Degenerate double covers of the projective plane

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Abstract

We prove that the set of canonical models of surfaces of general type which are double covers of $\mathbb{P}^2$ branched over a plane curve of degree $2h$ is a connected component of the moduli space if and only if h is even. To get a connected component when h is odd, we must add some special surfaces called *degenerate double covers of* $\mathbb{P}^2$.

Moreover, we show that the theory of simple iterated double covers (cf. [Ma3]) "works" for every degenerate double cover of $\mathbb{P}^2$; this allows us to construct many examples of connected components of the moduli space having simple iterated double covers of $\mathbb{P}^2$ as generic members.

0 Introduction

Double covers of rational surfaces play an important role in the theory of minimal surfaces of general type, especially those with small c_1^2 (cf. [Hor1]). For example, if S is a smooth minimal surface with $K_S^2 = 2$ and $p_g(S) = 3$, then its canonical model is a double cover of $\mathbb{P}^2$, while if $K_S^2 = 8$ and $p_g(S) = 6$ then S is either a deformation of a double cover of $\mathbb{P}^2$, or a deformation of a double cover of $\mathbb{P}^1 \times \mathbb{P}^1$ ([B–P–V, p. 231]).

One of the main goals of this paper is to determine all smooth surfaces of general type that are (smooth) deformations in the large of a double cover of $\mathbb{P}^2$. In other words, if $\mathcal{M}$ denotes the coarse moduli space of surfaces of general type ([Gi]) and $S_0 \to \mathbb{P}^2$ is a double cover (branched over a nonsingular curve), we want to describe all surfaces S whose classes $[S] \in \mathcal{M}$ are in the same connected component as $[S_0]$. It is important to say that, as we consider only smooth deformations, the minimal resolution of double covers of $\mathbb{P}^2$ branched over very singular curves are not in the same connected component of $[S_0]$.

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For every $h \geq 4$, let $N(\mathbb{P}^2, \mathcal{O}(h)) \subset \mathcal{M}$ be the (irreducible) subset of surfaces whose canonical model is a double cover of $\mathbb{P}^2$ branched over a plane curve $D \subset \mathbb{P}^2$ of degree $2h$. Equivalently $N(\mathbb{P}^2, \mathcal{O}(h)) \subset \mathcal{M}$ is the set of double covers of $\mathbb{P}^2$ branched over a curve of degree $2h$ with at most simple singularities [B–P–V, II.8]. It is not difficult to prove that $N(\mathbb{P}^2, \mathcal{O}(h))$ is an open subset of the moduli space, but the above examples show that it is not closed in general. Our first result is the following:

Theorem A *Let $h \geq 4$ be a fixed integer and set $N = N(\mathbb{P}^2, \mathcal{O}(h))$.*

(1) N is open in the moduli space.

(2) N is closed in the moduli space if and only if h is even.

(3) If h is odd then the closure of N in the moduli space is open.

(4) The closure of $N(\mathbb{P}^2, \mathcal{O}(h))$ in the moduli space is a connected component.

We recall that the local analytic structure of $\mathcal{M}$ at a point $[S]$ is isomorphic to the quotient of the base space of the semiuniversal deformation of the canonical model S^{can} by the (finite) group of automorphisms of S (cf. [Gi], [Ma4]) and the subset $\mathcal{M}_{K^2,\chi} \subset \mathcal{M}$ of minimal surfaces with fixed K^2, χ is a quasiprojective variety. (1) is therefore an easy application of well-known theorems about deformations of double covers of smooth surfaces, while (4) is an immediate consequence of (1), (2) and (3).

The idea of the proof of (2) and (3) is the following: let $\{S_t\}$, $t \in \Delta$ be a flat family of minimal surfaces of general type such that $[S_t] \in N(\mathbb{P}^2, \mathcal{O}(h))$ for every $t \neq 0$. Denoting by Y_0 the canonical model of S_0, we use the results of [Ma1] to prove that either $[S_0] \in N(\mathbb{P}^2, \mathcal{O}(h))$, or Y_0 is a double cover of W_0, the projective cone over the rational normal curve of degree 4 in $\mathbb{P}^4$, nonflat over the vertex $w_0 \in W_0$; in the latter case we call Y_0 a *degenerate double cover* of $\mathbb{P}^2$. (As the referee points out, degenerate double cover of $\mathbb{P}^2$ and their deformations can also be described easily in terms of subvarieties of the weighted projective space $\mathbb{P}(1, 1, 1, 2, d)$.)

It is then clear that (3) is a consequence of the local irreducibility of the moduli space $\mathcal{M}$ at every degenerate double cover Y_0 of $\mathbb{P}^2$; this is proved in §4 by giving an explicit description of the Kuranishi family of Y_0.

We also prove the vanishing of some Ext groups on Y_0; these results allow us to apply the machinery of [Ma3] of simple iterated double covers to give a large number of examples of connected components of the moduli space. A finite map between normal algebraic surfaces $p\colon X \to Y$ is called a *simple iterated double cover* associated to a sequence of line bundles $L_1, \ldots, L_n \in \operatorname{Pic} Y$ if the following conditions hold:

(1) There exist normal surfaces $X = X_0, \dots, X_n = Y$ and flat double covers $\pi_i\colon X_{i-1} \to X_i$ such that $p = \pi_n \circ \cdots \circ \pi_1$.

(2) If $p_i = \pi_n \circ \cdots \circ \pi_{i+1}\colon X_i \to Y$ is the composite of the π_j for $j > i$, then we have the eigensheaf decomposition $\pi_{i*}\mathcal{O}_{X_{i-1}} = \mathcal{O}_{X_i} \oplus p_i^*(-L_i)$ for each $i = 1, \dots, n$.

We say that a simple iterated double cover $\pi\colon X \to Y$ is *smooth* if each X_i is smooth. By [Ma3, 2.1], $\pi\colon X \to Y$ is smooth if and only if X is smooth.

For any sequence $L_1, \dots, L_n \in \operatorname{Pic}\mathbb{P}^2$, define $N(\mathbb{P}^2, L_1, \dots, L_n)$ to be the locus in the moduli space $\mathcal{M}$ of surfaces of general type whose canonical model is a simple iterated double cover of $\mathbb{P}^2$ associated to $L_1, \dots, L_n$. The subset $N(\mathbb{P}^2, L_1, \dots, L_n)$ is parametrized by a Zariski open subset of the space of sections of a decomposable vector bundle of rank $2^n - 1$ over $\mathbb{P}^2$ (cf. [Ma3]), and therefore it is irreducible and unirational.

Theorem B *Set $N = N(\mathbb{P}^2, L_1, \dots, L_n)$ with $l_i = \deg L_i$, and write $\overline{N}$ for the closure of N in the moduli space.*

(1) If $l_n \geq 4$ and $l_i > 2l_{i+1}$ for $i = 1, \dots, n-1$ then N is an open subset of the moduli space.

(2) Assume in addition to (1) that $l_1, \dots, l_{n-1}$ are even integers, l_n is odd and $[S] \in \overline{N} \setminus N$; then the canonical model of S is a simple iterated double cover of a degenerate double cover of $\mathbb{P}^2$ and has unobstructed deformations.

(3) If $l_n \geq 5$ is odd, l_i is even and $l_i > 2l_{i+1}$ for each $i = 1, \dots, n-1$ then $\overline{N}$ is a connected component of the moduli space $\mathcal{M}$.

In the last section of this paper we see, using Theorem B, (3) that simple iterated double covers of $\mathbb{P}^2$ give examples of distinct connected components of moduli space whose general members are smooth simple iterated double covers of $\mathbb{P}^2$ with fixed numerical invariants.

This paper is considered as the ideal continuation of [Ma1] and [Ma3]. To avoid the excessive number of pages necessary for a selfcontained proof of Theorems A and B, we often use the results of the above papers.

An earlier version of this paper formed part of the author's thesis [Ma4]. It is a pleasure to thank my advisor F. Catanese, and also L. Badescu and C. Ciliberto for their interest in this work and many fruitful discussions. Thanks also to the referee for several useful remarks.

Notation

We work exclusively over the complex field $\mathbb{C}$; for any algebraic variety X, we write Ω^1_X for the sheaf of Kähler differentials and $\theta_X = (\Omega^1_X)^\vee$ for the sheaf of tangent vector fields. If X is normal we denote by K_X its Weil canonical divisor.

By a *deformation* we mean any flat family over a connected base; a *small* deformation means a deformation over a germ of complex space. Thus to say that a class $\mathcal{C}$ of surfaces is *stable under small deformations* means that for every deformation $f\colon X \to B$ and every point $b \in B$ such that $f^{-1}(b) \in \mathcal{C}$, there exists an analytic neighbourhood $b \in U \subset B$ such that $f^{-1}(u) \in \mathcal{C}$ for every $u \in U$.

We denote by Def_X the functor of infinitesimal deformations of X from the category of local Artinian $\mathbb{C}$-algebras to pointed sets ([Sch]), by $T^1_X = \mathrm{Def}_X(\mathbb{C}[t]/(t^2))$ its tangent space, and, if T^1_X is finite dimensional, by $\mathrm{Def}\, X$ the base space of the semiuniversal deformation of X (also called Kuranishi family). We define in a similar way $\mathrm{Def}_{(X,0)}$, $T^1_{(X,0)}$ and $\mathrm{Def}(X, 0)$ for any isolated singularity $(X, 0)$.

According to [Ca3] a surface singularity $(X, 0)$ is called a *half rational double point* (or half R.D.P.) if it is the quotient of a rational double point by an analytic involution. The half rational double points are completely classified in [Ca3].

For every $q \geq 0$ we denote by $\mathbb{F}_q = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(q))$ the Segre–Hirzebruch surface; if $p\colon \mathbb{F}_q \to \mathbb{P}^1$ is the natural fibration we denote by $\sigma_\infty, f, \sigma_0 \in \mathrm{Pic}\, \mathbb{F}_q$ the classes of the unique section of p with negative selfintersection, of the fibre of p and of a section disjoint from σ_∞. It is well known that σ_0, f are generators of the Picard group, $\sigma_0 \sim \sigma_\infty + qf$, $\sigma_0^2 = -\sigma_\infty^2 = q$, and the canonical bundle is $K = -\sigma_0 - \sigma_\infty - 2f$.

1 Degenerations of double covers of $\mathbb{P}^2$

Throughout this paper we denote by $W_0 \subset \mathbb{P}^5$ the projective cone over the rational normal curve of degree 4 in $\mathbb{P}^4$ and by $w_0 \in W_0$ its singular point. The minimal resolution of W_0 is the Segre–Hirzebruch surface $\mathbb{F}_4$. It is well known that W_0 is a degeneration of $\mathbb{P}^2$ and, according to [Ca3, §2] and [Ma1, Theorem 15 and Theorem 8], $\mathbb{P}^2$ and W_0 are the only degenerations of the projective plane with at worst half R.D.P.s

Lemma–Definition 1.1 *Let $\sigma \subset W_0 \subset \mathbb{P}^5$ be a generic hyperplane section. Then σ is a generator of* $\mathrm{Pic}\, W_0 = \mathbb{Z}$.

If $W \to \Delta$ is a deformation of W_0 such that $W_t = \mathbb{P}^2$ for every $t \neq 0$, then every line bundle on W_0 extends to a line bundle on W. Moreover, if L

is a line bundle on W such that $L_0 = a\sigma$ then $L_t = \mathcal{O}_{\mathbb{P}^2}(2a)$ for $t \neq 0$.

Proof Let $\gamma\colon X = \mathbb{F}_4 \to W_0$ be the minimal resolution. Since σ doesn't contain the vertex w_0 of the cone and $\sigma^2 = 4$, $\gamma^{-1}(\sigma)$ must be a section σ_0. The singularity at w_0 is rational, which identifies $\operatorname{Pic} W_0$ with the set of line bundle L_0 on X such that $L_0 \cdot \sigma_\infty = 0$. Since $q(W_0) = p_g(W_0) = 0$, the restriction $\operatorname{Pic} W \to \operatorname{Pic} W_0$ is an isomorphism by ([Ma1, Lemma 2]). After a possible restriction of the family $W \to \Delta$ to an open disk $0 \in \Delta' \subset \Delta$ of smaller radius we can assume W embedded in $\mathbb{P}^5 \times \Delta$ (cf. [Ma1, Prop. 3]) and the restriction of $\mathcal{O}_{\mathbb{P}^5}(a)$ to W_t for $t \neq 0$, is a very ample line bundle with selfintersection $4a^2$. The conclusion is now trivial. □

Lemma 1.2 *Let $f\colon \mathcal{Y} \to \Delta$ be a proper flat family of normal surfaces such that Y_t is a smooth surface for every $t \neq 0$, and Y_0 has at worst R.D.P.s.*

Let $\tau\colon \mathcal{Y} \to \mathcal{Y}$ be an involution preserving f such that $Y_t/\tau = \mathbb{P}^2$ for every $t \neq 0$. Then either:

(i) $Y_0/\tau = \mathbb{P}^2$, or

(ii) $Y_0/\tau = W_0$. The double cover $\pi\colon Y_0 \to W_0$ is branched exactly over the vertex $w_0 \in W_0$ and over a divisor $D' \sim (2a-1)\sigma$ with $w_0 \notin D'$. For $t \neq 0$, $Y_t \to Y_t/\tau = \mathbb{P}^2$ is branched over $D'_t \sim \mathcal{O}(4a-2)$ and the divisibility of the canonical class $r(Y_t)$ is even.

Proof The fact that the quotient family $\mathcal{Y}/\tau \to \Delta$ has normal fibres follows from the general fact [L–W, 5.6] that smoothings of normal two dimensional singularities are preserved under finite group actions which are free in codimension 3. We also note that in our particular case, if $\pi\colon \mathcal{Y} \to \mathcal{Y}/\tau$ is the projection, then $\pi_*\mathcal{O}_{\mathcal{Y}} = \mathcal{O}_{\mathcal{Y}/\tau} \oplus M$, where M is the $\mathcal{O}_\Delta$ module of anti-invariant functions, and this decomposition commutes with base change $B \to \Delta$.

Y_0/τ is a normal degeneration of $\mathbb{P}^2$ with at worst half R.D.P.s, and therefore either $Y_0/\tau = \mathbb{P}^2$ or $Y_0/\tau = W_0$. Assume $Y_0/\tau = W_0$; then, since (W_0, w_0) is not a rational double point, $y_0 = \pi^{-1}(w_0)$ is a fixed point of the involution τ. By [Ma3, Prop. 3.2 and Table 2], the singularity (Y_0, y_0) is a simple node defined by the equation $x_0^2 + x_1^2 + x_2^2 = 0$ and the involution τ is conjugate to $x_i \mapsto -x_i$ for $i = 0, 1, 2$. In particular y_0 is an isolated fixed point of the involution.

Let $\delta\colon S \to Y_0$ be the resolution of the node (Y_0, y_0) and $E = \delta^{-1}(y_0) \subset S$ the corresponding nodal curve, i.e., a smooth rational curve with selfintersection $E^2 = -2$. The action of τ can be lifted to an action on S, and it is easy to see that $S/\tau = X = \mathbb{F}_4$. Moreover the flat double cover $\pi\colon S \to \mathbb{F}_4$ is branched over $D = \sigma_\infty \cup D'$, where $\sigma_\infty \cap D' = \emptyset$, and since this divisor must be 2-divisible in $\operatorname{Pic} \mathbb{F}_4$, $D' \sim (2a-1)\sigma_0$ and $\frac{1}{2}(\sigma_\infty \cup D') = a\sigma_0 - 2f$ where f denote the fibre of $\mathbb{F}_4$. $\square$

We now recall some results of [Ma3] about flat double covers of surfaces and their deformations. Let $\pi\colon X \to Y$ be a flat double cover of normal surfaces and $L \to Y$ the line bundle such that $\pi_*\mathcal{O}_X = \mathcal{O}_Y \oplus \mathcal{O}_Y(-L)$.

The surface X can be described as a hypersurface of L defined by the equation $z^2 = f$, where z is a coordinate in the fibres of L and $f \in H^0(\mathcal{O}_Y(2L))$. Clearly $R = \{z = 0\} \subset X$ and $D = \{f = 0\} \subset Y$ are exactly the ramification and the branch divisors of π. The line bundle L and the branch divisor D determine the double cover uniquely up to isomorphism. In this situation we say that X is a flat double cover of Y associated to the line bundle L. If $\varphi \in H^0(\mathcal{O}_Y(2L))$ the surface of equation $z^2 = f + \varphi$ is called a *natural deformation* of X.

Applying the functor $\operatorname{Hom}_{\mathcal{O}_X}(-, \mathcal{O}_X)$ to the exact sequence of sheaves

$$0 \to \pi^*\Omega^1_Y \to \Omega^1_X \to \mathcal{O}_R(-R) \to 0 \tag{1.3}$$

on X gives the exact sequence

$$\begin{aligned}\operatorname{Ext}^1_{\mathcal{O}_X}(\mathcal{O}_R(-R), \mathcal{O}_X) &\xrightarrow{\varepsilon} \operatorname{Ext}^1_{\mathcal{O}_X}(\Omega^1_X, \mathcal{O}_X)\\ &\to \operatorname{Ext}^1_{\mathcal{O}_Y}(\Omega^1_Y, \mathcal{O}_Y) \oplus \operatorname{Ext}^1_{\mathcal{O}_Y}(\Omega^1_Y, -L). \qquad (1.3')\end{aligned}$$

Now there exists an isomorphism $\operatorname{Ext}^1_{\mathcal{O}_X}(\mathcal{O}_R(-R), \mathcal{O}_X) = H^0(\mathcal{O}_R(\pi^*D)) = H^0(\mathcal{O}_D(D))$, and if $q(Y) = 0$ then the image of ε is exactly the space of first order natural deformations of X. In particular if $Y = \mathbb{P}^2$ and the degree of L is ≥ 4 then ε is surjective, so that by the Kodaira–Spencer criterion the family of natural deformations is complete, and then double covers of $\mathbb{P}^2$ are stable under small deformations.

This proves that $N(\mathbb{P}^2, \mathcal{O}(h))$ is open in the moduli space for every $h \geq 4$ and that

$$N_0(\mathbb{P}^2, \mathcal{O}(h)) = \big\{[S] \in N(\mathbb{P}^2, \mathcal{O}(h)) \bigm| K_S \text{ is ample}\big\}$$

is an open dense subset of it. More generally, the following holds:

Proposition 1.4 ([Ma3], 2.3) *In the above notation, let $\widetilde{X} \to \widetilde{Y} \to H$ be a deformation of the map π parametrized by a smooth germ $(H, 0)$ and let $r_X\colon (H, 0) \to \operatorname{Def} X$, $r_Y\colon (H, 0) \to \operatorname{Def} Y$ be the induced maps. Assume:*

1. *r_Y is smooth.*

2. *The image of r_X contains the natural deformations.*

3. $\mathrm{Ext}^1_{\mathcal{O}_Y}(\Omega^1_Y, -L) = 0$ *and* $H^1(\mathcal{O}_Y) = 0$.

Then the morphism r_X *is smooth.*

Definition 1.5 Let $a \geq 3$ be an integer and $\pi\colon S \to \mathbb{F}_4$ the double cover associated to $L = a\sigma_0 - 2f$ branched over the disjoint union of σ_∞ and a divisor $D' \sim (2a-1)\sigma_0$ with at worst simple singularities ([B–P–V, II.8]). $E = \pi^{-1}(\sigma_\infty)$ is a nodal curve; taking its contraction $\delta\colon S \to Y_0$ we get a surface with at worst rational double points which is a double cover of the cone W_0. We say that Y_0 is a *degenerate double cover of* $\mathbb{P}^2$. The number a determines $K^2_{Y_0} = 8(a-2)^2$ and is called the *discrete building data* of Y_0.

If $a = 2$ the above construction still makes sense. In this case, we obtain a well-studied class of K3 surfaces [Sai, §5], [Hor2].

Theorem 1.6 *For even* $h \geq 4$, *the set* $N = N(\mathbb{P}^2, \mathcal{O}(h))$ *is a connected component of moduli space. If* h *is odd then the set* $\overline{N} \setminus N$ *is contained in the set of degenerate double covers of* $\mathbb{P}^2$ *with discrete building data* $a = \frac{1}{2}(h+1)$.

Proof Note first that N and N_0 have the same closure in the moduli space. If $[S_0] \in \overline{N}_0$ then by valuative criterion, there exists a deformation $f\colon S \to \Delta$ of S_0 with $[S_t] \in N_0$ for every $t \neq 0$, and an involution τ acting on the punctured family $S^* \to \Delta^*$ such that $S_t/\tau = \mathbb{P}^2$ for every $t \neq 0$. Let $Y \to \Delta$ be the relative canonical model of $S \to \Delta$. Then Y_0 is a normal surface with at worst rational double points and ample canonical bundle and $Y_t = S_t$ for every $t \neq 0$. It is now an immediate consequence of [F–P, Prop. 4.4] (cf. also [Ma5]) that τ extends to a biregular involution of Y. As is well known, in general τ does not necessarily extend to a biregular involution of S (cf. [Ca3]).

The theorem now follows from Lemma 1.2. $\square$

Remark 1.7 In the notation of 1.4, if Def_X is prorepresentable (e.g., if $H^0(\theta_X) = 0$), then it can be proved that Proposition 1.4 holds without the assumption that $(H, 0)$ is smooth. Philosophically, this means that if $X \to Y$ is a flat double cover of normal surfaces ramified over a sufficiently ample divisor then $\mathrm{Def}\, X$ is isomorphic to the product of $\mathrm{Def}\, Y$ with a smooth germ $(\mathbb{C}^n, 0)$.

One of the main goals of the next sections is to prove that every degenerate double cover Y_0 has unobstructed deformations. The natural double cover $\pi\colon Y_0 \to W_0$ is not flat and so we cannot apply the above result (fortunately, since $\mathrm{Def}\, W_0$ is not irreducible, cf. [Rie]).

2 Vanishing theorems for degenerate double covers of $\mathbb{P}^2$

Before proving the main results of this section we need an explicit description of all cotangent vector fields with fixed poles on a Segre–Hirzebruch surface $\mathbb{F}_q$, $q > 1$. For this, we use the description

$$\mathbb{F}_q = (\mathbb{C}^2 - \{0\}) \times (\mathbb{C}^2 - \{0\}) / \sim$$

where $(l_0, l_1, t_0, t_1) \sim (\lambda l_0, \lambda l_1, \lambda^q \mu t_0, \mu t_1)$ for any $\lambda, \mu \in \mathbb{C}^*$.

From now on, by the standard torus action on $\mathbb{F}_q$ we mean the faithful $(\mathbb{C}^*)^2$ action given by

$$(\mathbb{C}^*)^2 \ni (\xi, \eta) \colon (l_0, l_1, t_0, t_1) \to (l_0, \xi l_1, \eta t_0, t_1).$$

$\mathbb{F}_q$ is covered by four affine planes $\mathbb{C}^2 \simeq U_{i,j} = \{l_i t_j \neq 0\}$ which are invariant for the standard torus action. In this affine covering, we choose local coordinates as follows:

$$\begin{aligned} U_{0,1}: &\quad z = l_1/l_0 \quad \text{and} \quad s = t_0/t_1 l_0^q, \\ U_{0,0}: &\quad z = l_1/l_0 \quad \text{and} \quad s' = t_1 l_0^q / t_0, \\ U_{1,0}: &\quad z' = l_0/l_1 \quad \text{and} \quad y' = t_1 l_1^q / t_0, \\ U_{1,1}: &\quad z' = l_0/l_1 \quad \text{and} \quad y = t_0/t_1 l_1^q. \end{aligned} \tag{2.1}$$

We call $U_{0,1}$ the principal affine subset and z, s principal affine coordinates. The other pairs of affine coordinates are related to s, z by

$$z' = z^{-1}, \quad s' = s^{-1} \quad y = s z^{-q} \quad \text{and} \quad y' = s^{-1} z^q = y^{-1}.$$

The map $\mathbb{F}_q \to \mathbb{P}^1$, $(l_0, l_1, t_0, t_1) \to (l_0, l_1)$ represents the Segre–Hirzebruch surface as a rational geometrically ruled surface, where $\sigma_\infty : \{t_1 = 0\}$, $\sigma_0 : \{t_0 = 0\}$ and $f : \{l_1 = 0\}$. Note that the rational function y gives the rational equivalence $\sigma_\infty \sim \sigma_0 - qf$.

Lemma 2.2 $h^0(\mathbb{F}_q, \Omega^1(p\sigma_0 + r\sigma_\infty)) = qp^2 - 1$ *for every* $p, q > 0$ *and* $r \geq 0$.

Proof $H^0(\Omega^1(p\sigma_0 + r\sigma_\infty))$ is the vector space of rational cotangent vector fields having at most poles of order p and r along σ_0 and σ_∞ respectively. The standard torus action induces an eigenspace decomposition

$$H^0(\Omega^1(p\sigma_0 + r\sigma_\infty)) = \bigoplus_{a,b \in \mathbb{Z}} M_{a,b}$$

where $\omega \in M_{a,b}$ if and only if

$$\omega = \alpha_{a,b} z^{a-1} s^b \mathrm{d}z + \beta_{a,b} z^a s^{b-1} \mathrm{d}s$$

on the open set $U_{0,1}$, for some complex numbers $\alpha_{a,b}, \beta_{a,b}$.

The same ω is written in $U_{0,0}$ as

$$\omega = \alpha_{a,b} z^{a-1} s'^{-b} \mathrm{d}z - \beta_{a,b} z^a s'^{-b-1} \mathrm{d}s'$$

and in $U_{1,1}$

$$\omega = -(\alpha_{a,b} + q\beta_{a,b}) z'^{-(a+1+qb)} y^b \mathrm{d}z' + \beta_{a,b} z'^{-(a+qb)} y^{b-1} \mathrm{d}y$$

Note that $\sigma_0 \cap U_{0,1} : \{s = 0\}$, $\sigma_\infty \cap U_{0,1} = \emptyset$, $\sigma_0 \cap U_{0,0} = \emptyset$, $\sigma_\infty \cap U_{0,0} : \{s' = 0\}$, $\sigma_0 \cap U_{1,1} : \{y = 0\}$ and $\sigma_\infty \cap U_{1,1} = \emptyset$. From the above local description of ω it follows immediately that $\omega \neq 0$ implies $b < 0$, and then there exists an isomorphism $H^0(\Omega^1(p\sigma_0 + r\sigma_\infty)) = H^0(\Omega^1(p\sigma_0))$. By reflexivity, every section of $\Omega^1(p\sigma_0)$ on $U_{0,1} \cup U_{0,0} \cup U_{1,1}$ extends to a unique section on $\mathbb{F}_q$, so that the following set of rational cotangent vector fields

$$\begin{cases} z^{a-1} s^b \mathrm{d}z & \text{for} \quad a \geq 1, \;\; 0 \geq b \geq -p, \;\; a + 1 + qb \leq 0, \\ z^a s^{b-1} \mathrm{d}s & \text{for} \quad a \geq 0, \;\; -1 \geq b \geq 1 - p, \;\; a + bq < 0, \\ -qz^{a-1} s^b \mathrm{d}z + z^a s^{b-1} \mathrm{d}s & \text{for} \quad -1 \geq b \geq 1 - p, \;\; a + bq = 0 \end{cases}$$

are $qp^2 - 1$ bihomogeneous sections of $\Omega^1(p\sigma_0)$; to prove that these form a basis is an easy calculation, that we omit. $\square$

Corollary 2.3 *For every $p, q, r > 0$, $h^1(\mathbb{F}_q, \theta) = q - 1$, $h^1(\mathbb{F}_q, \Omega^1(p\sigma_0)) = 1$ and $h^2(\mathbb{F}_q, \theta) = h^2(\mathbb{F}_q, \Omega^1(p\sigma_0)) = h^1(\mathbb{F}_q, \Omega^1(p\sigma_0 + rf)) = 0$.*

Proof The equality $h^1(\mathbb{F}_q, \theta) = q - 1$ is well known ([Ko]). By the Hodge decomposition and Serre duality we have $h^0(\mathbb{F}_q, \Omega^1) = h^2(\mathbb{F}_q, \theta(K)) = 0$ and $h^2(\mathbb{F}_q, \Omega^1) = 0$, and since both $-K$ and $p\sigma_0$ are effective divisors also $h^2(\mathbb{F}_q, \theta)$ and $h^2(\mathbb{F}_q, \Omega^1(p\sigma_0))$ vanish. By Riemann–Roch and Lemma 2.2 we then get $h^1(\mathbb{F}_q, \Omega^1(p\sigma_0)) = 1$.

For every $p, r > 0$ it follows from standard exact sequences that

$$h^1(\mathbb{F}_q, \Omega^1(p\sigma_0 + rf)) \leq h^1(\mathbb{F}_q, \Omega^1(\sigma_0 + f)) = h^0(\mathbb{F}_q, \Omega^1(\sigma_0 + f)) - q,$$

and using the method of proof of Lemma 2.2 we easily see that $z^{a-1}s^{-1}\mathrm{d}z$, for $0 \leq a \leq q - 1$ is a basis of $H^0(\mathbb{F}_q, \Omega^1(\sigma_0 + f))$, and the right-hand side above is 0. $\square$

Proposition 2.4 *In the surface $\mathbb{F}_q$, $q > 0$ we have:*

(i) $H^0(a\sigma_0 + bf) \neq 0$ *if and only if* $a \geq 0$ *and* $aq + b \geq 0$.

(ii) The linear system $|a\sigma_0 + bf|$ *contains a reduced divisor if and only if either* $a > 0, b \geq -q$ *or* $a = 0, b > 0$.

(iii) $H^1(a\sigma_0 + bf) = 0$ *if and only if either* $a = -1$ *or* $a \geq 0, b \geq -1$ *or* $a \leq -2, b \leq q - 1$.

(iv) For every pair of positive integers p, r*, the natural map*

$$H^0(p\sigma_0) \otimes H^0(r\sigma_0) \to H^0((p+r)\sigma_0)$$

is surjective; in particular the image of $\mathbb{F}_q$ *by the complete linear system* $|\sigma_0|$ *is projectively normal.*

(v) $P_{-1}(\mathbb{F}_q) = h^0(-K_{\mathbb{F}_q}) = \max(9, q+6)$.

Proof (i) and (ii) are clear since $|\sigma_0|$, $|f|$ are base point free and $\sigma_\infty \in |\sigma_0 - qf|$. By Serre duality it is sufficient to study the vanishing of h^1 only for $a \geq -1$. Using standard exact sequences and induction on $|b|$ we have

$$h^1(-\sigma_0 + bf) = h^1(-\sigma_0) = 0 \quad \text{for every } b \in Z,$$

and if $b \geq -1$, by induction on $a \geq 0$ we have

$$h^1(a\sigma_0 + bf) \leq h^1(-\sigma_0 + bf) = 0.$$

If $a \geq 0$ and $b \leq -2$ then we can write $a\sigma_0 + bf = \sigma_\infty + D$ where by (i) and Serre duality $h^2(D) = 0$, thus

$$h^1(a\sigma_0 + bf) \geq h^1(\mathcal{O}_{\sigma_\infty}(a\sigma_0 + bf)) = h^1(\mathcal{O}_{\mathbb{P}^1}(b)) > 0.$$

In the principal affine coordinates z, s a bihomogeneous basis of $H^0(p\sigma_0)$ is given by the monomials $s^{-a}z^b$ with $0 \leq a \leq p$ and $0 \leq b \leq aq$, so that (iv) follows immediately.

For every $q \geq 0$ we have $-K = \sigma_0 + \sigma_\infty + 2f$ and $K^2 = 8$. If $q \leq 3$ by (iii) and Serre duality $H^1(-K) = H^2(-K) = 0$ and $P_{-1} = 9$ by Riemann–Roch. If $q \geq 3$ then $-K \cdot \sigma_\infty < 0$ and $P_{-1} = h^0(\sigma_0 + 2f) = q + 6$. $\square$

Throughout the rest of this section a is a fixed integer ≥ 3. Let X be the Segre–Hirzebruch surface $\mathbb{F}_4$ and $\pi\colon S \to X$ the double cover ramified over $D = \sigma_\infty \cup D'$ with D' a reduced divisor linearly equivalent to $(2a-1)\sigma_0$. We assume that S has at worst rational double points and write $R \subset S$ for the ramification divisor.

We have $\pi_*\mathcal{O}_S = \mathcal{O}_X \oplus \mathcal{O}_X(-L)$ where $L = a\sigma_0 - 2f$ and $E = \pi^{-1}(\sigma_\infty)$ is a nodal curve. Denote by $\delta\colon S \to Y_0$ the contraction of E; then Y_0 is a surface with at worst rational double points and ample canonical bundle. We call $\delta(E) = y_0$ the *vertex* of the degenerate double cover Y_0.

By abuse of notation we use the same letter σ to denote the line bundles $\sigma_0 \in \operatorname{Pic} X$, $\pi^*\sigma_0 \in \operatorname{Pic} S$ and $\delta_*\pi^*\sigma_0 \in \operatorname{Pic} Y_0$. By the Hurwitz formula $K_S = \pi^*(K_X + L) = (a-2)\sigma$.

Lemma 2.5 *$H^1(Y_0, p\sigma) = 0$ for every integer p.*

Proof By the Leray spectral sequence we have

$$H^1(Y_0, p\sigma) = H^1(S, p\sigma) = H^1(X, p\sigma) \oplus H^1(X, (p-a)\sigma + 2f)$$

and the lemma follows from Proposition 2.4, (iii). $\square$

Lemma 2.6 *For every smooth curve C contained in a smooth surface S, $H^1_C(\Omega^1_S) \neq 0$.*

Proof There exists an inclusion $H^0(\mathcal{F} \otimes \mathcal{O}_C(C)) \subset H^1_C(\mathcal{F})$ for any locally free sheaf $\mathcal{F}$ on S (this is proved in [B–W, 1.5] for the tangent sheaf, but the same proof works for any locally free sheaf) and according to the exact sequence of differentials $H^0(\mathcal{O}_C) \subset H^0(\Omega^1_S \otimes \mathcal{O}_C(C))$. $\square$

Lemma 2.7 *If $p \geq 2a$ then $h^1(S, \Omega^1_S(K_S + p\sigma)) \leq 1$.*

Proof We consider the exact sequence on S (cf. (1.3))

$$0 \to \pi^*(\Omega^1_X(K_X + L + p\sigma)) \to \Omega^1_S(K_S + p\sigma) \to \mathcal{O}_R(\pi^*(K_X + p\sigma)) \to 0,$$

where $R \subset S$ is the ramification divisor.

Using the previous results, we get

$$\begin{aligned} &h^1(\mathcal{O}_D(K_X + p\sigma)) \\ &\qquad \leq h^1(X, (p-2)\sigma + 2f) + h^2(X, K_X + (p-2a)\sigma + 4f) = 0 \\ &h^1(\pi^*\Omega^1_X(K_X + L + p\sigma)) = h^1(\Omega^1_X(K_X + L + p\sigma)) + h^1(\Omega^1_X(K_X + p\sigma)) = 1 \end{aligned}$$

for $p \geq 2a$, and the proof follows from the equality $h^1(\mathcal{O}_R(\pi^*(K_X + p\sigma))) = h^1(\mathcal{O}_D(K_X + p\sigma))$. $\square$

Theorem 2.8 *In the above notation, $\operatorname{Ext}^1_{Y_0}(\Omega^1_{Y_0}, -p\sigma) = 0$ for every $p \geq 2a$.*

Proof Y_0 is a Gorenstein surface, in particular $K_{Y_0}+p\sigma$ is a Cartier divisor. By Serre duality ([Ha1, p. 243])

$$\mathrm{Ext}^1_{Y_0}(\Omega^1_{Y_0}, -p\sigma)^\vee = \mathrm{Ext}^1_{Y_0}(\Omega^1_{Y_0}(K_{Y_0}+p\sigma), K_{Y_0})^\vee = H^1(\Omega^1_{Y_0}(K_{Y_0}+p\sigma)).$$

We use the following exact sequence of sheaves on Y_0 ([Kas], [Pi2])

$$0 \to \Omega^1_{Y_0} \to \delta_*\Omega^1_S \xrightarrow{\alpha} \mathbb{C}_{y_0} \to 0,$$

where for every open subset $E \subset U \subset S$ and every $\omega \in H^0(U, \Omega^1_S)$, $\alpha(\omega) = 0$ if and only if the holomorphic 2-form $\mathrm{d}\omega$ vanishes in E. It is immediate to observe that $\Omega^1_{Y_0}$, being locally generated by closed 1-form, is contained in the kernel of α; the converse inclusion requires some computation ([Kas, p. 55]). Note moreover that, according to ([Ste], [Pi1, App.]), the sheaf $\delta_*\Omega^1_S$ is reflexive, and then the exactness of the above sequence is equivalent to the equality $H^1_{\{y_0\}}(Y_0, \Omega^1_{Y_0}) = \mathbb{C}$.

Twisting the above exact sequence by $K_{Y_0} + p\sigma = \delta_*(K_S + p\sigma)$ we get

$$0 \to \Omega^1_Y(K_{Y_0}+p\sigma) \to \delta_*\Omega^1_S(K_S+p\sigma) \xrightarrow{\alpha} \mathbb{C}_{y_0} \to 0.$$

Our first step is to prove that $H^1(\Omega^1_{Y_0}(K_{Y_0}+p\sigma)) = H^1(\delta_*\Omega^1_S(K_S+p\sigma))$ for $p \geq 2a$, that is, that α is surjective on global sections. Actually the following stronger result holds:

Lemma 2.9 *In the above notation if $p \geq 2$ then the composite of $H^0(\alpha)$ with the pullback map $\pi^*\colon H^0(\Omega^1_X(K_X+p\sigma)) \to H^0(\Omega^1_S(K_S+p\sigma))$ is surjective.*

Proof Let s, z be the principal affine coordinates on $X = \mathbb{F}_4$ and consider $\omega = s^{-2}\mathrm{d}z(\mathrm{d}z \wedge \mathrm{d}s) \in H^0(\Omega^1_X(K_X+p\sigma))$. In the open set $U_{0,0} \subset X$ with coordinates z, s', $\omega = \mathrm{d}z(\mathrm{d}s' \wedge \mathrm{d}z)$, $\sigma_\infty = \{s' = 0\}$ and locally S is the double cover of X defined by equation $\xi^2 = s'$ and then $\pi^*\omega = 2\xi\mathrm{d}z(\mathrm{d}\xi \wedge \mathrm{d}z)$.

Now $\mathrm{d}\xi \wedge \mathrm{d}z$ extends to a holomorphic invertible section of K_S in a neighbourhood of E and then, up to nonzero scalar multiplication, $\alpha(\pi^*\omega) = \alpha(\xi\mathrm{d}z) \neq 0$ since $\mathrm{d}(\xi\mathrm{d}z) = \mathrm{d}\xi \wedge \mathrm{d}z$. □

The Leray spectral sequence gives an exact sequence

$$0 \to H^1(\delta_*\Omega^1_S(K_S+p\sigma)) \to H^1(\Omega^1_S(K_S+p\sigma)) \xrightarrow{r} H^0(R^1\delta_*\Omega^1_S(K_S+p\sigma)),$$

and if $r \neq 0$ then by Lemma 2.7 the proof is complete.

For any open set $E \subset U \subset S$ there exists an exact sequence

$$0 \to H^0(U, \Omega^1_S(K_S+p\sigma)) \xrightarrow{\beta} H^0(U \setminus E, \Omega^1_S(K_S+p\sigma))$$
$$\xrightarrow{d} H^1_E(\Omega^1_S(K_S+p\sigma)) \xrightarrow{r_U} H^1(U, \Omega^1_S(K_S+p\sigma)).$$

On the open set $V = \delta(U) \subset Y$, the coherent sheaf $\delta_* \Omega^1_S(K_S + p\sigma)$ is reflexive, in particular the above map β is an isomorphism and the map r_U is injective.

Since $H^1_E(\Omega^1_S(K_S + p\sigma)) = H^1_E(\Omega^1_S) \neq 0$ the above inclusion factors as

$$H^1_E(\Omega^1_S) \subset H^1(S, \Omega^1_S(K_S + p\sigma)) \xrightarrow{r_U} H^1(U, \Omega^1_S(K_S + p\sigma)),$$

and then $r = \varinjlim r_U \neq 0$. $\square$

3 Deformations of degenerate double covers locally trivial at the vertex

Let a, $\pi \colon S \to X$ and $\delta \colon S \to Y_0$ be as in Definition 1.5. Then π is a flat double cover, and there exists a family of natural deformations of S obtained by deforming the branch divisor $D = \sigma_\infty \cup D' \sim 2a\sigma - 4f$. Since σ_∞ is a fixed part of the linear system $|D|$, the natural deformations are parametrized by $H^0(X, (2a-1)\sigma)$.

The singularity (Y_0, y_0) is rational, so that, as in [B–W], we can define the blowdown morphism $\beta \colon \mathrm{Def}_S \to \mathrm{Def}_{Y_0}$. It is clear that every (infinitesimal) natural deformation of S is trivial in a neighbourhood of E and its blowdown is a deformation of Y_0, locally trivial at y_0.

Thus taking first order deformations gives a commutative diagram

$$\begin{array}{ccc} H^0(X, (2a-1)\sigma) & \xrightarrow{\mathrm{Nat}} & T^1_S \\ \rho \downarrow & & \downarrow \beta \\ T^1 LT(Y_0, y_0) & \longrightarrow & T^1_{Y_0} \end{array}$$

where β is the blowdown map and $T^1 LT(Y_0, y_0)$ the kernel of the natural restriction map $T^1_{Y_0} \to T^1_{(Y_0, y_0)}$. Note that natural deformations never give a complete family of deformations of S, because the nodal curve E contributes to the space T^1_S ([B–W]).

Theorem 3.1 *The above map ρ is surjective and the blowdown of the family of natural deformations of S is a complete family of deformations of Y_0, locally trivial at the vertex, with smooth base space.*

Proof The exact sequence $(1.3')$ in this particular case becomes

$$H^0(\mathcal{O}_R(\pi^* D)) \xrightarrow{\varepsilon} \mathrm{Ext}^1_S(\Omega^1_S, \mathcal{O}_S) \xrightarrow{\sigma} H^1(\theta_X) \oplus H^1(\theta_X(-L)),$$

and the image of ε is the set of first order natural deformations. Given an open subset $V \subset X$, the inclusion $\pi^*\Omega^1_X \to \Omega^1_S$ induces a commutative diagram

$$\begin{array}{ccc} \mathrm{Ext}^1_S(\Omega^1_S, \mathcal{O}_S) & \longrightarrow & \mathrm{Ext}^1_{\pi^{-1}(V)}(\Omega^1_{\pi^{-1}(V)}, \mathcal{O}_{\pi^{-1}(V)}) \\ \sigma\downarrow & & \downarrow \\ H^1(\theta_X) \oplus H^1(\theta_X(-L)) & \xrightarrow{\gamma_V} & H^1(\theta_V) \oplus H^1(\theta_V(-L)). \end{array}$$

Lemma 3.2 *In the above set-up, if $\sigma_\infty \subset V$, then γ_V is injective.*

Proof of 3.2 It is clearly sufficient to prove that the two natural maps

$$\gamma_1 \colon H^1(\theta_X) \to H^1(\theta_X \otimes \mathcal{O}_{\sigma_\infty}), \quad \gamma_2 \colon H^1(\theta_X(-L)) \to H^1(\theta_X(-L) \otimes \mathcal{O}_{\sigma_\infty})$$

are isomorphisms.

Note first that $h^1(\theta_X \otimes \mathcal{O}_{\sigma_\infty}) = 3$, $h^1(\theta_X(-L) \otimes \mathcal{O}_{\sigma_\infty}) = 1$ and by Corollary 2.3, $h^1(\theta_X) = 3$, $h^1(\theta_X(-L)) = h^1(\Omega^1_X((a-2)\sigma_0)) = 1$, $h^2(\theta_X(-\sigma_\infty)) = h^0(\Omega^1_X(-\sigma_0 - 2f)) = 0$.

Hence γ_1 is surjective and then it is an isomorphism. To show that γ_2 is surjective, we prove that the natural map $H^2(\theta_X(-L-\sigma_\infty)) \to H^2(\theta_X(-L))$ or its Serre dual $H^0(\Omega^1_X((a-2)\sigma_0)) \to H^0(\Omega^1_X((a-2)\sigma_0 + \sigma_\infty))$ is an isomorphism, but this is exactly the result of Lemma 2.2. $\square$

Returning to the proof of Theorem 3.1, we note that the open sets $\pi^{-1}(V)$, $\sigma_\infty \subset V$ are a fundamental system of neighbourhoods of E. Thus from Lemma 3.2 it follows that for every open subset $U \subset S$ with $E \subset U$, the kernel of the natural map

$$\alpha \colon \mathrm{Ext}^1_S(\Omega^1_S, \mathcal{O}_S) \to \mathrm{Ext}^1_U(\Omega^1_U, \mathcal{O}_U)$$

is contained in the set of first order natural deformations $\ker\sigma = \mathrm{im}\,\varepsilon$.

We now apply this fact to a smooth open subset $E \subset U$ such that $\delta(U)$ is an affine open neighbourhood of y_0. According to the Cartesian diagram ([B–W])

$$\begin{array}{ccc} T^1_S & \xrightarrow{\alpha} & H^1(U, \theta_U) \\ \beta\downarrow & & \downarrow\beta_U \\ T^1_{Y_0} & \xrightarrow{r} & T^1_{Y_0, y_0} \end{array}$$

we have $\beta(\ker\alpha) = \ker r = T^1LT(Y_0, y_0)$ and since $\rho = \beta \circ \varepsilon$, the first part of the theorem is proved.

For the second part, we introduce the functor on Artin rings $LT(Y_0, y_0)$ of deformations of Y_0 which are locally trivial at the point y_0. More generally, for every complex space Z with isolated singularities and for every finite subset $\{z_1, \dots, z_n\} \subset Z$, we can define the functor D of deformations of Z which are locally trivial at the points $z_1, \dots, z_n$. This functor has been studied by several authors (cf. [F–M]); for example in [G–K, §1], it is proved that:

1. D satisfies the Schlessinger conditions [Sch, H1–H3].
2. There exists a closed analytic subgerm (possibly nonreduced) V of $\operatorname{Def} Z$ such that the restriction of the semiuniversal deformation of Z to V is a complete family of deformations locally trivial at $z_1, \dots, z_n$.
3. The Zariski tangent space of V is the kernel of the differential of the natural morphism $\operatorname{Def} Z \to \prod_i \operatorname{Def}(Z, z_i)$.

Applying these results to the functor $LT(Y_0, y_0)$ concludes the proof. □

4 The Kuranishi family of a degenerate double cover

Let $\pi \colon Y_0 \to W_0$ be a degenerate double cover of $\mathbb{P}^2$ ramified over the union of the vertex w_0 and a divisor $D' \sim (2a-1)\sigma$ with $a \geq 3$. Here we construct explicitly a smooth complete family of deformations of Y_0. This will imply in particular that the moduli space at Y_0 is locally irreducible and then the closure in the moduli space of the set $N(\mathbb{P}^2, \mathcal{O}(h))$ is a connected component for every $h \geq 4$.

The idea is to describe deformations of Y_0 as canonical coverings of suitable deformations of the cone W_0 and then prove that they give a complete family.

We first recall some well-known facts about cyclic coverings associated to $\mathbb{Q}$-Cartier divisors. For every normal complex space X we denote by $\mathcal{M}_X$ the sheaf of meromorphic functions on X and for every analytic Weil divisor $D \subset X$ by $\mathcal{O}_X(D)$ the reflexive subsheaf of $\mathcal{M}_X$ of meromorphic functions f such that $\operatorname{div}(f) + D \geq 0$. We keep this explicit description of $\mathcal{O}_X(D)$ throughout this section.

Let L be a Weil divisor on a normal irreducible variety X such that nL is Cartier and let $s \in H^0(X, nL)$ be a meromorphic function such that the divisor $D = \operatorname{div}(s) + nL$ is reduced and is contained in the set of points where L is Cartier.

Multiplication by s gives a morphism of $\mathcal{O}_X$-modules $\mathcal{O}_X(-nL) \to \mathcal{O}_X$ and we may define in a natural way a coherent analytic reflexive $\mathcal{O}_X$-algebra (cf. [Reid, 3.6], [E–V, 1.4]):

$$\mathcal{A}(L, s) = \bigoplus_{i=0}^{n-1} \mathcal{A}_i = \bigoplus_{i=0}^{n-1} \mathcal{O}_X(-iL).$$

If (X, x) is a normal analytic singularity, its local analytic class group is by definition the quotient of the free Abelian group generated by the germs of analytic Weil divisors modulo the subgroup of principal divisors. For a 2-dimensional rational singularity, it is a finite group naturally isomorphic to the first homology group of the link of X ([Bri]).

Lemma 4.1 *Let n, L, s, D be as above. If $x \notin D$ then up to isomorphism, the local analytic $\mathcal{O}_x$-algebra $\mathcal{A}_x(L, s)$ only depends on the class of L in the local analytic class group of the analytic singularity (X, x).*

Proof Let n, L', s', D' be another set of data with $x \notin D'$ and assume that $L - L'$ is principal at x. This means that there exists an analytic open neighbourhood U of x and a meromorphic function f on U such that $L = L' + \mathrm{div}(f)$ and $\mathrm{div}(s)_{|U} = -nL$, $\mathrm{div}(s')_{|U} = -nL'$.

Therefore $s^{-1}s'f^{-n}$ is an invertible holomorphic function on U and, possibly shrinking U, we may assume that it admits an nth root g. Thus $s = s'(fg)^n$ and the multiplication map $(fg)^i \colon \mathcal{O}_U(-iL') \to \mathcal{O}_U(-iL)$ gives the required isomorphism. □

The cyclic group μ_n acts on the algebra $\mathcal{A}$ by

$$\mu_n \times \mathcal{A}_i \ni (\xi, h) \mapsto \xi^{-i}h \in \mathcal{A}_i,$$

and then the finite map

$$\pi \colon Z = \mathrm{Specan}_X(\mathcal{A}(L, s)) \to X$$

is a cyclic covering of normal varieties (here Specan ([Fi, 1.14]) is the analytic spectrum; if X is projective then by GAGA principles it is the same as the usual algebraic spectrum ([Ha1, II, Ex. 5.17)]).

According to Lemma 4.1, if $x \notin \mathrm{div}(s) + nL$, the germ of the covering over the point x is independent of s.

Corollary 4.2 *In the above set-up, assume X compact and let T be a sufficiently small analytic open neighbourhood of s in $H^0(X, nL)$. Let $\pi \colon Z_T \to X \times T$ be the cyclic covering of degree n associated to the Weil divisor $L \times T$ and multiplication given by $s(x, t) = t(x)$, $t \in T$.*

If $X \to S$ is a flat map such that the composite $Z \to X \to S$ is flat then also the composite $Z_T \to X \times T \to S \times T$ is flat.

Proof Let $U \subset X$ be the open subset where L is Cartier. If T is sufficiently small then $s_t(x) = 0$ for some $t \in T$ implies that $x \in U$. Hence if $x \notin U$ then by Lemma 3.1 the germ of Z_T over (x, s) is locally isomorphic to $Z \times T$. On the other hand the map $U \times T \to S \times T$ is flat and the restriction of the algebra $\mathcal{A}$ over $U \times T$ is locally free and then the restriction of π over $U \times T$ is a flat map. $\square$

Therefore, in case $S =$ point, we have a morphism from deformations of s to deformations of Z. Consider for example the hypersurface $Z \subset \mathbb{P}^3 \times \mathbb{C}$ with equation $z_1z_2 - z_3^2 = tz_0^2$ for $t \in \mathbb{C}$, and the involution $\tau\colon Z \to Z$ given by $\tau(t, z_0, z_1, z_2, z_3) = (t, z_0, -z_1, -z_2, -z_3)$.

Let $t\colon Z \to \mathbb{C}$ be the projection on the coordinate t and Z_t the projective subvariety of Z of points with fixed t. It is immediate to observe that Z_t is a smooth quadric for $t \neq 0$, whereas Z_0 is the cone over a nonsingular conic, and t gives the semiuniversal deformation of the isolated singularity $(Z_0, (1, 0, 0, 0, 0))$.

The quotient Z/τ is the variety $W \subset \mathbb{P}^5 \times \mathbb{C}$ defined by the equation

$$\operatorname{rank}\begin{pmatrix} x_1 & x_2 & x_3 + tx_0 \\ x_2 & x_3 & x_4 \\ x_3 + tx_0 & x_4 & x_5 \end{pmatrix} \leq 1, \tag{4.3}$$

where $x_0 = z_0^2$, $x_1 = z_1^2$, $x_2 = z_1z_3$, $x_3 = z_3^2$, $x_4 = z_2z_3$, $x_5 = z_2^2$.

The quotient family $W \to \mathbb{C}$, $(x, t) \to t$ is a deformation of W_0 and is exactly the degeneration of $\mathbb{P}^2$ obtained by sweeping out the cone over the Veronese surface $V \subset \mathbb{P}^5$. To see this, let $C(V, v) \subset \mathbb{P}^6$ be the projective cone over the image of the map $\mathbb{P}^2_u \to \mathbb{P}^5_x$, $x_1 = u_0^2$, $x_2 = u_0u_1$, $x_3 = u_1^2$, $x_4 = u_1u_2$, $x_5 = u_2^2$, $x_6 = u_0u_2 - u_1^2$. It is defined by the equation

$$\operatorname{rank}\begin{pmatrix} x_1 & x_2 & x_3 + x_6 \\ x_2 & x_3 & x_4 \\ x_3 + x_6 & x_4 & x_5 \end{pmatrix} \leq 1. \tag{4.4}$$

V is the intersection of $C(V, v)$ with the hyperplane $x_0 = 0$ and the vertex v is the point with homogeneous coordinates $(1, 0, 0, 0, 0, 0, 0)$.

Let $H_t \subset \mathbb{P}^6$, $t \in \mathbb{C}$ be the hyperplane of equation $x_6 - tx_0 = 0$. Then $H_t \cap V = V \cap \{x_6 = 0\}$ is a smooth hyperplane section and the surface $W_t = C(V, v) \cap H_t$ is exactly the surface defined in (4.3).

Let $H \subset W$ be the Weil divisor defined by the equation $x_2 = x_3 = x_4 = 0$. Then $\mathcal{O}_W(-H)$ is the ideal sheaf of H, and $2H$ is the hyperplane section $x_3 = 0$ of W. In fact the closed subset $\{x_1 = x_3 = x_5 = 0\}$ has codimension 3 in W and then it is sufficient to prove the equality $2H = \operatorname{div}(x_3)$ on its complement. An easy computation shows that on every affine subset $W \cap \{x_i \neq 0\}$ $i = 1, 3, 5$ the equality of ideals $(x^2x_i^{-1}, x^3x_i^{-1}, x^4x_i^{-1})^2 = (x^3x_i^{-1})$ holds.

Note that $\pi_*\mathcal{O}_Z = \mathcal{O}_W \oplus (z_0/z_3)\mathcal{O}_W(-H)$ and then there exists an isomorphism of $\mathcal{O}_W$-algebras $\pi_*\mathcal{O}_Z = \mathcal{O}_W \oplus \mathcal{O}_W(-H)$, where the algebra structure on the right is induced by multiplication $x_0/x_3\colon \mathcal{O}_W(-2H) \to \mathcal{O}_W$.

Now let $\pi_0\colon Y_0 \to W_0 \subset \mathbb{P}^5$ be a fixed degenerate double cover. Then, according to Proposition 2.4, W_0 is projectively normal in $\mathbb{P}^5$ and then there exists a section $s_0 \in H^0(\mathbb{P}^5, \mathcal{O}(2a-1))$ such that π_0 is ramified over the union of $\{w_0\}$ with the divisor of the restriction of s_0 to W_0.

Let T be a small open neighbourhood of s_0 and consider the double cover

$$Y_T = \mathrm{Specan}_{W\times T}\Big(\mathcal{O}_{W\times T} \oplus \mathcal{O}_{W\times T}(-(2a-1)H \times T)\Big) \to W \times T,$$

where the algebra structure is induced by the section $s(x,t) = s_t(x)$ for $s_t \in T$ and $x \in W$. This makes sense since $2H \times T$ is a Cartier divisor linearly equivalent to $\{s(x,t) = 0\}$.

By previous results (4.1, 4.2) it follows that:

(i) The map $Y_T \to T$ is a deformation of the space

$$Y = \mathrm{Specan}_W\Big(\mathcal{O}_W \oplus \mathcal{O}_W(-(2a-1)H)\Big)$$

with the algebra structure induced by s_0.

(ii) Over the vertex w_0 the space Y is isomorphic to the above space Z and then the composite $Y \to W \to \mathbb{C}$ gives a complete deformation of the node (Y_0, y_0).

It is now easy to prove the following

Theorem 4.5 *In the above notation the composite*

$$f\colon Y_T \to W \times T \to \mathbb{C} \times T$$

is a smooth complete family of deformations of Y_0.

Proof We need to prove that $f^{-1}(0, s_0) = Y_0$ and that the Kodaira–Spencer map of the family is surjective.

By definition $f^{-1}(0,s_0) = \mathrm{Spec}_{W_0}\big(\mathcal{O}_{W_0} \oplus (\mathcal{O}_W(-(2a-1)H) \otimes \mathcal{O}_{W_0})\big)$ while by definition and from the normality of Y_0 we have $Y_0 = \mathrm{Spec}_{W_0}(\mathcal{O}_{W_0} \oplus \mathcal{O}_{W_0}(-L))$ where $L = a\sigma - 2l$, and $l \subset W_0$ is a line through w_0.

Note that all lines through w_0 are linearly equivalent, L is linearly equivalent to $(4a-2)l$, the intersection $H_0 = H \cap W_0$ is the union of the two lines $l_1 = \{x_1 = x_2 = x_3 = x_4 = 0\}$, $l_2 = \{x_5 = x_2 = x_3 = x_4 = 0\}$ and then the natural map $j_n\colon \mathcal{O}_W(nH) \otimes \mathcal{O}_{W_0} \to \mathcal{O}_{W_0}(2nl)$ is an isomorphism over $W_0 \setminus \{w_0\}$ for every integer n.

In a neighbourhood of the vertex w_0, since the sheaf $\mathcal{O}_W(nH)$ is reflexive on W and invertible for even n, by [E–V, 2.1], the map j_n is injective for every n and an isomorphism for even n. Moreover, the ideal of $H_0 \subset W_0$ is generated by $x_2x_0^{-1}, x_3x_0^{-1}, x_4x_0^{-1}$ and then j_{-1} is also surjective. Tensoring with the line bundle $\mathcal{O}_W(2pH)$, $p \in \mathbb{Z}$, we get the surjectivity of j_n for every integer n. In particular, since j_{1-2a} is an isomorphism, Y_0 is a fibre of f.

By (ii) the composite of the Kodaira–Spencer map of f with the natural restriction map $r\colon T^1(Y_0) \to T^1(Y_0, y_0)$ is surjective, therefore it is sufficient to prove that Y_T contains every deformation locally trivial at the vertex. But this is an immediate consequence of Theorem 3.1 and the surjectivity of the map $H^0(\mathbb{P}^5, \mathcal{O}(2a-1)) \to H^0(W_0, (2a-1)\sigma) = H^0(\mathbb{F}_4, (2a-1)\sigma)$. □

Corollary 4.6 *Every degenerate double cover deforms to a smooth double cover of $\mathbb{P}^2$; in particular, for every odd integer $h \geq 5$, the subset $N(\mathbb{P}^2, \mathcal{O}(h))$ is not closed in the moduli space.*

Corollary 4.7 *The line bundle σ of Y_0 can be extended to every deformation of Y_0.*

Proof The pullback of the hyperplane section $2H$ to Y_T is an extension of σ to a complete family. □

Proof of Theorem A (1) and (2) follow from Theorem 1.6. According to Theorem 4.5 the set of (possibly degenerate) double covers of $\mathbb{P}^2$ is stable under small deformations, therefore $\overline{N}$ is locally open in the moduli space, proving (3) and (4). □

5 Simple iterated double covers of $\mathbb{P}^2$ and their deformations

This section is almost entirely devoted to the proof of Theorem B of the introduction. The first preliminary result we need is the following

Lemma 5.1 *Let $\pi\colon X \to Y$ be a simple iterated double cover associated to a sequence $L_1, \ldots, L_n \in \operatorname{Pic} Y$. Assume that Y and $L_1, \ldots, L_n$ satisfy the conditions:*

(a) $\operatorname{Def} Y$ *is smooth.*

(b) $L_1, \ldots, L_n$ extends to a complete deformation of Y.

(c) For every $h \geq 0$ and $0 < i < j_1 < \cdots < j_h \leq n$,

$$H^1(Y, 2L_i - \sum_{s=1}^{h} L_{j_s}) = 0.$$

(d) $H^1(\mathcal{O}_Y) = 0$.

(e) For every $h \geq 0$ *and every sequence* $1 \leq j_1 < j_2 < \cdots < j_h \leq n$,

$$\operatorname{Ext}^1_{\mathcal{O}_Y}(\Omega^1_Y, \sum_{s=1}^{h} -L_{j_s}) = 0 \quad \text{and} \quad H^1(Y, \sum_{s=1}^{h} -L_{j_s}) = 0.$$

(f) $H^0(L_i) \neq 0$ *for every* i, *and* $H^0(Y, 2L_i - L_j) = 0$ *for every* $j < i$.

Then $\operatorname{Def} X$ *is smooth, every deformation of* X *is a simple iterated double cover of a deformation of* Y *and if* $M \in \operatorname{Pic} Y$ *extends to a complete deformation of* Y *then* $\pi^* M$ *extends to a complete deformation of* X.

Proof Using the computation (involving (f)) about Ext groups in the proof of [Ma3, 2.7] and induction on n, we can reduce the proof of the lemma to the case $n = 1$. In this case the cohomological conditions (c), (d) and (e) become

$$H^1(\mathcal{O}_Y) = H^1(Y, -L) = H^1(Y, 2L) = \operatorname{Ext}^1_{\mathcal{O}_Y}(\Omega^1_Y, -L) = 0.$$

Then X is defined in the total space of the line bundle L by the equation $z^2 = v_0$ with $v_0 \in H^0(Y, 2L)$. Let $\widetilde{Y} \to \operatorname{Def} Y$ be the Kuranishi family of Y and $\widetilde{L} \in \operatorname{Pic} \widetilde{Y}$ the extension of L (which is unique by (d)). Since $H^1(Y, 2L) = 0$, by semicontinuity and base change there exists a subspace $V \subset H^0(\widetilde{Y}, 2\widetilde{L})$ such that the natural restriction $V \to H^0(Y, 2L)$ is an isomorphism.

Taking $H \subset V$ a small neighbourhood of v_0 we can consider the flat double cover $\widetilde{\pi}\colon \widetilde{X} \to \widetilde{Y} \times V$ defined by equation

$$z^2 = v(y) \quad \text{with} \quad y \in \widetilde{Y} \text{ and } v \in V.$$

By construction the flat maps

$$\widetilde{X} \xrightarrow{\tilde{\pi}} \widetilde{Y} \times V \to \operatorname{Def} Y \times V$$

are deformations of the double cover $\pi\colon X \to Y$ and satisfy the assumptions of Proposition 1.4. Therefore $\widetilde{X}$ is a complete deformation of X, $\operatorname{Def} X$ is smooth and it is clear that if $\widetilde{M} \in \operatorname{Pic} \widetilde{Y}$ extends M then $\widetilde{\pi}^* \widetilde{M}$ extends $\pi^* M$. Note that if the generic deformation of Y is smooth and the linear system $|2L|$ is base point free then a generic small deformation of X is also smooth. □

We recall that $\operatorname{Ext}^1_{\mathcal{O}_{\mathbb{P}^2}}(\Omega^1_{\mathbb{P}^2}, \mathcal{O}_{\mathbb{P}^2}(-h)) = 0$ for every $h \neq 3$ and therefore we have the result:

Corollary 5.2 *Let* $\pi\colon X \to \mathbb{P}^2$ *be a simple iterated double cover associated to the sequence of line bundles* $L_1, \ldots, L_n$ *with* $\deg L_i = l_i$. *If* $l_i > 2l_{i+1}$ *and* $l_n \geq 4$ *then every deformation of* X *is a simple iterated double cover of* $\mathbb{P}^2$ *and the set* $N(\mathbb{P}^2, L_1, \ldots, L_n)$ *is open in the moduli space.*

Proof Immediate consequence of Lemma 5.1. □

Next we want to classify all the possible degenerations of simple iterated double covers of $\mathbb{P}^2$. For reasons that will be clear later, we consider only the case where the degrees l_i satisfy certain numerical conditions.

Definition 5.3 A sequence of line bundles $L_1, \ldots, L_n$, $L_i = \mathcal{O}_{\mathbb{P}^2}(l_i)$ is called a *good sequence* if satisfies the following 3 conditions:

(1) $l_i \geq 4$ for every $i = 1, \ldots, n$.

(2) $l_i > 2l_{i+1}$ for every $i = 1, \ldots, n-1$.

(3) l_n is odd and l_i is even for $i = 1, \ldots, n-1$.

A *good* simple iterated double cover of $\mathbb{P}^2$ is, by definition, a simple iterated double cover associated to a good sequence.

Proposition 5.4 *Let $L_1, \ldots, L_n \in \operatorname{Pic}\mathbb{P}^2$ be a good sequence, $L_i = \mathcal{O}(l_i)$ and let X_0 be the canonical model of a surface belonging to the closure of $N(\mathbb{P}^2, L_1, \ldots, L_n)$. Then either X_0 is a simple iterated double cover of $\mathbb{P}^2$ associated to $L_1, \ldots, L_n$, or there exists a degenerate double cover Y_0 of $\mathbb{P}^2$ with discrete building data $a = \frac{1}{2}(l_n + 1)$ such that X_0 is a simple iterated double cover of Y_0 associated to the sequence $M_1, \ldots, M_{n-1}$, $M_i = (l_i/2)\sigma$.*

Proof Note that if $\pi\colon X \to \mathbb{P}^2$ is a smooth simple iterated double cover associated to a good sequence $\mathcal{O}(l_i)$ then $K_X = \pi^*\mathcal{O}(\sum l_i - 3)$ is ample and then the subset $N_0 \subset N$ of surfaces with smooth canonical models is a Zariski open dense subset, in particular the closure of N_0 is the same as the closure of N in the moduli space.

Let $f\colon X \to \Delta$ be a deformation of X_0 such that X_t is a smooth simple iterated double cover of $\mathbb{P}^2$ associated to $L_1, \ldots, L_n$ for $t \neq 0$.

We now prove by induction on n that, up to base change, there exists a factorization

$$f\colon X \xrightarrow{p} Y \xrightarrow{g} \Delta,$$

where g is a deformation of a (possibly degenerate) double cover Y_0 of the projective plane with Y_t a smooth double cover associated to L_n for $t \neq 0$, and p is a simple iterated double cover of Y associated to $\widetilde{M}_1, \ldots, \widetilde{M}_{n-1}$ with $\widetilde{M}_i$ the unique extension of M_i to Y (if Y_0 is not degenerate we set $M_i = g_0^* L_i$).

This holds trivially if $n = 1$, so we assume $n > 1$. As in the proof of Theorem 1.6, after a possible base change, the action of the trivial involution

τ of the surfaces X_t extends to an action over X, and we have a quotient family

$$f'\colon Z = X/\tau \to \Delta.$$

Now Z_t is a simple iterated double cover of $\mathbb{P}^2$ associated to the sequence $L_2, \ldots, L_n$ for $t \neq 0$. By the Hurwitz formula, for $t \neq 0$ the canonical bundle of Z_t is the pullback of $\mathcal{O}_{\mathbb{P}^2}(l_2 + \cdots + l_n - 3)$ and then its divisibility in $H^2(Z_t, \mathbb{Z})$ is even. By [Ma3, 3.1], Z_0 has at worst rational double points and the double cover $\pi\colon X \to Z$ is flat. Now by the Brieskorn–Tyurina simultaneous resolution and semicontinuity, the minimal resolution of Z_0 is a smooth minimal surface of general type and, taking the relative canonical model of Z we have, up to a base change, a factorization

$$X \xrightarrow{\pi} Z = X/\tau \xrightarrow{\delta} Z_{\mathrm{can}} \xrightarrow{p} Y \xrightarrow{g} \Delta,$$

with p, g and Y as in the induction hypothesis.

Thus, to conclude the proof we only need to show that δ is an isomorphism, i.e., that the canonical line bundle of Z_0 is ample. Since π is flat there exists a decomposition

$$\pi_* \mathcal{O}_X = \mathcal{O}_Z \oplus p^* \mathcal{O}_Z(-H),$$

where H is a line bundle over Z such that H_t is the pullback of L_1 for every $t \neq 0$. If $\widetilde{M}_1 \in \operatorname{Pic} Y$ is the extension of $\frac{1}{2}(l_n + 1)\sigma$ then, $\widetilde{M}_{1,t}$ is the pullback of L_1 over Y_t and then, since $q(Z_0) = 0$, by injectivity of the extension ([Ma3, 3.8]) we have $H = \delta^* p^* \widetilde{M}_1$.

Assume that K_{Z_0} is not ample; then δ contracts some irreducible curve C with $K_{Z_0} \cdot C = 0$, $H_0 \cdot C = 0$. If $C' \subset X_0$ is the strict transform of C then by the Hurwitz formula $K_{X_0} \cdot C' = 0$, which is a contradiction. $\square$

Proof of Theorem B Theorem B, (1) is exactly Lemma 5.1 applied to the case $Y = \mathbb{P}^2$, $L_i = \mathcal{O}_{\mathbb{P}^2}(l_i)$. In B.2, $L_1, \ldots, L_n$ is a good sequence (Definition 5.3), so that by Proposition 5.4, if $[S] \in \overline{N} \setminus N$ then the canonical model of S is a simple iterated double cover of Y, a degenerate double cover of $\mathbb{P}^2$ with discrete building data $a = \frac{1}{2}(l_n + 1)$, associated to the sequence $M_1, \ldots, M_{n-1} \in \operatorname{Pic} Y$, $M_i = (l_i/2)\sigma$ and the smoothness of $\operatorname{Def} X$ follows from Lemma 5.1 and the vanishing theorems of §2.

B.3 follows immediately from B.1, B.2 and the local properties of the moduli space of surfaces of general type. $\square$

6 Some examples

For every smooth surface of general type S we denote by $I_S = K_S^2 - 8\chi(\mathcal{O}_S)$ its index and by $r(S)$ the divisibility of the canonical class:

$$r(S) = \max\{r \in \mathbb{N} \mid k_S = rc \text{ for some } c \in H^2(S, \mathbb{Z})\}.$$

If S is a smooth simple iterated double cover of $\mathbb{P}^2$ associated to a good sequence $L_1, \ldots, L_n$ with $\deg(L_i) = l_i$, then using the formulas of [Ma3] we have $\pi_1(S) = 0$, and

$$K_S^2 = 2^n(\sum l_i - 3)^2, \quad I_S = 2^n(1 - \sum l_i^2), \quad r(S) = (\sum l_i - 3),$$

so that the invariants K^2, I and r only depend on the two positive integers $\sum l_i$, and $\sum l_i^2$.

Example 6.1 For $n = 3$ we can consider the two sequences

$$(l_1, l_2, l_3) = (3T - 24, T, 5) \quad \text{and} \quad (l_1', l_2', l_3') = (3T - 22, T - 6, 9).$$

Then $\sum l_i = \sum l_i'$, $\sum l_i^2 = \sum l_i'^2$ and for every even $T \geq 26$, both $L_i = \mathcal{O}(l_i)$ and $L_i' = \mathcal{O}(l_i')$ are good sequences.

For $T = 26$ the associated simple iterated double covers have $K^2 = 53792$, $I = -28928$, $c_2 = 70288$, $r = 82$. It is not difficult to prove that every two distinct good sequences $L_1, \ldots, L_n$, $M_1, \ldots, M_m$ give distinct connected components $N(\mathbb{P}^2, L_1, \ldots, L_n)$ and $N(\mathbb{P}^2, M_1, \ldots, M_m)$. To see this, it is enough to show that for the generic $[S] \in N(\mathbb{P}^2, L_1, \ldots, L_n)$ we have $\operatorname{Aut} S = \mathbb{Z}/2\mathbb{Z} = \{1, \tau\}$. In fact S/τ must necessarily belongs to $N(\mathbb{P}^2, L_2, \ldots, L_n)$, and then we use induction on n.

We don't prove here the above statement about automorphisms. However, this is a straightforward generalization of the analogous result for simple iterated double covers of $\mathbb{P}^1 \times \mathbb{P}^1$ proved in [Ma3], as well an immediate application of a general result about automorphisms of generic simple cyclic covers proved in [Ma5].

Example 6.2 Let $X \to \mathbb{P}^2$, $Y \to \mathbb{P}^1 \times \mathbb{P}^1$ be simple iterated double covers associated to $L_1 = \mathcal{O}(26), L_2 = \mathcal{O}(12), L_3 = \mathcal{O}(5)$ and $L_1' = \mathcal{O}(20, 40), L_2' = \mathcal{O}(22, 2)$. A calculation shows that X and Y have the same invariants K^2, I, r, and it is not difficult to see that X, Y do not belong to the same connected component of the moduli space.

In fact the equation of a generic Y is

$$\begin{cases} z^2 = f + wh & \text{with } f \in H^0(\mathcal{O}(40, 80)) \text{ and } h \in H^0(\mathcal{O}(18, 78)), \\ w^2 = g & \text{with } g \in H^0(\mathcal{O}(44, 4)), \end{cases}$$

where f, g, h are generic, and the arguments of [Ma3] show that the only automorphisms of Y are the identity and the "trivial" involution $z \mapsto -z$. Its quotient is the surface $Y_1 = \{w^2 = g\}$. Since the invariants of Y_1 are different from those of surfaces in $N(\mathbb{P}^2, L_2, L_3)$, the surface Y cannot belong to $N(\mathbb{P}^2, L_1, L_2, L_3)$.

Although explicitly finding simple iterated double covers of $\mathbb{P}^2$ having the same invariants is not easy, it is not difficult to use these surfaces to obtain again a lower bound of the form $\delta \geq (K^2)^{c \log K^2}$ for the number of connected components of moduli space, where c is a positive constant. In fact for sufficiently large n, if q_n is the number of sequences $l_1, \ldots, l_n$ such that $\sum l_i = T_n = 8 \cdot 3^n + 3$, $l_n \geq 5$ odd, l_i even for $i < n$ and $l_i > 2l_{i+1}$ then $\log q_n \geq an^2$ for a positive constant a independent of n. For each of these q_n sequences, its quadratic sum $\sum l_i^2$ is smaller than T_n^2, and then there exists at least q_n/T_n^2 good sequences giving simple iterated double covers with the same invariants $K^2 = 2^n T_n^2$ and $I = 2^n(1 - \sum l_i^2)$. An easy computation along the lines of [Ma3, §5] gives the claimed lower bound for δ.

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