Bounds for Seshadri constants

Oliver Küchle* Andreas Steffens*

Introduction

This paper develops a new approach to bounding Seshadri constants of nef and big line bundles at a general point of a complex projective variety. A modification of this approach even allows us to give bounds valid at arbitrary points.

The Seshadri constant $\varepsilon(L, x)$, introduced by Demailly [De92], measures the local positivity of a nef line bundle L at a point $x \in X$ of a complex projective variety X; it can be defined as

$$\varepsilon(L, x) := \inf_{C \ni x} \left\{ \frac{L \cdot C}{\operatorname{mult}_x(C)} \right\},$$

where the infimum is taken over all reduced irreducible curves $C \subset X$ passing through x. The interest in Seshadri constants comes in part from the fact that they govern an elementary method for producing sections in adjoint line bundles $\mathcal{O}_X(K_X + rL)$ with certain properties. This connects the theory of Seshadri constants to the famous conjectures of Fujita on global generation and very ampleness of such bundles (cf. §1 below for further characterizations and properties of Seshadri constants; see also [De92] and [EKL]).

There has been considerable progress in the study of Seshadri constants in recent years, starting with Ein and Lazarsfeld's result (cf. [EL]) that the Seshadri constant of an ample line bundle on a smooth surface is ≥ 1 for all except possibly countably many points. On the other hand, examples by Miranda (cf. [EKL, 1.5]) show that for any integer $n \geq 2$ and any real number $\delta > 0$ there is a smooth n-fold X, an ample line bundle L on X and a point $x \in X$ with $\varepsilon(L, x) < \delta$; in other words, there does not exist a universal lower bound for Seshadri constants valid for all X and ample L at every point $x \in X$.

Then it was proved by Ein, Küchle and Lazarsfeld [EKL] that, for a nef and big line bundle L on a projective n-fold, the Seshadri constant at very

*Supported by the Deutsche Forschungsgemeinschaft

general points (that is, outside a countable union of proper subvarieties) is bounded below by $1/n$, which implies the existence of a lower bound at general points depending only on n.

Finally, we want to mention the recent papers by Lazarsfeld [La] and Nakamaye [Na] dealing with Seshadri constants on Abelian varieties, as well as variants due to Küchle [Kü], Xu [Xu95] and Paoletti [Pa] concerning Seshadri constants along several points and higher dimensional subvarieties respectively.

Our approach to bounding Seshadri constants is based on considering families of divisors with high multiplicity at assigned points. The method itself relies upon ideas of Demailly's paper [De93] as explained and translated into the language of algebraic geometry by Ein, Lazarsfeld and Nakamaye [ELN]. Very roughly, the idea is to start with an effective divisor E in a linear system $|kL|$ (for $k \gg 0$) with large multiplicity at x, and to consider the locus V of points where the singularities of E are "concentrated" in a certain way. The possibility that V is zero dimensional imposes constraints on the local positivity of L at x in a sense. Otherwise one uses variational techniques to give a lower bound on the degree of V.

In contrast to [De93] and [ELN], instead of making a positivity assumption on the tangent bundle of the manifold in question to be able to "differentiate", we apply the strategy of differentiation in parameter directions in the spirit of [EKL]. The result of this method is the following theorem on linear system, which might also be of interest in other contexts:

Theorem 1 *Let X be a smooth projective n-fold, L a nef and big line bundle on X and $\alpha > 0$ a rational number such that $L^n > \alpha^n$. Let*

$$0 = \beta_1 < \beta_2 < \cdots < \beta_n < \beta_{n+1} = \alpha$$

be any sequence of rational numbers and $x \in X$ a very general point. Then one of the following holds:

(a) there exist $k \gg 0$ and a divisor $E \in |kL|$ having an isolated singularity with multiplicity at least $k(\beta_{n+1} - \beta_n)$ at x, or

(b) there exists a proper subvariety $V \subset X$ through x of codimension $c < n$ such that

$$\begin{aligned} \deg_L V = L^{n-c} \cdot V &\le \frac{1}{(\beta_{c+1} - \beta_c)^c} \left(1 - \sqrt[n]{\left(1 - \frac{\alpha^n}{L^n}\right)^c}\right) L^n \\ &< \frac{\alpha^n}{(\beta_{c+1} - \beta_c)^c} \,. \end{aligned}$$

To pass from Theorem 1 to actually bounding the Seshadri constant, we use a rescaling trick (cf. Remark 3.2) in combination with the well-known characterization of Seshadri constants via the generation of s-jets by certain adjoint linear systems (cf. Theorem 1.5). The bound we get does not improve that of [EKL] in general, although the method at hand may give better results in certain cases, since our bound can be expressed more flexibly in terms of the degrees of subvarieties with respect to the line bundle in question:

Theorem 2 *Let L be a nef line bundle on an irreducible projective n-fold X, and $x \in X$ a very general point. Suppose given positive rational numbers $\alpha_1, \ldots, \alpha_n$, and set $\gamma = 1 + \sum_{i=1}^{n-1} \alpha_i$. For some real number $\varepsilon > 0$, suppose that every d-dimensional subvariety $V \subset X$ containing x (for any d with $1 \leq d \leq n$) satisfies*

$$\deg_L V = L^d \cdot V \geq \varepsilon^d \gamma^n \alpha_d^{d-n}.$$

Then $\varepsilon(L, x) \geq \varepsilon$.

After writing a first draft of this paper, we realized that the rescaling argument mentioned above can also be applied in the context of [ELN], leading, somewhat surprisingly, to bounds for Seshadri constants valid at *arbitrary* points; however, these bounds depend on the line bundle L and the manifold X, or rather, its tangent bundle T_X:

Corollary 3 *Let X be a smooth projective n-fold, $x \in X$ any point, A an ample line bundle and $\delta \geq 0$ a real number such that $T_X(\delta A)$ is nef. Then*

$$\varepsilon(A, x) \geq \min\left\{\frac{1}{(n-1)^{n-1}(2n-1)^n}, \frac{1}{\delta}\right\}.$$

This gives, in particular, bounds valid at arbitrary points for the Seshadri constants of the canonical line bundle $K_X = (\bigwedge^n T_X)^*$ or its inverse if these are ample, or for any ample line bundle in case K_X is trivial (cf. Remark 4.4).

Corollary 3 accords with, and should be compared to, bounds following from recent work of Angehrn and Siu [AS] on the basepoint freeness of adjoint linear system (cf. 4.5), and Demailly's original very ampleness criteria [De93].

The paper is organized as follows. After fixing notation and establishing the general setup, we recall some basic facts about Seshadri constants and collect some auxiliary statements in §1. Then, in §2, we prove the main technical result, Theorem 1. Finally we give the applications to bounding Seshadri constants at general points in §3, and at arbitrary points in §4.

Acknowledgments

We are grateful to Rob Lazarsfeld, both for having drawn our attention to Seshadri constants, and for sharing his insights in this topic with us. The reader will notice that our treatment relies on ideas and prior works due to Demailly [De93], Ein, Küchle and Lazarsfeld [EKL], Fujita [Fu94], and especially Ein, Lazarsfeld and Nakamaye [ELN], to whom we are indebted.

Notation and the general setup

0.1 Throughout this paper we work over the field $\mathbb{C}$ of complex numbers. Given a variety Y, a statement valid for a *very general* point $y \in Y$ is a statement which holds for all points in the complement of some countable union of proper subvarieties of Y.

0.2 Given a smooth variety Y, an integer $m \geq 0$ and a subvariety $W \subset Y$ we denote by $\mathcal{I}_W^{(m)}$ the symbolic power sheaf of all functions vanishing to order at least m along W. Then $\mathcal{I}_W^{(1)} = \mathcal{I}_W$ is the ideal sheaf of W, and $\mathcal{I}_Z^{(m)} = \mathcal{I}_Z^m$ for a smooth subvariety $Z \subset Y$.

0.3 Let M be a line bundle on a smooth variety Y. Given a divisor $E \in |kM|$ we call the normalized multiplicity

$$\mathrm{ind}_y(E) = \frac{\mathrm{mult}_y(E)}{k}$$

the *index* of E at a point $y \in Y$.

0.4 For a line bundle L and a coherent sheaf of ideals $\mathcal{J}$ on Y, we denote by $|L \otimes \mathcal{J}|$ the linear subsystem of the complete linear system $|L|$ corresponding to sections in $L \otimes \mathcal{J}$. Given such a system $|L \otimes \mathcal{J}| \neq \emptyset$ on Y, the *base locus* $\mathrm{Bs}\,|L \otimes \mathcal{J}|$ is the support of the intersection of all members of $|L \otimes \mathcal{J}|$.

0.5 We will be concerned with the following setup: let X be a smooth irreducible n-dimensional projective variety, T a smooth irreducible affine variety and $g\colon T \to X$ a quasi-finite dominant morphism with graph Γ; examples of this situation are provided by Zariski open subsets $T \subset X$. Let pr_X and pr_T denote the projections from $Y = X \times T$ to its factors. Note that these restrict to dominant maps from Γ to X respectively T. Given a Zariski closed subset (or subscheme) $Z \subset X \times T$, we consider the fibre Z_t of pr_T over $t \in T$ as a subset (or subscheme) of X. Similarly, $Z_x \subset T$ is the fibre of Z over $x \in X$. Given a sheaf $\mathcal{F}$ on $X \times T$ we write $\mathcal{F}_t$ for the induced sheaf on X.

1 Preliminaries

In this section we collect some preliminary results needed in the sequel. We start with some remarks concerning multiplicity loci in a family. Let E be an effective divisor on a smooth variety Y. Then the function $y \mapsto \mathrm{mult}_y(E)$ is Zariski upper-semicontinuous on Y. For any given irreducible subvariety $Z \subset Y$ we write $\mathrm{mult}_Z(E)$ for the value of $\mathrm{mult}_z(E)$ at a general point $z \in Z$. The following lemma allows us to calculate multiplicities fibrewise.

Lemma 1.1 (cf. [EKL, 2.1]) *Let X and T be smooth irreducible varieties, and suppose that $Z \subset X \times T$ is an irreducible subvariety which dominates T. Let $E \subset X \times T$ be an effective divisor. Then for a general point $t \in T$, and any irreducible component $W_t \subset Z_t$ of the fibre Z_t, we have*

$$\mathrm{mult}_{W_t}(E_t) = \mathrm{mult}_Z(E). \quad \Box$$

The next two elementary lemmas give a way of detecting irreducible components of base loci, and show that being an irreducible component is well-behaved in families.

Lemma 1.2 *Let m be a positive integer, M a line bundle on a smooth variety Y and $V \subset W \subset Y$ subvarieties such that V is an irreducible component of W. Suppose that*

(1) $\mathcal{I}_W \otimes M$ is generated by global sections, and

(2) $\mathcal{I}_W \subset \mathcal{I}_V^{(m)}$.

Then V is an irreducible component of $\mathrm{Bs}\,|\mathcal{I}_V^{(m)} \otimes M|$.

Proof Consider the inclusions

$$V \subset \mathrm{Bs}\,|\mathcal{I}_V^{(m)} \otimes M| \subset \mathrm{Bs}\,|\mathcal{I}_W \otimes M| \subset W, \qquad (*)$$

where the first inclusion is obvious, the second follows from (2), and the third from the fact that $\mathrm{Bs}\,|\mathcal{I}_W \otimes M| = W$ because $\mathcal{I}_W \otimes M$ is globally generated according to (1).

Now let $Z \subset \mathrm{Bs}\,|\mathcal{I}_V^{(m)} \otimes M|$ be any irreducible component containing V. Then $(*)$ and the assumption that $V \subset W$ is an irreducible component imply $V = Z$, hence the claim follows. $\Box$

Lemma 1.3 *Let $f\colon Y \to Z$ be a morphism between irreducible varieties, and $V \subset W \subset Y$ subvarieties such that V is an irreducible component of W, and f restricts to a dominant map from V to Z.*

Then over a general point $z \in Z$, every irreducible component U_z of V_z has dimension $\dim V - \dim Z$ *and is an irreducible component of W_z.*

Proof For the first part we refer to [Ha, II, Ex 3.22]. The second assertion comes down to an easy dimension count as follows: write

$$W = V \cup V'$$

with $V' \subset Y$ a subvariety not containing V, so that $\dim V > \dim(V \cap V')$. If the map from $V \cap V'$ to Z is not dominant, then V_z and V'_z do not meet in a general fibre and the claim follows.

Otherwise, over general $z \in Z$, we obtain:

$$\begin{aligned}\dim V_z &= \dim V - \dim Y \\ &> \dim(V \cap V') - \dim Y = \dim(V \cap V')_z,\end{aligned}$$

and therefore $\dim U_z > \dim(V \cap V')_z \geq \dim(U_z \cap V'_z)$, where $U_z \subset V_z$ is any irreducible component. Considering the decomposition $V_z = U_z \cup U'_z$, where $U_z \not\subset U'_z$, we conclude $\dim U_z > \dim(U_z \cap (U'_z \cup V'_z))$, which shows that U_z is an irreducible component of $W_z = U_z \cup (V'_z \cup U'_z)$. $\square$

For the reader's convenience we recall some well-known facts concerning Seshadri constants. The next lemma deals with the Seshadri constant at general points versus that at very general points (compare [EKL, 1.4]).

Lemma 1.4 *Let X be a smooth projective variety and L a nef and big line bundle on X. Suppose $\varepsilon(L, y) = \varepsilon$ for a point $y \in X$. Then for any real $\delta > 0$ there exists a Zariski open neighbourhood $U(\delta) \subset X$ of y such that*

$$\varepsilon(L, x) \geq \varepsilon - \delta \quad \textit{for all } x \in U(\delta). \quad \square$$

Finally we recall the relations between isolated singularities, generation of higher jets, and Seshadri constants (cf. [ELN, 1.1], [EKL, 1.3]).

Theorem 1.5 *Let X be a smooth projective n-fold and L a nef and big line bundle on X.*

(1) Suppose there exists a divisor $E \in |kL|$ having an isolated singularity of index $\geq n + s$ at $x \in X$. Then $H^1(X, \mathcal{O}_X(K_X + L) \otimes \mathcal{I}_x^{s+1}) = 0$. In particular, $|K_X + L|$ generates s-jets at x, i.e., we have a surjective evaluation map

$$H^0(X, \mathcal{O}_X(K_X + L)) \to H^0(X, \mathcal{O}_X(K_X + L) \otimes \mathcal{O}_X/\mathcal{I}_x^{s+1}).$$

(2) Let $\varepsilon(L, x)$ be the Seshadri constant of L at x. If

$$r > \frac{s}{\varepsilon(L, x)} + \frac{n}{\varepsilon(L, x)},$$

then $|K_X + rL|$ generates s-jets at $x \in X$.

(3) Conversely, if $\varepsilon(L,x) < \alpha$, then for all $s_0 > 0$ and all real c there exists an $s \geq s_0$ and an $r > \dfrac{s}{\alpha} + c$ such that $|K_X + rL|$ does not generate s-jets at x. $\square$

2 Proof of Theorem 1

2.1

Let $\alpha \in \mathbb{Q}^+$ be such that $L^n > \alpha$, and $0 = \beta_1 < \beta_2 < \cdots < \beta_{n+1} = \alpha$ a partition of the interval $[0, \alpha]$ with rational β_i. In the applications in §3 we will use a clever choice of the β_i, which we call "rescaling" of the interval $[0, \alpha]$."

2.2 Families of divisors

Pick an arbitrary point $y \in X$ and a smooth affine neighbourhood $T \subset X$ of y in X. Then the embedding $g \colon T \hookrightarrow X$ satisfies the properties of 0.5. Note that (very) general points of T correspond to (very) general points of X. We will use the notations introduced in 0.5 henceforth.

Arguing as in [EKL, 3.8], for $k \gg 0$ with $\alpha k \in \mathbb{Z}$ we obtain divisors $\mathcal{E}_k \in |\operatorname{pr}_X^*(kL)|$ in $X \times T$ satisfying

$$\operatorname{ind}_\Gamma(\mathcal{E}_k) > \alpha.$$

The argument is, in brief, that for any $x \in X$ one finds a divisor $E \in |kL|$ with $\operatorname{mult}_x(E) > \alpha k$ by using Riemann-Roch and a parameter count. Hence the torsion free $\mathcal{O}_T$-module

$$R = \operatorname{pr}_{T*}\left(\operatorname{pr}_X^*(kL) \otimes \mathcal{I}_\Gamma^{(\alpha k)}\right)$$

has positive rank, and is globally generated since T is affine. Therefore via the evaluation map $\operatorname{pr}_T^* R \to \operatorname{pr}_X^*(kL) \otimes \mathcal{I}_\Gamma^{(\alpha k)}$, a nonzero section Φ of R gives the desired divisor $\mathcal{E}_k$.

2.3 The multiplicity schemes $\mathcal{Z}_\sigma(\mathcal{E})$

For $k \geq 0$ with $\alpha k \in \mathbb{Z}$, we set

$$A_k = \left| \mathcal{I}_\Gamma^{(\alpha k)} \otimes \operatorname{pr}_X^*(kL) \right|.$$

Then by 2.2, A_k is nonempty for sufficiently large k. For nonzero $\mathcal{E}_k \in A_k$ and any rational number $\sigma > 0$, we define

$$\mathcal{Z}_\sigma(\mathcal{E}_k) = \left\{ y = (x,t) \in \mathcal{E}_k \;\middle|\; \operatorname{ind}_y(\mathcal{E}_k) \geq \sigma \right\}.$$

Note that $\mathcal{Z}_\sigma(\mathcal{E}_k)$ is a Zariski closed subset of $X \times T$. Its natural scheme structure is given locally by the vanishing of all partial derivatives of order $< k\sigma$ of a local equation of $\mathcal{E}_k$. We are only interested in $\mathcal{Z}_\sigma(\mathcal{E}_k)$ as an algebraic set, and for a general choice of $\mathcal{E}_k$.

The following lemma, which is an analog of [ELN, 3.8], says that the multiplicity loci $\mathcal{Z}_\sigma(\mathcal{E}_k)$ are independent of k and the choice of a general $\mathcal{E}_k \in A_k$ as soon as k is sufficiently large.

Lemma 2.3.1 *For fixed σ there is a positive integer k_0 such that $\mathcal{Z}_\sigma(\mathcal{E}_{k_1}) = \mathcal{Z}_\sigma(\mathcal{E}_{k_2})$ for all $k_1, k_2 \geq k_0$.*

Proof (compare [ELN]) For simplicity, as σ and α are fixed, we write $\mathcal{Z}(k)$ for $\mathcal{Z}_\sigma(\mathcal{E}_k)$. Choose an integer $m \geq 2$ such that $A_k \neq \emptyset$ for $k \geq m$. Fixing an integer $a \geq m$, we claim that there exists a positive integer $k(a)$ such that

$$\mathcal{Z}(r) \subset \mathcal{Z}(a) \quad \text{whenever} \quad r \geq k(a). \qquad (*)$$

To prove the claim, suppose that $y \notin \mathcal{Z}(a)$, so that there exists $\eta > 0$ satisfying

$$\mathrm{mult}_y(\mathcal{E}_a) \leq a\sigma - \eta.$$

Since the index is a discrete invariant, η is bounded below independently of y; in fact if $m\sigma \in \mathbb{Z}$ then $\eta \geq 1/m$. Suppose $b \geq m$ is an integer coprime to a. Then any integer $r \geq ab$ can be expressed as $r = \alpha a + \beta b$ with $\alpha, \beta \in \mathbb{Z}$ and $0 \leq \beta \leq a$. Consider the divisor $\mathcal{E}'_r = \alpha\mathcal{E}_a + \beta\mathcal{E}_b \in A_r$. Then

$$\begin{aligned}\mathrm{mult}_y(\mathcal{E}_r) \leq 1 + \mathrm{mult}_y(\mathcal{E}'_r) &= 1 + \alpha \cdot \mathrm{mult}_y(\mathcal{E}_a) + \beta \cdot \mathrm{mult}_y(\mathcal{E}_b) \\ &\leq 1 + \alpha a\sigma - \alpha\eta + \beta \cdot \mathrm{mult}_y(\mathcal{E}_b) \\ &= r\left(\sigma - \frac{\eta\left(1 - \frac{\beta b}{r}\right)}{a} + \frac{1 + \beta\left(\mathrm{mult}_y(\mathcal{E}_b) - \sigma b\right)}{r}\right),\end{aligned}$$

where the first inequality is a consequence of a general version of Bertini's Theorem for the general $\mathcal{E}_r \in A_r$ (see, for example, [Xu96]). Since η, β and b are bounded independently of r, it follows that $\mathrm{mult}_y(\mathcal{E}_r) < r\sigma$ for $r \gg 0$. Hence $y \notin \mathcal{Z}(r)$ for all sufficiently large r, as claimed.

If $\mathcal{Z}(r) = \mathcal{Z}(a)$ for all $r \gg 0$ we are finished. Otherwise by $(*)$, there exists $a' > 0$ such that $\mathcal{Z}(a') \subsetneq \mathcal{Z}(a)$. The argument can then be repeated with a' instead of a. Since this process must stop after a finite number of steps, the lemma follows. □

The next lemma is an adaptation of [ELN, 1.5, 1.6] to our situation. We present a sketch proof for the reader's convenience.

Lemma 2.3.2 (Gap Lemma) *Let $\mathcal{E} \subset X \times T$ be a family of effective divisors on X with $\operatorname{ind}_\Gamma(\mathcal{E}) > \alpha$ along the graph $\Gamma \subset X \times T$ of $g\colon T \to X$. Define*

$$\mathcal{Z}_0 = X \times T \quad \textit{and}$$
$$\mathcal{Z}_j = \mathcal{Z}_{\beta_j}(\mathcal{E}) = \left\{ y \in \mathcal{E} \;\middle|\; \operatorname{ind}_y(\mathcal{E}) \geq \beta_j \right\} \quad \textit{for } 1 \leq j \leq n+1$$

(with β_i as above). Then there exists an index c with $1 \leq c \leq n$, and an irreducible subvariety $\mathcal{V} \subset X \times T$ such that:

(1) $\operatorname{codim}(\mathcal{V}) = c$,

(2) $\Gamma \subset \mathcal{V}$, *and*

(3) $\mathcal{V}$ *is an irreducible component of both* $\mathcal{Z}_c$ *and* $\mathcal{Z}_{c+1}$.

This means that the index of $\mathcal{E}$ "jumps" by at least $\beta_{c+1} - \beta_c$ along $\mathcal{V}$, that is, $\operatorname{ind}_y(\mathcal{E}) \geq \beta_{c+1}$ for every $y \in \mathcal{V}$ and there is an open set $U \subset X \times T$ meeting $\mathcal{V}$ such that $\operatorname{ind}_v(\mathcal{E}) < \beta_c$ for every $v \in U \setminus \mathcal{V}$.

Sketch of Proof The sets $\mathcal{Z}_i$ lie in a chain

$$\Gamma \subset \mathcal{Z}_{n+1} \subset \cdots \subset \mathcal{Z}_1 = \mathcal{E} \subsetneq \mathcal{Z}_0 = X \times T.$$

Starting with $\mathcal{Z}_{n+1}$ and working up in dimension, we can choose irreducible components $\mathcal{V}_j$ of $\mathcal{Z}_j$ containing Γ such that $\mathcal{V}_{j+1} \subset \mathcal{V}_j$. So we arrive at a chain of irreducible varieties

$$\Gamma \;\subset\; \mathcal{V}_{n+1} \;\subset\; \mathcal{V}_n \;\subset \ldots \subset \mathcal{V}_1 \;\subsetneq \mathcal{V}_0 \;=\; X \times T,$$

and since $X \times T$ is irreducible of dimension $2n = \dim(\Gamma) + n$, at least two consecutive links in the chain must coincide, say $\mathcal{V}_c = \mathcal{V}_{c+1}$, and we take $\mathcal{V} = \mathcal{V}_c$. Using elementary combinatorial arguments one shows that also the condition $\operatorname{codim}(\mathcal{V}) = c$ can be achieved. For details we refer to [ELN], proof of Lemma 1.6. $\square$

2.4 The "jumping" locus as base locus of a certain linear system

By Lemma 2.3.1, there exists an integer k_0 such that the multiplicity loci $\mathcal{Z}_{\beta_i}(\mathcal{E}_k)$, for $1 \leq i \leq n+1$, are independent of k as soon as $k \geq k_0$ and $\mathcal{E}_k \in A_k$ is general. Therefore also the multiplicity "jumping" loci $\mathcal{V}$ obtained by the Gap Lemma 2.3.2 can be chosen independently of $\mathcal{E} = \mathcal{E}_k$ and k up to the above restrictions. Fix such a $\mathcal{V}$ and put $\beta = \beta_{c+1} - \beta_c$.

Proposition 2.4.1 *For all sufficiently divisible $k \gg 0$ the jumping locus $\mathcal{V}$ is an irreducible component of the base locus of the linear system*

$$\left| \mathcal{I}_{\mathcal{V}}^{(\beta k)} \otimes \mathrm{pr}_X^*(kL) \right| .$$

Here sufficiently divisible means that $\beta_i k \in \mathbb{Z}$ for all $i = 1, \ldots, n+1$.

We start by recalling some general facts concerning differentiating sections of line bundles $\mathrm{pr}_X^*(kL)$ in parameter directions and its connection with certain multiplicity loci (cf. also [EKL, §2] and [ELN, §2]).

Let $\mathcal{D}^{\ell}_{X\times T}(\mathrm{pr}_X^*(kL))$ be the sheaf of differential operators of order $\leq \ell$ on $\mathrm{pr}_X^*(kL)$ and $\mathcal{D}^{\ell}_T$ the sheaf of differential operators of order $\leq \ell$ on T. Since there is a canonical inclusion of vector bundles

$$\mathrm{pr}_T^*(\mathcal{D}^{\ell}_T) \hookrightarrow \mathcal{D}^{\ell}_{X\times T}(\mathrm{pr}_X^*(kL)),$$

the sections of $\mathcal{D}^{\ell}_T$ act naturally on the space of sections of $\mathrm{pr}_X^*(kL)$. A section $\psi \in \Gamma(X \times T, \mathrm{pr}_X^*(kL))$ determines a sheaf homomorphism

$$\mathfrak{d}_\ell(\psi)\colon\ \mathrm{pr}_T^*(\mathcal{D}^{\ell}_T) \to \mathrm{pr}_X^*(kL).$$

If we represent ψ locally by a function f, then $\mathfrak{d}_\ell(\psi)$ just takes a differential operator D to the function $D(f)$. Since $\mathrm{pr}_X^*(kL)$ is a line bundle, there exists a sheaf of ideals $\mathcal{I}_{\Sigma_\ell(\psi)}$ such that

$$\mathrm{Im}\Big(\mathfrak{d}_\ell(\psi)\colon\ \mathrm{pr}_T^*(\mathcal{D}^{\ell}_T) \to \mathrm{pr}_X^*(kL)\Big) = \mathcal{I}_{\Sigma_\ell(\psi)} \otimes \mathrm{pr}_X^*(kL).$$

Let ψ be a defining section for a divisor $\mathcal{E} \in |\,\mathrm{pr}_X^*(kL)|$. Then we claim that

$$\Sigma_\ell(\psi) = \big\{(x,t) \in X \times T \ \big|\ \mathrm{mult}_t(\mathcal{E}_x) > \ell\big\}. \tag{2.4.2}$$

Indeed, the scheme structure on the right hand side is given locally by the vanishing of all partial derivatives of order $\leq \ell$ in the T direction of a local equation for $\mathcal{E}$.

Note also that the sheaves $\mathcal{I}_{\Sigma_\ell(\psi)} \otimes \mathrm{pr}_X^*(kL)$ are generated by global sections, because they are quotients of the globally generated sheaf $\mathrm{pr}_T^*\left(\mathcal{D}^{\ell}_T\right)$.

Proof of Proposition 2.4.1 The plan is to apply Lemma 1.2. By assumption,

$$\alpha k, \quad p = \beta_c k \quad \text{and} \quad q = \beta k \quad \text{are integers.}$$

Let $\mathcal{E} \in A_k$ be a divisor determining $\mathcal{V}$ and ψ a section defining $\mathcal{E}$. For integers ℓ put $\Sigma_\ell = \Sigma_\ell(\psi)$. We claim that

$$\mathcal{I}_{\Sigma_{p-1}} \subset \mathcal{I}_{\mathcal{V}}^{(q)}. \tag{2.4.3}$$

To prove this, let f be a local equation for $\mathcal{E}$ over some open set U. Then $\mathcal{I}_{\Sigma_{p-1}}$ is locally generated by all functions

$$\left\{D(f) \mid D \in (\mathrm{pr}_T^* \mathcal{D}_T^{p-1})(U)\right\}.$$

On the other hand we have $\widetilde{D}(f) \in \mathcal{I}_{\mathcal{V}}$ for every $\widetilde{D} \in \mathcal{D}_{X\times T}^{p+q-1}(\mathrm{pr}_X^*(kL))(U)$, since $\mathcal{V}$ is an irreducible component of $\mathcal{Z}_{\frac{p+q}{k}}(\mathcal{E})$. And in particular $\widetilde{D}(f) \in \mathcal{I}_{\mathcal{V}}$ for every $\widetilde{D} \in (\mathrm{pr}_T^* \mathcal{D}_T^{p+q-1})(U)$. Hence for all $D \in \mathrm{pr}_T^* \mathcal{D}_T^{p-1}$ the function $D(f)$ vanishes to order $\geq q$ on $\mathcal{V}$, which proves (2.4.3).

Since $\mathcal{I}_{\Sigma_{p-1}} \otimes \mathrm{pr}_X^*(kL)$ is globally generated, applying Lemma 1.2 to our situation will give the desired result once we show that $\mathcal{V}$ is also an irreducible component of Σ_{p-1}. This is the content of Lemma 2.4.4 below. □

Lemma 2.4.4 *For σk a positive integer, let $\mathcal{E} \in |\mathrm{pr}_X^*(kL)|$ be an effective divisor on $X \times T$ and $V \subset \mathcal{Z}_\sigma(\mathcal{E})$ an irreducible component dominating X. Then V is also an irreducible component of $\Sigma_{k\sigma-1}(\mathcal{E})$.*

Proof By definition $\mathcal{Z}_\sigma(\mathcal{E}) \subset \Sigma_{k\sigma-1}(\mathcal{E})$. Let $W \subset \Sigma_{k\sigma-1}(\mathcal{E})$ be an irreducible component containing V. If we can show that $W \subset \mathcal{Z}_\sigma(\mathcal{E})$, then we are done, because that implies $V = W$. Lemma 1.1 shows that $\mathrm{ind}_W(\mathcal{E}) = \mathrm{ind}_{W'_x}(\mathcal{E})$ for general $x \in X$ and any irreducible component W'_x of W_x. Hence the assertion follows from $\mathrm{ind}_{W_x}(\mathcal{E}_x) \geq \sigma$, where we used (2.4.2). □

Corollary 2.4.5 *For all sufficiently divisible $k \gg 0$, we have*

$$\left|\mathcal{I}_\Gamma^{(\alpha k)} \otimes \mathrm{pr}_X^*(kL)\right| \subset \left|\mathcal{I}_{\mathcal{V}}^{(\beta k)} \otimes \mathrm{pr}_X^*(kL)\right|.$$

Proof By the above, any sufficiently general $\mathcal{E} \in A_k$ determines the same $\mathcal{V}$, in particular, such an $\mathcal{E}$ satisfies $\mathrm{ind}_{\mathcal{V}}(\mathcal{E}) \geq \beta_c$, and this implies that $\mathcal{E} \in |\mathcal{I}_{\Sigma_{p-1}} \otimes \mathrm{pr}_X^*(kL)|$ by (2.4.2), where again we assume that αk, $p = \beta_c k$ and $q = \beta k$ are integers. The claim then follows from (2.4.3). □

Proposition 2.4.6 *For all sufficiently divisible $k \gg 0$ and for very general $t \in T$, the following hold:*

(1) There exists an irreducible subvariety $V \subset X$ of codimension c containing $g(t)$ which is an irreducible component of the base locus of the linear system

$$|\mathcal{J}_k| := \left|\left(\mathcal{I}_{\mathcal{V}}^{(\beta k)}\right)_t \otimes kL\right|$$

on X such that $\mathrm{mult}_V D \geq k\beta$ for all $D \in |\mathcal{J}_k|$. In particular, if $c = n$, i.e., $\mathcal{V} = \Gamma$, then by Bertini's Theorem there exists a divisor $D \in |kL|$ having an isolated singularity of index $\geq \beta$ at the point $x = g(t)$.

(2) $\dim H^0(X, \mathcal{I}_{g(t)}^{\alpha k} \otimes kL) \leq \dim H^0(X, \mathcal{J}_k)$.

Proof To begin with, we study the situation for a fixed k. First we note that after possibly shrinking T, we can assume that the coherent sheaf $\mathcal{F} = \mathcal{I}_{\mathcal{V}}^{(\beta k)} \otimes \mathrm{pr}_X^*(kL)$ is flat over T. In fact $\mathrm{pr}_T \colon X \times T \to T$ is projective and T is affine and integral, hence the assertion follows by considering the Hilbert polynomials of the $\mathcal{F}_t$ (cf. also [Ha, III.9.9]): these do not depend on t for t in an open dense subset of T.

After possibly shrinking T more, it follows from semicontinuity that there is a natural isomorphism

$$H^0(X \times T, \mathcal{F}) \otimes k(t) \simeq H^0(X, \mathcal{F}_t). \tag{$*$}$$

(cf. [Ha, III.12.9]). In other words, taking global sections of $\mathcal{F}$ commutes with restricting to fibres over general $t \in T$, and therefore $(\mathrm{Bs}\,|\mathcal{F}|)_t = \mathrm{Bs}\,|\mathcal{J}_k|$ for such t.

Now we can prove (1). By Proposition 2.4.1 we have $\Gamma \subset \mathcal{V} \subset \mathrm{Bs}\,|\mathcal{F}|$ with $\mathcal{V}$ an irreducible component of $\mathrm{Bs}\,|\mathcal{F}|$, hence Lemma 1.3 shows that any irreducible component V of $\mathcal{V}_t$ is an $(n-c)$-dimensional irreducible component of $(\mathrm{Bs}\,|\mathcal{F}|)_t = \mathrm{Bs}\,|\mathcal{J}_k|$. It remains to show that $\mathrm{mult}_V D \geq k\beta$ for all $D \in |\mathcal{J}_k| = |\mathcal{F}_t|$. But this follows from $(*)$ and Lemma 1.2.

Assertion (2) follows in the same way from Corollary 2.4.5 and the fact that

$$\left(\mathcal{I}_\Gamma^{(\alpha k)}\right)_t = \left(\mathcal{I}_\Gamma^{\alpha k}\right)_t = \mathcal{I}_{g(t)}^{\alpha k}.$$

To complete the proof of the proposition we only have to remark that, since $\mathcal{V}$ does not depend on k, the above arguments work simultaneously for all divisible $k \gg 0$ if we replace the general $t \in T$ by a very general $t \in T$. $\square$

2.5 Bounding the degree of irreducible components of base loci

In this section we complete the proof of Theorem 1 by bounding the degree of the irreducible component V of Proposition 2.4.6, using a strategy essentially due to Fujita (see [Fu82], [Fu94]). Alternatively one could carry out an approach via graded linear system as in [ELN], leading to slightly weaker bounds.

Let $k \gg 0$ be sufficiently large and divisible, and fix a very general $t \in T$. Let $\mathcal{J}_k = (\mathcal{I}_{\mathcal{V}}^{(\beta k)})_t \otimes kL$ and $V \subset X$ be as in Proposition 2.4.6; recall that V depends on t but *not* on k, and that V is an $(n-c)$-dimensional irreducible component of $\mathrm{Bs}\,|\mathcal{J}_k|$. We may assume that $\dim V > 0$, since otherwise the assertion of Theorem 1 follows from Proposition 2.4.6, (1).

2.5.1 Resolving the base locus of $\Lambda := |\mathcal{J}_k|$, we can find a sequence

$$X' = X_s \to \cdots \to X_r \to X_{r-1} \to \cdots \to X_1 \to X_0 = X$$

of birational morphisms $\tau_i \colon X_i \to X_{i-1}$ together with linear systems Λ_i on X_i such that

(1) $\Lambda_0 = \Lambda$.

(2) $\tau_i \colon X_i \to X_{i-1}$ is the blowup of a smooth subvariety C_i of X_{i-1}.

(3) $\tau_i^* \Lambda_{i-1} = \Lambda_i + m_i E_i$ for some nonnegative integers m_i, where E_i is the exceptional divisor on X_i lying over C_i and $E_i \not\subset \operatorname{Bs} \Lambda_i$.

(4) $\operatorname{Bs} \Lambda_s = \emptyset$.

Let $\tau = \tau_s \circ \cdots \circ \tau_1$ be the composite, E_i^* the pullback of E_i to X', and $Y_i = \tau(E_i^*)$, so that Y_i coincides with the image of C_i in X.

Let F_i denote the pullback to X' of the general member of the linear system Λ_i on X_i. Finally, let H be a general member of Λ_s, $F = \tau^*(kL)$, and E the fixed part of $\tau^*\Lambda$, so that $H = F - E$, where $E = \sum_{i=1}^{s} m_i E_i^*$.

By assumption there exists an index r with $Y_r = V$, and

$$m_r \geq k\beta \tag{2.5.2}$$

since $\operatorname{mult}_V(D) \geq k\beta$ for all $D \in |\mathcal{J}_k|$. We also may and will assume that the resolution τ is chosen in such a way that $\dim Y_i < \dim V = n - c$ for all $i < r$, and in particular $V = Y_r$ is birational to C_r.

Lemma 2.5.3 $F^{n-c} \cdot E \cdot H^{c-1} \geq k^{n-c} m_r^c \deg_L V$.

Proof First of all note that, since F and H are nef and $E - m_r E_r^*$ is an effective divisor, we have

$$F^{n-c} \cdot E \cdot H^{c-1} \geq m_r F^{n-c} \cdot E_r^* \cdot H^{c-1}. \tag{$*$}$$

Then one proceeds exactly as in the proof of [Fu82, Lemma 3.2] by first showing

$$F^{n-c} \cdot F_r^c \geq F^{n-c} \cdot F_r \cdot H^{c-1}. \tag{$**$}$$

This inequality is proved inductively, and reduces to checking the inequality $H^{c-b-1} F_r^b F^{n-c}(F_r - H) \geq 0$ for $b \geq 0$, which follows from the semipositivity of F_r^b on the class representing $F^{n-c} \cdot E'$ for all components $E' \subset E$.

Finally one computes both sides of the inequality $(**)$. Since $\dim Y_i < n-c$ for all $i < r$, we have

$$F^{n-c} \cdot F_r^c = F^{n-c} \cdot (F - \sum_{i \le r} m_i E_i^*)^c = F^{n-c} \cdot (F - m_r E_r^*)^c,$$

hence $F^{n-c} \cdot F_r^c = F^n - m_r^c \deg_{kL}(V)$ by the birationality of the morphism $C_r \to Y_r = V$. A similar argument shows $F^{n-c} \cdot F_r \cdot H^{c-1} = F^n - m_r E_r^* \cdot H^{c-1} \cdot F^{n-c}$. Combining this with $(*)$ and $(**)$ proves the lemma. $\square$

Lemma 2.5.4 *For any $\varepsilon > 0$ there exist a sufficiently large and divisible k and a resolution $\tau \colon X' \to X$ of the rational map given by $|\mathcal{J}_k|$ satisfying the properties in 2.5.1 and such that $H^n = (F-E)^n \ge k^n(L^n - \alpha^n - \varepsilon)$.*

Proof The proof follows closely the proof of [Fu94, Theorem]. We therefore only give an outline, indicating the necessary modifications. For varying k, consider $|\mathcal{J}_k|$ and denote by (X'_k, H_k) the pair (X', H) obtained as in 2.5.1. We derive a contradiction assuming that $H_k^n < k^n(L^n - \alpha^n - \varepsilon)$ on X'_k for all large and divisible k.

Letting ε grow if necessary, we may assume that

$$H_\ell^n \ge \ell^n \left(L^n - \alpha^n - \varepsilon - \frac{\varepsilon}{(2n)!} \right)$$

on X'_ℓ for one fixed large and divisible ℓ. Now, for any integer $s > 0$, we claim that

$$h^0(X, \mathcal{J}_{s\ell}) \le h^0(X'_\ell, sH_\ell) + \frac{n(s\ell)^n \varepsilon}{(2n)!},$$

which is proved exactly as in [Fu94] by using the lower bound on H_ℓ^n and considering an appropriate resolution of $\Lambda = |\mathcal{J}_{s\ell}|$. From Proposition 2.4.6, (2) and asymptotic Riemann–Roch we then obtain

$$\begin{aligned} \frac{(s\ell)^n}{n!} (L^n - \alpha^n) + \varphi(s) &\le h^0(X, I_x^{\alpha s\ell} \otimes s\ell L) \le h^0(X, \mathcal{J}_{s\ell}) \\ &\le h^0(X'_\ell, sH_\ell) + \frac{n(s\ell)^n \varepsilon}{(2n)!} \\ &\le \frac{s^n}{n!} \ell^n (L^n - \alpha^n - \varepsilon) + \frac{n(s\ell)^n \varepsilon}{(2n)!} + \psi(s), \end{aligned}$$

where φ and ψ are functions with $\lim_{s\to\infty} \frac{\varphi(s)}{s^n} = \lim_{s\to\infty} \frac{\psi(s)}{s^n} = 0$. This gives the desired contradiction. $\square$

Before stating the main result of this section which will complete the proof of Theorem 1, we need to recall some well-known facts (cf. [De93, 5.2], [Fu82, 1.2]).

Lemma 2.5.5 *Let F, H be nef divisors on a smooth projective n-fold Y. Then:*

(1) $F^d \cdot H^{n-d} \geq \sqrt[n]{(F^n)^d}\,\sqrt[n]{(H^n)^{n-d}}$ *holds for all* $0 \leq d \leq n$.

(2) If $E = F - H$ *is effective, then* $F^a \cdot H^{n-a} \geq F^b \cdot H^{n-b}$ *for any* $a \geq b$. $\square$

Proposition 2.5.6 *In the above notation the degree of V satisfies*

$$\deg_L V \leq \frac{1}{\beta^c}\left(1 - \sqrt[n]{\left(1 - \frac{\alpha^n}{L^n}\right)^c}\,\right) L^n.$$

Proof By Lemma 2.5.3 we have

$$\deg_L V \leq \frac{1}{k^{n-c} m_r^c} F^{n-c} \cdot E \cdot H^{c-1}.$$

Note that $F^{n-c} \cdot E \cdot H^{c-1} = (F^{n-c+1} \cdot H^{c-1} - F^{n-c} \cdot H^c)$. So if we bound the first term using Lemma 2.5.5, (2) and the second using (1), we find

$$\begin{aligned}
\deg_L V &\leq \frac{1}{k^{n-c} m_r^c}\left(F^{n-c+1} \cdot H^{c-1} - F^{n-c} \cdot H^c\right) \\
&\leq \frac{1}{k^{n-c} m_r^c}\left(F^n - (F^n)^{\frac{n-c}{n}} (H^n)^{\frac{c}{n}}\right) \\
&\leq \frac{k^n}{k^{n-c} m_r^c}\left(L^n - (L^n)^{\frac{n-c}{n}} (L^n - \alpha^n)^{\frac{c}{n}}\right) \\
&\leq \frac{1}{\beta^c}\left(1 - \sqrt[n]{\left(1 - \frac{\alpha^n}{L^n}\right)^c}\,\right) L^n,
\end{aligned}$$

where the last steps are Lemma 2.5.4 plus the fact that $\deg_L V$ is integral, and (2.5.2). $\square$

3 Applications

Theorem 3.1 *Let X be a smooth projective n-fold, $x \in X$ a very general point and L a nef and big line bundle. Let r and s be positive integers, $\gamma > 1$ a rational number satisfying $(rL)^n > (\gamma(n+s))^n$, and $\alpha_1, \ldots, \alpha_{n-1}$ positive rational numbers with $\sum_{i=1}^{n-1} \alpha_i = \gamma - 1$. Suppose that $|K_X + rL|$ does* not *generate s-jets at x.*

Then there is a proper subvariety $V \subset X$ of positive dimension d containing x with

$$\deg_L V = L^d \cdot V \leq \frac{(rL)^n}{\alpha_d^{n-d}(n+s)^{n-d} r^d}\left(1 - \sqrt[n]{\left(1 - \frac{\gamma^n (n+s)^n}{(rL)^n}\right)^{n-d}}\,\right).$$

Proof Put $L' = rL$, $\alpha = \beta_{n+1} = \gamma(n+s)$, $\beta_n = (\gamma-1)(n+s)$, and recursively downwards $\beta_i = \beta_{i+1} - \alpha_{n-i}(n+s)$ for $i = n-1, \dots, 1$. Let $x \in X$ be a very general point and apply Theorem 1. In case (a) there exists $k \gg 0$ and a divisor $E \in |kL'|$ having an isolated singularity of index $\geq n+s$ at x, hence by Theorem 1.5, (1) the linear system $|K_X + L'| = |K_X + rL|$ generates s-jets at x. Therefore there is a subvariety $V \subset X$ with the claimed properties. $\square$

Remark 3.2 Up to the assumption on the positivity of T_X and the genericity of $x \in X$, Theorem 3.1 looks similar to [ELN, Theorem 4.1]. Note however that in the estimate of the degree $\deg_L V$ we have $(n+s)^d$ compared to $(n+s)^n$ in [ELN], which turns out to be crucial when bounding the Seshadri constant. This improvement is achieved by "rescaling" the interval $[0, \alpha]$ as in 2.1.

3.3 Proof of Theorem 2

First we note that, since we are only considering general points, there is no loss of generality in supposing that X is smooth (see [EKL, 3.2] for the precise argument).

Suppose that $\varepsilon(L, x) < \varepsilon$. Then by Theorem 1.5, (3) there exist positive integers s and r with $r > \frac{s+n}{\varepsilon}$ such that $|K_X + rL|$ does not separate s-jets at x. By assumption we have

$$(rL)^n \geq r^n(\varepsilon\gamma)^n > (\gamma(s+n))^n .$$

So we apply Theorem 3.1 to obtain a subvariety $V \ni x$ of positive dimension d satisfying a degree bound which, due to the trivial estimate $1 - \sqrt[n]{(1-a)^{n-d}} < 1$ for $0 < a < 1$ can be replaced by

$$L^d \cdot V < \varepsilon^d \gamma^n \alpha_d^{d-n},$$

leading to a contradiction. $\square$

Remark 3.4 From Theorem 1 one can deduce easily various boundedness statements by specifying the α_i:

(a) Setting $\alpha_1 = \cdots = \alpha_n = 1$, we get the universal bound $\varepsilon(L, x) \geq n^{-n}$ for the Seshadri constant.

(b) With the following choice, one comes closer to the bound of [EKL]: put $\alpha_i = (n-1)(2^{-i}(1-2^{1-n})^{-1}$, and define $\mu(d) := \min_V \left\{L^d \cdot V\right\}$, where the minimum runs over all d-dimensional subvarieties $V \subset X$ containing very general points. Then

$$\varepsilon(L, x) \geq \min_{1 \leq d \leq n} \left\{ \sqrt[d]{\mu(d) \frac{(n-1)^{n-d}}{n^n[2^d(1-2^{1-n})]^{n-d}}} \right\}.$$

4 Bounds at an arbitrary point

In this section, we show how to apply the strategy of §3 to obtain certain bounds for Seshadri constants at arbitrary points using the following result of [ELN]:

Theorem 4.1 (Ein–Lazarsfeld–Nakamaye) *Let $x \in X$ be any point of a smooth n-fold X. Let A be an ample line bundle on X and $\delta \geq 0$ a real number such that $T_X(\delta A)$ is nef. Suppose that*

$$0 = \beta_1 < \cdots < \beta_{n+1} < \sqrt[n]{L^n}$$

are rational numbers. Then either

(a) there exists $E \in |kA|$ (for $k \gg 0$) with an isolated singularity at x of index

$$\frac{\beta_{n+1} - \beta_n}{1 + \delta\beta_n}; \quad or$$

(b) there exists an irreducible subvariety $V \ni x$ of codimension $c \neq n$ with

$$A^{n-c}V = \deg_A V \leq \left(\frac{1 + \delta\beta_c}{\beta_{c+1} - \beta_c}\right)^c \cdot \beta_{n+1}^n.$$

Theorem 4.1 follows from [ELN, Theorem 3.9] together with [ELN, 1.5, 1.6] and the remark (from the proof of Theorem 3.9) that V is in any event an irreducible component of the base locus of $|\mathcal{I}_V^{(k\varepsilon)} \otimes \mathcal{O}_X(k(1+\delta\sigma)A))|$ on X; when V is 0-dimensional, this gives the divisor in (a) by Bertini's Theorem.

Then the argument proceeds as in §3:

Corollary 4.2 *Let X, A and δ be as in (4.1), and moreover, let r, s be positive integers, and $\gamma > 1$ a rational number such that $(rA)^n > (\gamma(n+s))^n$. Let $\alpha_1, \ldots, \alpha_{n-1}$ be positive rational numbers with*

$$1 + \left(1 + \frac{\delta}{r}(n+s)\right) \cdot \sum_{i=1}^{n-1} \alpha_i = \gamma.$$

Then for any $x \in X$ either $K_X + rA$ separates s-jets at x, or there exists a subvariety $V \subset X$ through x of dimension $d \geq 1$ and degree

$$A^{n-c} \cdot V \leq \frac{\gamma^n (n+s)^d}{r^d} \left(\frac{1 + \frac{\delta}{r}(n+s)\sum_{i=d+1}^{n-1} \alpha_i}{\alpha_d}\right)^{n-d}.$$

Proof Put $A' = rA, \beta_{n+1} = \gamma(n+s), \beta_n = (n+s) \cdot \sum_{i=1}^{n-1} \alpha_i$, and $\beta_i = \beta_{i+1} - \alpha_{n-i}(n+s)$. Then apply Theorem 4.1 to A' and $\delta' = \frac{\delta}{r}$, and use Theorem 1.5, (1). □

Theorem 4.3 *Let X, A and δ be as in 4.1, moreover let $\varepsilon > 0$ be real and $\alpha_1, \ldots, \alpha_n$ positive rational numbers. Let $x \in X$ be any point and suppose that, for any $1 \leq d \leq n$, any d-dimensional subvariety $V \subset X$ containing x satisfies*

$$\deg_A V = A^d \cdot V \geq \varepsilon^d \left(\frac{1 + \delta\varepsilon \sum_{i=d+1}^{n-1} \alpha_i}{\alpha_d} \right)^{n-d} \cdot \left(1 + (1+\delta\varepsilon) \sum_{i=1}^{n-1} \alpha_i \right)^n .$$

Then $\varepsilon(A, x) \geq \varepsilon$.

Proof Fix $x \in X$ and suppose $\varepsilon(A, x) < \varepsilon$. Then there exist positive integers r, s with $r > (s+n)/\varepsilon$ such that $|K_X + rA|$ does not generate s-jets at x. Put $\gamma = 1 + \left(1 + \frac{\delta}{r}(n+s)\right) \sum_{i=1}^{n-1} \alpha_i$. Then by assumption

$$\begin{aligned} (rA)^n &\geq r^n \varepsilon^n \left(1 + (1+\delta\varepsilon) \sum_{i=1}^{n-1} \alpha_i \right)^n \\ &> (s+n)^n \left(1 + \left(1 + \frac{\delta}{r}(s+n) \right) \sum_{i=1}^{n-1} \alpha_i \right)^n = ((s+n)\gamma)^n . \end{aligned}$$

Then we arrive at a contradiction because Corollary 4.2 gives the existence of $V \ni x$ of dimension $d \geq 1$ with

$$\begin{aligned} A^d \cdot V &\leq \frac{\gamma^n (n+s)^d}{r^d} \left(\frac{1 + \frac{\delta}{r}(n+s) \sum_{i=d+1}^{n-1} \alpha_i}{\alpha_d} \right)^{n-d} \\ &< \varepsilon^d \gamma^n \left(\frac{1 + \delta\varepsilon \sum_{i=d+1}^{n-1} \alpha_i}{\alpha_d} \right)^{n-d} \\ &< \varepsilon^d \left(\frac{1 + \delta\varepsilon \sum_{i=d+1}^{n-1} \alpha_i}{\alpha_d} \right)^{n-d} \cdot \left(1 + (1+\delta\varepsilon) \sum_{i=1}^{n-1} \alpha_i \right)^n . \quad \square \end{aligned}$$

Setting $\alpha_1 = \cdots = \alpha_n = 1$ one obtains Corollary 3 of the introduction.

Remark 4.4 It is well known that, for a very ample line bundle H on X, the twisted tangent bundle $T_X(K_X + nH)$ is globally generated, and in particular nef (cf. [De93, 12.1]). In case $A = K_X$ is ample on X one therefore can use one of the available effectivity statements for very ampleness of ample line bundles (e.g. [De93]) to determine explicit values for δ, making Theorem 4.3 or Corollary 3 effective, in the sense that the bounds for the Seshadri constant of K_X at any $x \in X$ only depend on the dimension n. The same argument works in case $A = -K_X$ is ample, or for any ample A in case K_X is trivial.

Remark 4.5 Finally, let us compare Corollary 3 with the bounds that can be obtained using Angehrn and Siu's basepoint-free Theorem. Namely, Angehrn and Siu prove that, for an ample line bundle A on X, the adjoint line bundles $mA + K_X$ are free for $m \geq \frac{1}{2}n(n+1)+1$. An elementary argument (see for example [Kü, 3.3]) shows that $\varepsilon(A,x) \geq 1$ for all ample basepoint-free line bundles A. Moreover, if $T_X(\delta A)$ is nef, then so is the $\mathbb{Q}$-line bundle $M := \det(T_X(\delta A)) = -K_X + n\delta A$. Seshadri constants by definition have the sublinearity property

$$\varepsilon(\lambda L + \mu M, x) \geq \lambda \cdot \varepsilon(L,x) + \mu \cdot \varepsilon(M,x)$$

for nef line bundles L, M and any rational $\lambda, \mu \geq 0$. This shows that

$$\varepsilon(A,x) \geq \frac{2}{n(n+2\delta+1)+2}\,.$$

References

[AS] U. Angehrn and Y.-T. Siu Effective freeness and point separation for adjoint bundles Invent. Math. 122 1995 291–308

[De92] J.-P. Demailly Singular Hermitian metrics on positive line bundles Hulek et al., eds. Complex Algebraic Varieties Proceedings, Bayreuth 1990, Lect. Notes in Math 1507 1992 87–104

[De93] J.-P. Demailly A numerical criterion for very ample line bundles J. Diff. Geom. 37 1993 323–374

[EKL] L. Ein, O. Küchle and R. Lazarsfeld Local positivity of ample line bundles J. Diff. Geom. 43 1995 193–219

[EL] L. Ein and R. Lazarsfeld Seshadri constants on smooth surfaces Astérisque 218 1993 177–185

[ELN] L. Ein, R. Lazarsfeld and M. Nakamaye Zero-estimates, intersection theory and a theorem of Demailly M. Andreatta and T. Peternell, eds. Higher dimensional algebraic varieties (Trento, 1994) de Gruyter 1996 183–208

[Fu82] T. Fujita Theorems of Bertini type for certain types of polarized manifolds J. Math. Soc. Japan 34, No. 4 1982 709–717

[Fu94] T. Fujita Approximating Zariski decomposition of big line bundles Kodai Math. J. 17 1994 1–3

[Ha] R. Hartshorne Algebraic geometry Graduate Texts in Mathematics 52 Springer

[Kü] O. Küchle Multiple point Seshadri constants and the dimension of adjoint linear series Ann. Inst. Four. 46 1996 63–71

[La] R. Lazarsfeld Lengths of periods and Seshadri constants of Abelian varieties Preprint alg-geom/9606012 Math. Res. Lett. 3 1996 439–447

[Na] M. Nakamaye Seshadri constants on Abelian varieties Amer. Jour. of Math. 118 1996 621–636

[Pa] R. Paoletti Seshadri constants, gonality of space curves, and restrictions of stable bundles J. Diff. Geom. 40 1994 475–504

[Xu95] G. Xu Ample line bundles on smooth surfaces Crelle's J. 469 1995 199–209

[Xu96] G. Xu On Bertini's theorem Man. Math 89 1996 237–244

Oliver Küchle and Andreas Steffens,
Mathematisches Institut, Universität Bayreuth,
D–95440 Bayreuth, Deutschland
Oliver.Kuechle@uni-bayreuth.de
Andreas.Steffens@freeway.de