational contraction $\Psi\colon X \to \overline{X}$, such that $\pm K_X$ descends to an ample $\mathbb{Q}$-divisor on $\overline{X}$, and thus the crepant effective cone is the cone $N_1(X/\overline{X}) \subset \mathrm{NE}(X)$ generated by classes of curves contracted by φ. In general, we define the crepant effective cone of X to be the subcone $\mathrm{NE}_{\mathrm{cr}}(X) \subset \mathrm{NE}(X)$ generated by those curves contracting under some projective crepant contraction. This is a locally finite rational polyhedral cone, whose edges correspond to primitive contractions, and when $\pm K_X$ is nef and big we have $\mathrm{NE}_{\mathrm{cr}}(X) = \mathrm{NE}(X/\overline{X})$. In this case, $\mathrm{NE}_{\mathrm{cr}}(X)$ is the intersection of $\mathrm{NE}(X)$ with the hyperplane defined by K_X in $N_1(X)$. At any rate, $\mathrm{NE}_{\mathrm{cr}}(X)$ is locally finite rational polyhedral, and we name the edges ℓ_i of this polyhedron the *crepant extremal rays* of X.

We remark that in Theorems 1.2, 1.3 and 1.4 below, the assumption on K_X is only needed to ensure that any given crepant extremal ray can be contracted. We could easily avoid this assumption by starting from an extremal contraction and its corresponding extremal ray.

We first look at the case that φ contracts a $\mathbb{P}^k$ to a point.

Theorem 1.2 *Let X be a smooth projective variety of dimension $n \geq 3$ for which $\pm K_X$ is nef and big or trivial. Suppose that $\ell \subset N_1(X)$ is an extremal ray, either crepant or of Mori type, with associated birational contraction $\varphi\colon X \to \overline{X}$. Suppose also that the exceptional locus $E \subset X$ of φ is isomorphic to $\mathbb{P}^k$, and that its normal bundle splits as a direct sum of line bundles: $N_{E/X} \cong \bigoplus_{i=1}^{n-k} \mathcal{O}_{\mathbb{P}^k}(-a_i)$. Then the ray ℓ is J-effective on all tamed almost* *complex deformations of X.*

For example, if X is 3-dimensional and a crepant birational contraction $\varphi\colon X \to \overline{X}$ has exceptional locus E isomorphic to $\mathbb{P}^1$, then any complex manifold which can be deformed to X through a family of tame almost complex structures has an effective curve homologous to E [Wl 2]. Example 1.6 below shows that in general the assumption that the normal bundle splits as a direct sum of line bundles may not be omitted.

Theorem 1.3 *Let X be a smooth projective variety and $\varphi\colon X \to \overline{X}$ a divisorial contraction of an extremal ray ℓ of Mori type (that is, $-K_X$ is φ-ample [KMM]). Suppose that φ contracts a smooth $\mathbb{P}^1$-bundle $E \to B$ whose fibres generate the ray ℓ. If A is the homology class of a fibre of φ, then A is J-effective for any tamed almost complex deformation of the holomorphic structure of X.*

By [A], Theorem 2.3, the above contraction is simply a blowdown. The situation is more subtle for crepant contractions:

Theorem 1.4 *Let X be a smooth projective n-fold for which $\pm K_X$ is nef and big or trivial, $\ell \subset N_1(X)$ an extremal ray and $\varphi\colon X \to \overline{X}$ its crepant divisorial contraction. Assume that the exceptional locus $E \subset X$ is quasi-ruled via φ over $Z = \varphi(E) \subset X$, and that Z is smooth.*

1. *If $c_1(\omega_{X|E}) \neq (\varphi_{|E})^* c_1(\omega_Z)$, then the ray ℓ remains effective on all tame almost complex deformations of the holomorphic structure of X.*
2. *Suppose that E does not deform generically in the Kuranishi family of X. If $\omega_{X|E} \cong (\varphi_{|E})^* \omega_Z$, then the ray ℓ is not effective on a general small holomorphic deformation of X.*

We then look at the specific case $n = 4$:

Theorem 1.5 *Let X be a smooth projective 4-fold for which $\pm K_X$ is nef and big or trivial and $h^2(X, \mathcal{O}_X) = 0$, and suppose that the crepant effective cone of X is not locally constant in the Kuranishi family of X. Then X admits a primitive crepant birational contraction $\varphi\colon X \to \overline{X}$. If φ is divisorial, then φ contracts the exceptional locus $E \subset X$ (necessarily irreducible) to a surface $S \subset \overline{X}$ which is smooth in codimension one. In this case the following two possibilities hold:*

(a) away from finitely many fibres, E is quasi-ruled over S, meaning that the fibres are either all smooth conics or all reduced line pairs; if $S^* \subseteq S$ is the smooth locus, then $\omega_{S^*} \cong \omega_{X|S}$; or

(b) *the general fibre of $E \to S$ is a reduced line pair, but there is a nonempty divisor $D \subset S$ over which the fibres of φ are double lines.*

If on the other hand φ is a small contraction, then its exceptional locus is 2-dimensional and is contracted to a finite set in $\overline{X}$; furthermore the normalization of each irreducible component of the exceptional locus is isomorphic to $\mathbb{P}^2$.

This is proved by combining (a slight variation of) an argument of Wisniewski [Ws 2] with the results of [Wl 1], [P 1] and [P 2].

We illustrate the theorem by some examples:

Example 1.6 We give an example of a 4-fold admitting a crepant birational contraction whose exceptional locus is isomorphic to $\mathbb{P}^2$ and for which the crepant effective cone is not locally constant in the Kuranishi family. Let $\mathcal{G}_t$ denote the rank 3 vector bundle on $\mathbb{P}^2$ obtained as the extension

$$0 \to \mathcal{O}_{\mathbb{P}^2} \to \mathcal{G}_t \to T_{\mathbb{P}^2} \to 0 \quad \text{for } t \in H^1(\mathbb{P}^2, \Omega^1_{\mathbb{P}^2}) \cong \mathbb{C}.$$

For $t \neq 0$, this is the Euler sequence, so that $\mathcal{G}_t \cong \mathcal{O}_{\mathbb{P}^2}(1)^{\oplus 3}$ for $t \neq 0$, while $\mathcal{G}_0 \cong \mathcal{O}_{\mathbb{P}^2} \oplus T_{\mathbb{P}^2}$. Set $X_t = \mathbb{P}(\mathcal{G}_t^*)$, the relative projective space of lines in the dual vector bundle $\mathcal{G}_t^*$. This defines a smooth projective family $\mathcal{X} \to \mathbb{C}$, with $X_t \cong \mathbb{P}^2 \times \mathbb{P}^2$ if $t \neq 0$, and $X_0 \cong \mathbb{P}\big(\mathcal{O}_{\mathbb{P}^2} \oplus \Omega^1_{\mathbb{P}^2}\big)$. Set $\xi_t = \mathcal{O}_{X_t}(1)$; then ξ_t is ample for $t \neq 0$ and globally generated for all $t \in \mathbb{C}$, since $\mathcal{G}_t$ is. Moreover, the Leray–Hirsch relation $\xi_t^3 - c_1(T_{\mathbb{P}^2})\xi_t^2 + c_2(T)\xi_t = 0$ gives $\xi_t^4 = 6$. The canonical line bundle of X_t is $\omega_{X_t} = \mathcal{O}_{X_t}(-3)$, and therefore X_t is a quasi-Fano 4-fold for all $t \in \mathbb{C}$, actually Fano for $t \neq 0$. It is, however, not Fano when $t = 0$: the direct summand $\mathcal{O}_{\mathbb{P}^2} \subset \mathcal{G}_0^*$ corresponds to a section $X_0 \supset F \cong \mathbb{P}^2$, such that $\mathcal{O}_F(\xi)$ is trivial. Large positive multiples of ξ_0 induce a crepant birational contraction of X_0, whose exceptional locus $F \cong \mathbb{P}^2$ is contracted to a point, and has normal bundle $N_{E/X_0} \cong \Omega^1_{\mathbb{P}^2}$. Note that by taking branched covers ramified along appropiate general multiples of ξ_t one can turn this into either a family of Calabi–Yau 4-folds or of 4-folds with nef and big canonical divisor.

Example 1.7 Let $P_i(t)$ for $i = 1, 2, 3$ be distinct points in the plane, in general position for $t \neq 0$ but collinear for $t = 0$, and denote by S_t the blowup of $\mathbb{P}^2$ in the $P_i(t)$; then set $X_t = S_t \times S$, where S is any fixed quasi-Fano surface. Then S_t is a del Pezzo surface for $t \neq 0$, while it is only quasi-Fano for $t = 0$, and if $\ell \subset \mathbb{P}^2$ is the line containing the $P_i(0)$ then its proper transform $\widetilde{\ell} \subset S_0$ is a -2-curve. Let $S_0 \to \overline{S}_0$ denote the contraction of $\widetilde{\ell}$ to a rational double point $Q \in \overline{S}_0$. Then X_t is Fano for $t \neq 0$, while X_0 is quasi-Fano but not Fano. Hence there is a discontinuity in the crepant effective cone, that is, some crepant extremal ray is effective for $t = 0$ but not for general t. In fact, the divisor $\widetilde{\ell} \times S$ is the exceptional locus of a crepant contraction, where the factors $\ell \times \{P\}$, with $P \in S$, are the fibres, and the image is $\{Q\} \times S \subset \overline{S}_0 \times S$, a surface of cA_1 singularities. Note that similar examples may be constructed using K3 surfaces or minimal surfaces of general type, and then the families of 3-folds thus constructed are either Calabi–Yau or have nef and big canonical divisor.

Example 1.8 Let $\mathbb{P}^3 \supset \Lambda \cong \mathbb{P}^2$ be a plane and $D \subset \Lambda$ a smooth conic. Let $P(t) \in \mathbb{P}^3 \setminus D$ be a point, such that $P(t) \notin \Lambda$ for $t \neq 0$, while $P(0) \in \Lambda$. Denote by Y_t the blowup of $\mathbb{P}^3$ along C and $P(t)$. Then Y_t is a Fano 3-fold for $t \neq 0$, and $-K_{Y_0}$ is nef and big but not ample. Thus the crepant effective cone is not locally constant at the origin in the family. In fact, if $\widetilde{\Lambda} \subset Y_0$ is the proper transform of Λ and $F \subset \widetilde{\Lambda}$ is the exceptional divisor of the blowup of Λ at $P(0)$, then $\widetilde{\Lambda}$ may be viewed as a rational ruled surface ($\cong \mathbb{F}_1$, with exceptional fibre F) of type $(0,2)$ in the notation of [P 1]; it is the exceptional locus of a crepant divisorial contraction $X \to \overline{X}$, under which Λ is contracted to $F \subset \overline{X}$. We now set $X_t = Y_t \times \mathbb{P}^1$; then X_t is a Fano 4-fold when $t \neq 0$, but it is only quasi-Fano for $t = 0$. The divisor $\widetilde{\Lambda} \times \mathbb{P}^1$ is the exceptional locus of a crepant morphism contraction, under which it contracts to $F \times \mathbb{P}^1 \subset \overline{X} \times \mathbb{P}^1$. Again, it is easy to modify this example so as to obtain families of 4-folds with $\pm K_X$ nef and big or trivial, for example, using a family of Calabi–Yau manifolds whose central fibre contains an elliptic ruled surface not deforming sideways in the family [Wl 1].

Notation We recall some notation from the theory of J-holomorphic curves; a more adequate reference is [MS]. Let (M, ω) be any compact symplectic manifold. We say that an almost complex structure J on M is *compatible* with ω if it preserves ω and if $g_x(v, w) \stackrel{\text{def}}{=} \omega_x(v, Jw)$ (for $x \in M$ and $v, w \in T_xM$) is a Riemannian metric on M. The almost complex structure J is *ω-tame* if $\omega(v, J(v)) > 0$ for all tangent vectors $\neq 0$, and it is *tame* if it is ω-tame with respect to some symplectic structure ω. The space of all almost complex structures on M compatible with ω is contractible and nonempty; it is denoted by $\mathcal{J}(M, \omega)$. Having fixed J, and for $A \in H_2(M, \mathbb{Z})$ a spherical class, we denote by $\mathcal{M}(A, J)$ the space of all unparametrized J-holomorphic rational curves representing A. The almost complex structure is called *A-good* if for all J-holomorphic curves $f\colon \mathbb{P}^1 \to M$ representing the class A we have $H^{0,1}(f^*T_M) = 0$ and the space of cusp A-spheres has codimension at least two [MS], [Ru]; if J is A-good, it may be used to actually compute the Gromov–Witten invariants Φ_A. A ray ℓ in $H_2(X, \mathbb{R})$ is called J-effective if it is generated by an integral class for which the space of J-holomorphic curves is nonempty. Occasionally we say that a J-effective ray is a *jumping ray* if it does not remain effective under arbitrary small deformations of the almost complex structure on X. Accordingly, a *discontinuity* of the crepant effective cone will mean a jumping ray belonging to the crepant effective cone, and similarly for the Mori cone.

Finally, following Wilson [Wl 2], we say that a projective surface E is *quasi-ruled* if it admits a conic bundle structure $E \to C$ over a smooth curve C, all of whose fibres are either smooth conics or all line pairs.

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2 Proof of Theorem 1.2

We deal with the case $n > 3$, the case $n = 3$ having being treated in [Wl 2] (see [P 2] for some remarks in the non Calabi–Yau case). The strategy of the proof is first to show that under small deformations of the almost complex structure the excess dimensional moduli space of unparametrized J-holomorphic curves in the given homology class is replaced by a tractable geometric object: essentially the zero locus of a general section of the obstruction bundle on the moduli space itself. We then use this identification to prove that some associated Gromov–Witten invariant is nonzero, and therefore that the given homology class remains effective under arbitrary tamed almost complex deformations; this strategy is inspired by the work of Ruan [Ru] and Wilson [Wl 2]. In the present context all the relevant moduli spaces are *a priori* compact and smooth projective varieties, although our varieties are not necessarily weakly monotone.

Notice first that, either by an elementary direct computation using adjunction or by [Ws 1], Theorem 1.2, we have $2k \geq n - a$ with $a = 0$ in the Mori case and $a = 1$ in the crepant case. (Theorem 1.2 is stated for Mori extremal rays, but the proof works essentially just as well for crepant ones; see the more detailed discussion below.) Given that $n > 3$, then $k \geq 2$ and certainly $a_i > 0$ for all i; this is because under the hypothesis, $H^1(E, N) = 0$, so that the Hilbert scheme of X is smooth at E, and thus if $h^0(E, N) > 0$ then E must move in X, which is absurd. Let $L \subset E \cong \mathbb{P}^k$ be any line, and let $A \in H_2(X, \mathbb{Z})$ denote the homology class of L. We then have a short exact sequence of normal bundles

$$0 \to N_{L/E} \to N_{L/X} \to N_{E/X}{}_{|L} \to 0.$$

Given that $N_{L/E} \cong \mathcal{O}_L(1)^{k-1}$ and $N_{E/X} \cong \bigoplus_{i=1}^{c} \mathcal{O}_E(-a_i)$, this extension splits, that is, $N_{L/X} \cong \mathcal{O}_L(1)^{k-1} \oplus \bigoplus_{i=1}^{c} \mathcal{O}_E(-a_i)$ (here $c = n - k$). Therefore, if J denotes the complex structure of X, then J is not good for the class A in the sense of Gromov–Witten theory unless $a_i = 1$ for all i. The moduli space $\mathcal{M}(A,J)$ of all unparametrized J-holomorphic curves coincides with the Hilbert scheme of lines in $\mathbb{P}^k$, that is, with the Grassmannian $G = \mathrm{Grass}(2,k+1)$. Consider the incidence correspondence $F \subset G \times \mathbb{P}^k$, and let $p\colon F \to G$ and $q\colon F \to \mathbb{P}^k$ denote the projections. Then via p we may identify F with the relative projective space $\mathbb{P}_G(\mathcal{S})$ of lines in the tautological rank 2 bundle on G, and via q with $\mathbb{P}_{\mathbb{P}^k}(T_{\mathbb{P}^k})$.

For any given J-holomorphic curve L in the family (a line in $E \cong \mathbb{P}^k$), let $u\colon \mathbb{P}^1 \to X$ be a parametrization, and $D_u\colon \mathcal{A}(u^*T_M) \to \mathcal{A}^{0,1}(u^*T_M)$ the usual Fredholm operator of Gromov–Witten theory [MS], where $\mathcal{A}$ stands for C^∞ sections, and $\mathcal{A}^{0,1}(u^*T_X) = \mathcal{A}(\Omega^{0,1}_{\mathbb{P}^1} \otimes u^*(T_X))$. Since the almost complex structure of X is integrable, in a natural way $\ker(D_u) \cong H^0(\mathbb{P}^1, u^*T_X)$ and $\mathrm{coker}(D_u) \cong H^1(\mathbb{P}^1, u^*T_X) \cong H^1(\mathbb{P}^1, N_{L/X})$ [MS]. Thus,

$$\dim \mathrm{coker}(D_u) = \sum_{i=1}^{n-k} h^1(L, \mathcal{O}_L(-a_i)) = \sum_{i=1}^{n-k} (a_i - 1).$$

Besides, we have on the one hand $\deg(N_{L/X}) = -K_X \cdot L - 2$, and on the other $\deg(N_{L/X}) = -\sum_1^{n-k} a_i + k - 1$, and therefore $\sum_1^{n-k} a_i = K_X \cdot L + k + 1$. Hence, $\dim \mathrm{coker}(D_u) = K_X \cdot L + 2k + 1 - n$. Thus $n - K_X \cdot L - 3 + \dim \mathrm{coker}(D_u) = 2k - 2 = \dim \mathrm{Grass}(2, k+1)$. Therefore [Ru], Proposition 5.7 may be applied; in fact, although the variety in question is not necessarily weakly monotone, the moduli space at hand (the Grassmannian) is a priori smooth and compact. For the purpose of computing Gromov–Witten invariants, the moduli space of unparametrized J'-holomorphic curves, for J' a generic small deformation of J, may be identified with the zero locus of a transverse section of the obstruction bundle $\mathcal{E}_{\mathrm{ob}}$ on $G = \mathcal{M}(A,J)$. The latter, however, may in this case be described purely algebraically: on F there is a short exact sequence

$$0 \to T_{F/G} \to q^*T_X \to \mathcal{N} \to 0,$$

and $\mathcal{E}_{\mathrm{ob}} \cong R^1 p_*(\mathcal{N})$. We have the exact commutative diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & T_{F/G} & \longrightarrow & q^*(T_{\mathbb{P}^k}) & \longrightarrow & \mathcal{R} & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & T_{F/G} & \longrightarrow & q^*(T_X) & \longrightarrow & \mathcal{N} & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & q^*(\oplus_1^{n-k} \mathcal{O}_{\mathbb{P}^k}(-a_i)) & = & q^*(\oplus_1^{n-k} \mathcal{O}_{\mathbb{P}^k}(-a_i)) & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

and since $R^i p_*(\mathcal{R}) = 0$ for all $i > 0$, we have

$$\mathcal{E}_{\mathrm{ob}} \cong R^1 p_* \Big(q^* \bigoplus_{i=1}^{n-k} \mathcal{O}_{\mathbb{P}^k}(-a_i) \Big) = \bigoplus_{i=1}^{n-k} R^1 p_* q^* \mathcal{O}_{\mathbb{P}^k}(-a_i).$$

On the other hand, with the identification $F = \mathbb{P}_G(\mathcal{S})$, we have $q^*\mathcal{O}_{\mathbb{P}^k}(-1) \cong \mathcal{O}_F(-1)$. Therefore,

$$\mathcal{E}_{\mathrm{ob}} = \bigoplus_{i=1}^{n-k} R^1 p_* \mathcal{O}_F(-a_i) \cong \bigoplus_{i=1}^{n-k} \mathrm{Sym}^{a_i-2}(\mathcal{S}) \otimes \det(\mathcal{S}).$$

Now let $s \in \mathcal{A}(\mathcal{E}_{\mathrm{ob}})$ be a transverse section. If s is nowhere zero, then $\mathcal{E}_{\mathrm{ob}}$ has vanishing top Chern class: $c_r(\mathcal{E}_{\mathrm{ob}}) = 0$, where $r = \sum_i (a_i - 1)$. This however implies that $c_r(\mathcal{E}^*_{\mathrm{ob}}) = 0$, that is, $c_r\big(\bigoplus_{i=1}^{n-k} \mathrm{Sym}^{a_i-2}(\mathcal{S}^*) \otimes \det(\mathcal{S}^*)\big) = 0$. Set $H = \det(\mathcal{S}^*)$; this is an ample line bundle on G. Then

$$c_r\Big(\bigoplus_{i=1}^{n-k} \mathrm{Sym}^{a_i-2}(\mathcal{S}^*) \otimes H\Big) = c_1(H)^r + \sum_{k=1}^{r} c_1(H)^{r-k} c_k\Big(\bigoplus_{i=1}^{n-k} \mathrm{Sym}^{a_i-2}(\mathcal{S}^*)\Big);$$

however, since $\bigoplus_{i=1}^{n-k} \mathrm{Sym}^{a_i-2}(\mathcal{S}^*)$ is globally generated, each of its Chern classes may be represented by an effective cycle ([F], Chap. 14). Thus it is clear that the above Chern class is not zero. More succintly, since $\mathcal{E}^*_{\mathrm{ob}}$ is ample and globally generated, and has rank strictly less than the dimension of G, if $s \in H^0(\mathrm{Grass}(2, k+1), \mathcal{E}^*_{\mathrm{ob}})$ is a general section then its zero locus $Z = Z(s)$ is a nonempty, connected, smooth algebraic variety [F], [L]. This shows that the ray ℓ remains effective under small deformations of the almost complex structure of X. We claim, however, the stronger statement that the ray ℓ remains effective under arbitrary tamed almost complex deformations. To complete the proof, we now show that the Gromov–Witten invariant associated to A is nonzero.

Consider the case of smallest possible dimension of the exceptional locus: $2k = n - 1$. In this case $K_X \cdot A = 0$, that is, φ is a crepant contraction, and the normal bundle of E in X is $N \cong \mathcal{O}_{\mathbb{P}^k}(-1)^{\oplus(k+1)}$. Therefore, any line L in E has normal bundle

$$N_{L/X} = \mathcal{O}_L(1)^{\oplus(k-1)} \oplus \mathcal{O}_L(-1)^{\oplus(k+1)}.$$

In particular, the complex structure of X is A-good, and may be used to compute the invariant directly. To get a nonzero result we need to use a 2-point invariant, that is, we set $p = 2$ in [MS], Formula 7.1 on p. 93, and get $d = 4k + 4 = 2(2k + 2)$. Now let $H_i, H'_i \subset X$ for $i = 1, \ldots, k$ be general hyperplane sections of X, to fix ideas, in the same very ample linear series on X, and set $V = H_1 \cap \cdots \cap H_k$ and $V' = H'_1 \cap \cdots \cap H'_k$. Then $[V] = [V'] \in H_{2k+2}(X, \mathbb{Z})$, and $V \cap E$, $V' \cap E$ are two reduced finite subschemes, consisting of points $\{p_i\}$ and $\{p'_j\}$, say. The invariant $\Phi_A([V],[V])$ then counts the number of lines in E joining some p_i with some p'_j, and this is a positive number.

In all other cases we may write $n = 2k - h$, with $0 \le h < k$. According to [MS], Formula 7.1, for $p = 2$ we have

$$d = 2(n-1) + 4 - 2c_1(A) = 2\Big(k - h + \sum_{i=1}^{k-h} a_i\Big).$$

Let $\Gamma \subset E$ be a linear subspace of dimension $c = n - k = k - h$. If $\Sigma \subset X$ is a general representative for the homology class $D \in H_{2(k-h)}(X, \mathbb{Z})$ of Γ, then Σ meets E in a finite number of points $\{p_i\}$, which counted with multiplicity is $(-1)^{k-h} \prod_{i=1}^{k-h} a_i$. Set $d_1 = 2(k-h)$ and $d_2 = d - d_1 = 2\sum_{i=1}^{k-h} a_i$. We need to produce a class of dimension d_2, that is, of codimension $2n - d_2 = 2\big(k - \sum_{i=1}^{k-h}(a_i - 1)\big)$. Let $s = k - \sum_{i=1}^{k-h}(a_i - 1)$, and $V = H_1 \cap \cdots \cap H_s$, where the H_i are general very ample divisors in X. Then $V \cap E$ is a smooth complete intersection subvariety of $E \cong \mathbb{P}^k$ of dimension $\sum_{i=1}^{k-h}(a_i - 1)$. If some $a_i > 1$, the structure is not good, and as above, we need to consider the zero locus of a transverse section s of the obstruction bundle. If $Z = Z(s)$ is such a zero locus, and $\widetilde{Z}$ the family of all parametrizations of lines in Z, there is a pseudo-cycle ([MS], Chap. 7) given by the 2-point evaluation map:

$$\mathrm{ev}_2 \colon \widetilde{Z} \times_{\mathbb{G}} (\mathbb{P}^1 \times \mathbb{P}^1) \to X \times X,$$

where $\mathbb{G} = \mathrm{PGL}(2, \mathbb{C})$. The Gromov–Witten invariant $\Phi_A([V],[D])$ may then be computed by taking suitable intersection numbers with this pseudo-cycle ([Ru], Proposition 5.7); in particular, Z may be replaced by any subvariety representing the same homology class. On the other hand, by changing the orientation of Z we change at most the sign of these intersection numbers, and therefore by identifying $\mathcal{E}_{\mathrm{ob}}$ (as a real bundle) with its dual by some choice of a Hermitian form, we may replace Z by the zero locus of a transversal section of $\mathcal{E}^*_{\mathrm{ob}}$. Then Z represents the top Chern class of $\mathcal{E}^*_{\mathrm{ob}}$, which decomposes as above as a sum of terms $c_1(H)^{r-k} c_k\big(\bigoplus_{i=1}^{n-k} \mathrm{Sym}^{a_i - 2}(\mathcal{S}^*)\big)$. The relevant intersection numbers do not change if we replace Z by a sum of pseudo-cycles Z_k, with each Z_k representing one such term. Each Z_k may be represented as an intersection of Schubert cycles, and upon choosing the underlying flag sufficiently general we may assume that all relevant intersections are transversal, and then that the contribution of each Z_k to the total intersection number is a nonnegative multiple of $(-1)^{k-h} \prod_{i=1}^{k-h} a_i$. Hence to prove that the total invariant is nonzero it is sufficient to show that the contribution due to Z_0 does not vanish. Now

Z_0 represents $c_1(A)^r$, where $r = \sum_{i=1}^{k-h}(a_j - 1)$. Define $W = \mathbb{C}^{k+1}$ so that $E = \mathbb{P}(W)$, and let

$$W_*^{(i)} = \{W_1^{(i)} \subset W_2^{(i)} \subset \cdots \subset W_{k+1}^{(i)} = W\}$$

be general complete flags for $i = 1, \ldots, \sum_{i=1}^{k-h}(a_i - 1)$. We may then set

$$Z_0 = \sigma_1(W_*^{(1)}) \cap \cdots \cap \sigma_1(W_*^{(r)}),$$

where for each i we have

$$\sigma_1(W_*^{(i)}) = \{L \in \mathrm{Grass}(2, W) \mid L \cap \mathbb{P}(W_{k-1}^{(i)}) \neq \emptyset\}.$$

For every given point $p \in E$, the lines through p lying in the family Z_0 sweep out a linear subspace $\Pi \subset E$ of dimension $k - \sum_{i=1}^{k-h}(a_j - 1)$. Therefore Π meets $V \cap E$ in a positive number of points $\{q_j^{(i)}\}$. The contribution of Z_0 toward the invariant $\Phi_A([V], [D])$ is then $(-1)^{k-h} \prod_{i=1}^{k-h} a_i$ times the number of lines joining p_i with some $q_j^{(i)}$, and is therefore nonzero. Q.E.D.

3 Proof of Theorem 1.3

Let $F \cong \mathbb{P}^1 \subset X$ be any fibre of E over its image $Z = \varphi(E) \subset \overline{X}$. Then we have the short exact sequence of normal bundles

$$0 \to \mathcal{O}_F^{n-2} \to N_{F/X} \to \mathcal{O}_F(E) \to 0;$$

on the other hand, letting $k = -K_X \cdot F > 0$, we have $\deg(N_{F/X}) = k - 2 \geq -1$. Hence, because $-E$ is φ-ample ([KMM], Cor. 0-3-5), we have $E \cdot F = -1$ and therefore the above sequence is split: $N_{F/X} \cong \mathcal{O}_F^{n-2} \cong \mathcal{O}_F(-1)$. In particular, $c_1(F) = 1$ and the holomorphic structure of X is A-good, where A is the homology class of F, and furthermore the moduli space of holomorphic curves in the class A is scheme theoretically isomorphic to the base manifold $\varphi(E)$. [MS], Formula 7.1 with $p = 2$ now gives $d = 2(n-1) + 2$. Let $H \subset X$ be an ample divisor on X; then $\Phi_A([F], [H])$ computes the number of triples (u, z_1, z_2), where $u \colon \mathbb{P}^1 \to X$ is J-holomorphic and $z_1, z_2 \in \mathbb{P}^1$, $u(z_1) \in E$ and $u(z_2) \in H$, taken with multiplicity and modulo the action of $\mathrm{PGL}(2, \mathbb{C})$. These add up to the product of the intersection numbers, that is, $(F \cdot E)(F \cdot H) \neq 0$. Q.E.D.

4 Proof of Theorem 1.4

Suppose that the crepant extremal ray $\ell \subset N_1(X)$ and its primitive contraction $\varphi \colon X \to \overline{X}$ are as in the statement of the theorem. Suppose first that the exceptional locus E is a $\mathbb{P}^1$-bundle over its image $Z = \varphi(E) \subset \overline{X}$, and let $A \in H_2(X, \mathbb{Z})$ denote the homology class of a fibre. If J_0 is the complex structure of X, then the moduli space $\mathcal{M}(A, J_0)$ of unparametrized holomorphic curves representing the class $[A]$ (that is, the component of the Hilbert scheme of X containing the point associated to a fibre of $E \to Z$) is isomorphic to Z. If, on the other hand, E is a conic bundle over Z all of whose fibres are reduced line pairs, then the normalization $\widetilde{E} \to E$ is a $\mathbb{P}^1$-bundle over an etale double cover $\widetilde{Z} \to Z$. In this case, let $A \in H_2(X, \mathbb{Z})$ be the homology class of the irreducible component of any fibre; then the moduli space $\mathcal{M}(A, J_0)$ is isomorphic to $\widetilde{Z}$. This is clear set theoretically, and from a scheme theoretic point of view it follows from the following:

Lemma 4.1 *Let $F \cong \mathbb{P}^1 \subset E$ be an irreducible component of any fibre of $\varphi_{|E}$. Then the normal bundle of F in X is $N_{F/X} \cong \mathcal{O}_F^{\oplus(n-2)} \oplus \mathcal{O}_F(-2)$.*

Proof Suppose first that E is a $\mathbb{P}^1$-bundle over $Z = \varphi(E)$, with Z smooth. Then we have the short exact sequence $0 \to N_{F/E} \to N_{F/X} \to N_{E/X}{}_{|F} \to 0$. Furthermore, $N_{F/E} \cong \mathcal{O}_F^{\oplus(n-2)}$ and $N_{E/X}{}_{|F} \cong \mathcal{O}_F(E) \cong \omega_F$; hence the above exact sequence is $0 \to \mathcal{O}_F^{\oplus(n-2)} \to N_{F/X} \to \mathcal{O}_F(-2) \to 0$, which is split. The argument in the case where all the fibres of E are reduced line pairs is similar. Q.E.D.

This shows that $h^0(N_{F/X}) = n - 2$ for all irreducible components F of a fibre of $\varphi_{|E}$, and therefore that $\mathcal{M}(A, J_0)$ is indeed isomorphic to the smooth variety Z (or $\widetilde{Z}$) with its reduced structure. Thus $\mathcal{M}(A, J_0)$ does not have the expected dimension, and in fact the holomorphic structure of X is not A-good in the sense of [MS]. However, as in the proof of Theorem 1.2, we can rely on [Ru], Proposition 5.7 to relate the moduli space $\mathcal{M}(A, J)$ for generic J to the obstruction bundle on $\mathcal{M}(A, J_0)$. This is because on the one hand each fibre F maps immersively into X, and on the other, since $\operatorname{coker}(D_u) \cong H^1(F, N_{F/X})$, we have

$$2c_1(A) + 2n - 6 + \dim \operatorname{coker}(D_u) = 2n - 6 + 2 = 2(n-2) = \dim_{\mathbb{R}} \mathcal{M}(A, J_0),$$

so that the assumptions of [Ru], Proposition 5.7 are satisfied. Therefore, for a generic nearby almost complex structure J on X, it follows that $\mathcal{M}(A, J)$ is oriented cobordant to the zero set of a transverse section of the obstruction bundle $\mathcal{E}$. So as above our next step is to identify the obstruction bundle on Z (or $\widetilde{Z}$); this is a rank 2 real vector bundle over $\widetilde{Z}$.

Lemma 4.2 *Let $\widetilde{Z} = Z$ when E is a $\mathbb{P}^1$-bundle over Z, and let $\widetilde{Z}$ be the etale double cover of Z which is the base variety of $\widetilde{E}$ when the fibres of E*

are all line pairs; let $f\colon \widetilde{E} \to X$ be the induced immersive morphism. Note that $f^*\omega_X$ naturally descends to a holomorphic line bundle on $\widetilde{Z}$, that we still denote by ω_X. The obstruction bundle on $\widetilde{Z}$ is then given by $\mathcal{E}_{\rm ob} = \omega_{\tilde{Z}} \otimes \omega_X^{-1}$.

Proof Let $T_{\tilde{E}/\tilde{Z}} \subset T_{\tilde{E}}$ denote the subbundle of vertical tangent vectors, that is, $T_{\tilde{E}/\tilde{Z}} = \ker\{d\pi\colon T_{\tilde{E}} \to \pi^* T_{\tilde{Z}}\}$, where $\pi\colon \widetilde{E} \to \widetilde{Z}$ denotes the projection. For every fibre $F \subset \widetilde{E}$, clearly $T_{\tilde{E}/\tilde{Z}}{}_{|F} \cong T_F$. Let also $\widetilde{N}$ be the rank $n-1$ bundle on $\widetilde{Z}$ defined by the exact sequence $0 \to T_{\tilde{E}/\tilde{Z}} \to f^*T_X \to \widetilde{N} \to 0$, so that for each F there is an isomorphism $\widetilde{N}_{|F} \cong N_{F/X}$. The obstruction bundle on $\widetilde{Z}$ is then $R^1\pi_*\widetilde{N}$. On the other hand, it is easily seen that there is an exact sequence $0 \to \pi^* T_Z \to \widetilde{N} \to \omega_E \otimes \omega_X^{-1} \to 0$, from which it follows that $R^1\pi_*\widetilde{N} = \omega_{\tilde{Z}} \otimes \omega_X^{-1}$. Q.E.D.

Thus if the first Chern class of the obstruction bundle is not zero, Ruan's result shows that the class A is represented by some J-holomorphic curve for a generic almost complex structure J on X which is sufficiently close to J_0. To complete the proof of (1), we show that a Gromov–Witten invariant associated to the ray ℓ is not zero. Since by assumption $c_1(\omega_Z) \neq c_1(\omega_{X|Z})$, there exists some irreducible curve $C \subset Z$ such that $c_1(\omega_Z)\cdot C \neq c_1(\omega_{X|Z})\cdot C$. Now let L be some sufficiently positive line bundle on Z such that both $\omega_Z \otimes L$ and $\omega_{X|Z} \otimes L$ are very ample, and pick smooth divisors

$$V_1 \in |\omega_Z \otimes L| \quad \text{and} \quad V_2 \in |\omega_{X|Z} \otimes L|;$$

moreover, let $\widetilde{V}_1$ and $\widetilde{V}_2 \subset \widetilde{\mathcal{M}}(A, J_0)$ be the family of all parametrizations of J_0-holomorphic curves contained in V_1 and V_2, respectively, and consider the surface $\widetilde{C} = \varphi^{-1}(C) \subset E$, whose homology class we denote by $D \in H_4(X, \mathbb{Z})$. Since $-E$ is φ-ample ([KMM], Cor. 0-3-5), a general representative $F \subset X$ of the homology class of the irreducible component of a fibre of E over Z meets E in a finite number of points, with multiplicities yielding a negative intersection number. If on the other hand we take a general representative of the homology class of $\widetilde{C}$, say $\Sigma \subset X$, then Σ meets E along a smooth subvariety C' of real dimension two, whose image in Z is homologous to a multiple of C, say $\varphi_*([C']) = a[C]$ for some $a \neq 0$. Consider the 2-point evaluation maps

$$\mathrm{ev}_2^{(i)}\colon \widetilde{V}_i \times_{\mathrm{PGL}(2)} (\mathbb{P}^1)^2 \to X \times X \quad \text{for } i = 1, 2,$$

each of which determines a pseudo-cycle in $X \times X$ ([MS], Chap. 7). Given that [MS], Formula 7.1 for $p = 2$ yields $d = 2(n-1) + 4$, we may consider the Gromov–Witten invariant $\Phi_A(D, [H])$, where A is the homology class of a fibre of E over Z, and $[H]$ denotes the class of some hyperplane section of X; our discussion shows that this invariant is determined by taking the difference of the intersection numbers related to the above pseudo-cycles. If $\overline{V}_i = \varphi^{-1}(V_i)$, then the image of $\mathrm{ev}_2^{(i)}$ is the fibre product $\overline{V}_i \times_Z \overline{V}_i$. The points of intersection between the pseudo-cycle $\mathrm{ev}_2^{(i)}$ and $\Sigma \times H$ are the pairs (z_1, z_2) such that $\varphi(z_1) = \varphi(z_2) \in V_i$ and $(z_1, z_2) \in C' \times H$. In particular, for a given $z_1 \in \overline{V}_i \cap C'$, the second component z_2 may range over the points of intersection of H with the fibre through z_2; furthermore, the intersection multiplicity at the pair (z_1, z_2) is given by the intersection multiplicity of Σ and $\overline{V}_i$, and this is the same as the intersection multiplicity (in E) of the Cartier divisor $\widetilde{V}_i$ and C'. Summing up, the value of the invariant is

$$\Phi_A(D, [H]) = a(F \cdot H)\Big(\big(c_1(\omega_Z) - c_1(\omega_{X|Z})\big) \cdot C\Big),$$

which is nonvanishing by construction.

To prove (2), one may argue as in [Wl1], proof of Proposition 4.4 and erratum. For simplicity, let us consider the case where E is a $\mathbb{P}^1$-bundle and the first obstruction to the deformation of E in a given 1-parameter family of deformations of X, say $\mathcal{X} \to \Delta$, occurs at first order. Then the Kodaira–Spencer class $\theta \in H^1(X, T_X)$ of the given family has a nonzero image in $H^1(E, N_{E/X}) = H^1(X, \omega_E \otimes \omega_X^{-1})$. Let $\mathbb{P}^1 \cong F \subset E$ denote an arbitrary fibre of $\varphi_{|E}$; the claim will be proved if we can show that θ does not map to zero in $H^1(F, N_{F/X})$. We have however already seen that $N_{F/X} \cong N_{F/E} \oplus \mathcal{O}_F(E)$, so that $H^1(F, N_{F/X}) \cong H^1(F, N_{E/X|F})$. Hence, the map $H^1(X, T_X) \to H^1(N_{F/X})$ is in fact the composite $H^1(X, T_X) \to H^1(E, N_{E/X}) \to H^1(F, N_{E/X|F})$. Given the hypothesis, then, it is sufficient to show that $H^1(E, N_{E/X}) \to H^1(F, N_{E/X|F})$. But $H^1(F, N_{E/X|F}) \cong \mathcal{E}_{\mathrm{ob}}(x)$, where $x = \varphi(F) \in Z$ and $\mathcal{E}_{\mathrm{ob}} \cong \omega_Z \otimes \omega_X^{-1}$ is the obstruction bundle, while on the other hand, by relative duality $H^1(E, N_{E/X}) \cong H^1(E, \omega_E \otimes \omega_X^{-1}) \cong H^0(Z, \omega_Z \otimes \omega_X^{-1})$, and the map

$$H^1(E, N_{E/X}) \cong H^0(Z, \mathcal{E}_{\mathrm{ob}}) \to H^1(F, N_{E/X|F}) \cong \mathcal{E}_{\mathrm{ob}}(x)$$

is section evaluation. If φ is critical, that is, $\omega_Z \otimes \omega_X^{-1}$ is trivial, this is an isomorphism. Q.E.D.

5 Proof of Theorem 1.5

Any 1-parameter family of deformations $\mathcal{X} \to \Delta$ of $X = X_0$ may be assumed to be projective, perhaps after restricting to a smaller open disc. In fact, by [KS], Theorem 15, after restricting Δ to a smaller open disc we may assume that every fibre X_t is Kähler. The condition that $H^{0,2}(X) = 0$ implies $H^2(X_t, \mathbb{R}) = H^{1,1}_{\mathbb{R}}(X_t)$ for all t; therefore an integral class of type $(1,1)$ remains of the same type on X_t, for $t \in \Delta$. If H is a very ample line bundle on X with $H^i(X, H) = 0$ for $i > 0$, by taking a local section in a suitable Jacobian bundle we may extend H to a family of line bundles $\mathcal{H}$ over $\mathcal{X}$. By openness of the ample condition, H_t is very ample on X_t for sufficiently small t, with a linear system of the same dimension.

Let ℓ be a crepant extremal ray of X and $\varphi\colon X \to \overline{X}$ its contraction. If $E \subset X$ is the exceptional locus of φ, by [K], Theorem 1, E is covered by rational curves. We then write $\ell = \mathbb{R}_{\geq 0}[C]$, where $C \subset X$ is an unbreakable rational curve (see e.g., [BS] for terminology). If furthermore $\overline{A}$ is an ample line bundle on $\overline{X}$, then $A = \varphi^*(\overline{A})$ is a nef and big line bundle on X supporting φ; since $\overline{X}$ has only canonical, hence rational, singularities, we may assume that $H^i(X, A) = 0$ for all $i > 0$. Then by the above argument A deforms to a family of line bundles $\mathcal{A}$ on $\mathcal{X}$. Restricting the base of the deformation if necessary we then see that the linear series $|A_t|$ may be assumed to have the same dimension as $|A|$ for all $t \in \Delta$. Hence there is a family of morphisms $\Phi\colon \mathcal{X} \to \overline{\mathcal{X}}$ over Δ, which for $t = 0$ yields $\varphi\colon X \to \overline{X}$. By openness of the ample condition, the ray ℓ is not effective on X_t for $t \neq 0$ if and only if φ_t is an isomorphism for $t \neq 0$.

Since a 1-parameter family of deformations $\mathcal{X} \to \Delta$ of $X = X_0$ is necessarily differentiably locally trivial, perhaps after restriction, we may assume that $H^j(\mathcal{X}, \mathbb{Z}) \cong H^j(X, \mathbb{Z})$ and $H_j(\mathcal{X}, \mathbb{Z}) \cong H_j(X, \mathbb{Z})$. Given that $H^2(X, \mathcal{O}_X) = 0$, furthermore, we have $N^1(X) \cong H^2(X, \mathbb{R})$ and so, dually, $N_1(X) \cong H_2(X, \mathbb{R})$. Hence, there are isomorphisms $N_1(X) \cong N_1(\mathcal{X})$ and $N^1(\mathcal{X}) \cong N^1(X)$; the former induces an inclusion $\mathrm{NE}(X) \subseteq \mathrm{NE}(\mathcal{X}/\Delta)$. In this situation, the *relative* crepant effective cone $\mathrm{NE}_{\mathrm{cr}}(\mathcal{X}/\Delta) \subset \mathrm{NE}(\mathcal{X}/\Delta)$ is the subcone generated by those relative crepant rays ℓ for which there exists a crepant relative contraction $\varphi\colon \mathcal{X} \to \overline{\mathcal{X}}$ over Δ, such that $\mathrm{NE}(\mathcal{X}/\overline{\mathcal{X}}) = \mathbb{R}_{\geq 0}[\ell]$. Clearly, the above shows that $\mathrm{NE}_{\mathrm{cr}}(\mathcal{X}/\Delta) \supseteq \mathrm{NE}_{\mathrm{cr}}(X)$.

Lemma 5.1 *Let X be a smooth projective manifold and $\pi\colon \mathcal{X} \to \Delta$ a one parameter family of deformations of X. If $\pm K_X$ is nef and big then, perhaps after restricting the base to a smaller open disc containing the origin,* $\mathrm{NE}_{\mathrm{cr}}(\mathcal{X}/\Delta) \subseteq \mathrm{NE}_{\mathrm{cr}}(X)$. *If K_X is trivial, the same conclusion holds over the complement of at most countably many points in $\Delta \setminus \{0\}$.*

Thus the crepant effective cone of X is generically constant in the Kuranishi family if and only if each of the crepant extremal rays of X remains effective on the generic small deformation of X; in fact "generic" may be omitted when $\pm K_X$ is nef and big, while it stands for "on the complement of a countable union of subvarieties of $B \setminus \{0\}$" if K_X is trivial.

Proof Suppose that $\pm K_X$ is nef and big. After replacing B by a smaller open neighbourhood of the origin, we may assume that $\pm K_{X_b}$ is semiample and big, and $N_1(X_b) \cong H_2(X, \mathbb{R})$ for all $b \in B$. Fix an ample divisor $H \subset X$. Since $\pm K_X$ is nef and big, if $n \gg 0$ there exists an effective divisor $\Delta \subset X$ such that $\pm nK_X \equiv H + \Delta$. Thus $\ell \cdot \Delta < 0$ for any crepant extremal ray. After replacing Δ by a suitably small positive rational multiple, (X, Δ) has log terminal singularities ([KMM], §0). By [KMM], §4, the effective cone $\mathrm{NE}(X)$ is locally finite rational polyhedral in $\Sigma = \{\ell : \ell \cdot (K_X + \Delta) < 0\}$, and therefore finite rational polyhedral in a suitable neighbourhood of the hyperplane Λ of $H_2(X, \mathbb{R})$ defined by $[K_X]$. Thus, it is bounded in this neighbourhood by rational faces $F_1, \dots, F_k$ of codimension one in $H_2(X, \mathbb{R})$. Each F_i is cut out on $\mathrm{NE}(X)$ by some nef divisor A_i; let $\Gamma_i = \{A_i \cdot z \geq 0\}$. Then $\mathrm{NE}_{\mathrm{cr}}(X) = \Lambda \cap \bigcap_i \Gamma_i$. Set $\widetilde{A}_i := A_i \pm rK_X$ for some $r \gg 0$; then $\widetilde{A}_i$ is big and semiample, and if $\widetilde{\Gamma}_i = \{\widetilde{A}_i \cdot z \geq 0\}$, then $\mathrm{NE}_{\mathrm{cr}}(X) = \Lambda \cap \bigcap_i \widetilde{\Gamma}_i$. For every i, a suitably large positive multiple of $\widetilde{A}_i$ extends to a family of relatively globally generated line bundles on the family. Hence for each i there is a relative crepant contraction $\varphi_i \colon \mathcal{X} \to \mathcal{X}_i$ over Δ extending φ. Therefore

$$\mathrm{NE}_{\mathrm{cr}}(\mathcal{X}/\Delta) \subseteq \bigcap_{i=1}^{k} H_i \cap \Lambda.$$

In particular, $\mathrm{NE}(X_b) \subset \widetilde{\Gamma}_i$ for all $b \in B$ sufficiently close to the origin, therefore $\mathrm{NE}_{\mathrm{cr}}(X_b) \subseteq \mathrm{NE}_{\mathrm{cr}}(X)$. The crepant effective cone is not locally constant in the family if and only if one of the edges (crepant extremal rays) $R_1, \dots, R_l$ of $\mathrm{NE}_{\mathrm{cr}}(X)$ is not effective on X_b when $b \in B$ is general.

Suppose next that K_X is trivial. The statement then simply follows from the fact that if a class $\ell \in H_2(X)$ is not effective, then the locus in Δ over which it is effective is at most countable. Q.E.D.

We need the following slight variant of [Ws 1], Theorem 1.2:

Proposition 5.2 *Let X be a projective n-fold for which $\pm K_X$ is nef and big or trivial, $\ell \subset N_1(X)$ a crepant extremal ray of X and $\varphi \colon X \to \overline{X}$ its crepant primitive birational contraction. Let $\mathcal{X} \to \Delta$ be a 1-parameter family of holomorphic deformations of X, and $\Phi \colon \mathcal{X} \to \overline{\mathcal{X}}$ a crepant birational contraction over Δ extending φ. Let $E \subset \mathcal{X}$ be the exceptional locus of Φ, and $E_1 \subseteq E$ an irreducible component. Suppose that $F \subseteq E_1$ is an irreducible component of a fibre of the restriction $\varphi_{|E_1}$. Then*

$$\dim E_1 + \dim F \geq n. \tag{1}$$

The proof is identical to the proof given by Ionescu [Io] and Wisniewski [Ws 1] of the corresponding inequality

$$\dim E_1 + \dim F \geq n + \operatorname{length} \ell$$

in Mori theory (here $n = \dim \mathcal{X} - 1$). Since however we need to tell when equality holds, we recall briefly the argument of [Ws 1], p. 144. Pick a general point $x \in F$ and let $F \supset C_0 \ni x$ be a rational curve minimal among the rational curves contained in F and passing through x, in the sense that it minimizes the intersection number with some fixed ample line bundle on X. Now let T be an irreducible variety parametrizing the embedded deformations of $C_0 \subset \mathcal{X}$, as constructed in [Io], and $V \subset \mathcal{X} \times T$ the incidence correspondence, with projections $p\colon V \to \mathcal{X}$ and $q\colon V \to T$. The standard Riemann–Roch estimate yields $\dim T \geq n-2$ and therefore $\dim V \geq n-1$. On the other hand, $\dim p^{-1}(x) \leq \dim F - 1$: as in [Ws 1], claim on p. 144, this follows from Mori's breaking lemma and the minimality assumption on C. Therefore, $\dim E_1 \geq \dim p(V) \geq \dim V - \dim p^{-1}(x) \geq n - \dim F$, and this is the claimed inequality. We see furthermore that equality holds in (1) if and only if $p(V) = E_1$ and the general fibre of $p\colon V \to X$ has dimension $\dim F - 1$.

We next recall [Ws 2], Key Lemma 1.1, which in the present context we may rephrase as follows:

Lemma 5.3 *Let X be a smooth projective variety for which $\pm K_X$ is nef and big or trivial and $h^2(X, \mathcal{O}_X) = 0$. Let ℓ be a jumping crepant extremal ray of X and $\varphi\colon X \to \overline{X}$ its crepant birational contraction, and let $E \subset X$ be any irreducible component of the exceptional locus of φ. Then*

$$2 \dim E - \dim \varphi(E) \leq n.$$

By Proposition 5.2 and Lemma 5.3, we have:

Corollary 5.4 *Let X be a smooth projective n-fold for which $\pm K_X$ is nef and big or trivial and $h^2(X, \mathcal{O}_X) = 0$. Let ℓ be a jumping extremal ray of X and $\varphi\colon X \to \overline{X}$ its crepant birational contraction. If $E \subset X$ is any irreducible component of the exceptional locus of φ, then $2 \dim E - \dim \varphi(E) = n$. In particular, if $n = 4$ then the exceptional locus of φ is either a union of surfaces mapping to a finite set, or a 3-fold mapping to a surface.*

Lemma 5.5 *Let X be a smooth projective 4-fold for which $\pm K_X$ is nef and big or trivial and $h^2(X, \mathcal{O}_X) = 0$. Let ℓ be a jumping ray on X, and assume that the associated crepant contraction $\varphi\colon X \to \overline{X}$ is small. Then the normalization of any irreducible component of the exceptional locus of φ is isomorphic to $\mathbb{P}^2$.*

Proof Let $F \subset E$ be any irreducible component and $x \in F$ a general point, lying in the smooth locus of F_{red}. Pick a rational curve $F \supset C_0 \ni x$ of minimal degree with respect to some fixed ample line bundle among curves in

F through x (see [K]). Since equality must hold in (1), in the notation used in the discussion following Proposition 5.2 we must have $\dim p^{-1}(x) = 1$, so that $\dim V = 3$. Setting $\delta_y = \dim p^{-1}(y)$ for $y \in F$, it thus follows that $\delta_y \geq 1$ for any $y \in F$. On the other hand, suppose that $\delta_y \geq 2$ for some $y \in F$; then $\dim V_y \geq 3$, and therefore every curve in the family T through x also passes through y, and by the nonbreaking lemma, this contradicts the assumption that C_0 is minimal. Thus $\delta_y = 1$ for every $y \in F$.

(I am indebted to the referee for the concise argument used in the rest of the proof.) Again referring to the notation used after Proposition 5.2, in this case $\dim T = 2$ and no point of F lies on all curves of the family. Let F', T', V' be the normalizations of F, T and V respectively, with the induced morphisms $q'\colon V' \to T'$ and $p'\colon V' \to F'$. If $F'' \to F'$ is a minimal desingularization of F, we have $H^i(F, \mathcal{O}_{F''}) = 0$ for $i = 1, 2$, so that algebraic and rational equivalence of divisors coincide on F''. Let $C' \subset F'$ be a curve from T' disjoint from the singular locus of F', and $C'' \subset F''$ its inverse image. By the above observation, the linear system $|C''|$ contains all deformations of $C' \subset F'$ disjoint from F'_{sing}; thus the map $|C''| \to T'$, which is birational, is an isomorphism, and therefore $|C''| \cong \mathbb{P}^2 \cong T'$. This linear system is base point free, and therefore its general representative is a smooth rational curve. By Riemann–Roch, since $h^i(F'', \mathcal{O}_{F''}(C'')) = 0$ for $i > 0$, $(C'', C'')_{F''} = 1$. Thus two general curves in the family T' are $\cong \mathbb{P}^1$ and meet at exactly one point. Now let $T'_x \subset T'$ be the subfamily parametrizing those curves through x; then $T'_x \cong |\mathcal{I}_x(C'')| \cong \mathbb{P}^1$. By minimality, the incidence correspondence $V' \supset V'_x \to T'_x$ is a $\mathbb{P}^1$-bundle. By the above, the induced map $V'_x \to F'$ is birational and contracts a section to x; since however x is a smooth point of F', we conclude $V'_x \cong \mathbb{F}_1$ and $F' \cong \mathbb{P}^2$. Q.E.D.

The proof of the theorem is then completed by the following:

Proposition 5.6 *Let X be a 4-fold for which $\pm K_X$ is nef and big or trivial and $h^2(X, \mathcal{O}_X) = 0$. Suppose that ℓ is a crepant extremal ray whose contraction $\varphi\colon X \to \overline{X}$ is divisorial. If φ is not as in the critical cases (a) and (b) of Theorem 1.5, ℓ is not a jumping ray (that is, it remains effective on all small deformations of X).*

Proof Let $E \subset X$ denote the exceptional divisor of φ, which is an irreducible 3-fold, and set $S = \varphi(E) \subset \overline{X}$. Then S is a surface of canonical singularities of $\overline{X}$. Given a 1-parameter deformation $\alpha\colon \mathcal{X} \to \Delta$ of X, we prove that if φ is not critical then ℓ is effective on X_t, for all t close to the origin. Let $\overline{A} \in \mathrm{Pic}(\overline{X})$ be a sufficiently positive very ample line bundle such that $H^i(\overline{X}, \overline{A}) = 0$ for all $i > 0$, and set $A = \varphi^*(\overline{A})$. Fix a general section $\overline{\sigma} \in H^0(\overline{X}, \overline{A})$, and set $\sigma = \varphi^*\overline{\sigma} \in H^0(X, A)$. Set $\overline{Y} = \mathrm{div}(\overline{\sigma})$, $Y = \varphi^*\overline{Y}$. Then Y is a smooth 3-fold, with nef and big canonical divisor

$K_Y = (K_X + A)_{|Y}$, and the restriction $\varphi_{|Y}$ is a crepant birational contraction, whose exceptional locus is the surface $Z = E \cap Y$ and is contracted to the curve $C = S \cap \overline{Y}$. The ray ℓ is then the image of an effective ray ℓ' on Y, which is represented by the irreducible components of the fibres of the conic bundle $Z \to C$. Arguing as at the start of the proof of Theorem 1.5, perhaps after restricting the base Δ, we may extend A to a family of globally generated line bundles $\mathcal{A}$ on $\mathcal{X}$, such that $\dim |A_t|$ is constant. By taking a suitable local section of $\alpha_*(\mathcal{A})$, and the corresponding divisor, we obtain a smooth family of 3-folds $\mathcal{Y} \subset \mathcal{X}$ with $Y_0 = Y$. Therefore, in order to prove the statement it suffices to note that, by the results of [P 2], if φ is not critical then the ray ℓ' remains effective on all sufficiently small deformations of Y. Q.E.D.

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