Algorithms for computing intersection numbers on moduli spaces of curves, with an application to the class of the locus of Jacobians

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The first purpose of this note is to explain how to compute intersection numbers of divisors on the moduli spaces $\overline{\mathcal{M}}_{g,n}$ of stable pointed curves. The Witten conjecture, proved by Kontsevich, gives a recipe to compute the intersection numbers of the n basic line bundles on $\overline{\mathcal{M}}_{g,n}$. As we will see, knowing these numbers allows one to compute all other intersection numbers of divisors as well. That this is possible was pointed out to me by Rahul Pandharipande. Earlier, Eduard Looijenga had made a remark that went a long way in the same direction.

After describing the various divisors on $\overline{\mathcal{M}}_{g,n}$, we proceed to discuss the algorithm computing their intersection numbers. We discuss our implementation of this algorithm and the results we obtain from it. For example, we compute all intersection numbers on $\overline{\mathcal{M}}_g$ for $g \leq 6$. (Copies of the program and some data computed with it are available from the author.)

A refined version of the algorithm requires us to take into account certain higher codimension classes, introduced by Mumford and Arbarello–Cornalba; it computes all intersection numbers of these classes and divisors. Recently, we realized that the Chern classes of the Hodge bundle can be taken along as well; hence all intersection numbers of Mumford's tautological classes and divisors can be computed. This has several applications. In §4 we discuss one application in detail: the calculation of the class of the locus of Jacobians in the moduli space of principally polarized abelian varieties of dimension g (projected in the tautological ring). This class was classically known for $g = 4$ and we computed it by ad hoc methods for $g = 5$; the new method allows us in principle to compute the class for all g, in practice currently for $g \leq 7$. Other applications may be found in the recent papers of Graber and Pandharipande [GP] and Kontsevich and Manin [KM].

1 Line bundles and divisors on $\overline{\mathcal{M}}_{g,n}$

For nonnegative integers g and n with $2g-2+n>0$, denote by $\overline{\mathcal{M}}_{g,n}$ the moduli space of stable n-pointed curves of genus g, over an algebraically closed field k. This is the Deligne–Mumford compactification of the moduli space $\mathcal{M}_{g,n}$ of smooth n-pointed curves $(C;x_1,\ldots,x_n)$ of genus g (with $x_i\neq x_j$ if $i\neq j$). We consider certain classes in the rational Picard group of $\overline{\mathcal{M}}_{g,n}$. First, for $1\leq i\leq n$, let ψ_i denote the first Chern class of the line bundle whose fiber at a stable n-pointed curve $(C;x_1,\ldots,x_n)$ is the cotangent space to C at x_i; that is, $\psi_i=c_1(\sigma_i^*(\omega_{\pi_{n+1}}))$, where $\pi_{n+1}\colon\overline{\mathcal{M}}_{g,n+1}\to\overline{\mathcal{M}}_{g,n}$ is the morphism obtained by forgetting the $(n+1)$st marked point (the universal curve, cf. [Kn 1]), $\omega_{\pi_{n+1}}$ is the relative dualizing sheaf, and $\sigma_1,\ldots,\sigma_n$ are the natural sections of π_{n+1} (the image of a stable n-pointed curve under σ_i is the stable $(n+1)$-pointed curve obtained by attaching a 3-pointed rational curve at the ith point and considering the remaining 2 points on that curve as the ith and $(n+1)$st point). Next, following [AC 2], §1, we define $\kappa_1=\pi_{n+1,*}(K^2)$, with $K=c_1(\omega_{\pi_{n+1}}(\sum_{i=1}^n D_i))$, where D_i is the divisor that is the image of the section σ_i, for $1\leq i\leq n$. Note that it is a consequence of results of Harer (cf. [Ha 1], [Ha 2], [AC 1]) that over $\mathbb{C}$ the restrictions to $\mathcal{M}_{g,n}$ of the classes κ_1 and $\psi_1,\ldots,\psi_n$ generate the rational Picard group of $\mathcal{M}_{g,n}$.

To get generators for the rational Picard group of $\overline{\mathcal{M}}_{g,n}$, we have to add the fundamental classes of the boundary divisors. Exactly when $g>0$, there is a boundary component whose generic point corresponds to an irreducible singular curve. It is the image of $\overline{\mathcal{M}}_{g-1,n+2}$ under the degree 2 map that identifies the $(n+1)$st and $(n+2)$nd point on each curve. Following [AC 2], we denote this locus by Δ_{irr} and its class in the Picard group by δ_{irr}. (For $g=0$ this class is 0 by definition.)

The other boundary components parametrize reducible singular curves. The generic point of such a component corresponds to a curve with two irreducible components C_1 and C_2 of genus g_1 and g_2 with $g_1+g_2=g$, and labelled by subsets N_1 and N_2 of $\underline{n}=\{1,2,\ldots,n\}$ satisfying $N_1\coprod N_2=\underline{n}$ that correspond to the marked points on the two components. All partitions of g and $\underline{n}$ that lead to a stable curve occur; this just translates as the condition $|N_i|\geq 2$ when $g_i=0$. Such a boundary component is the image of $\overline{\mathcal{M}}_{g_1,|N_1|+1}\times\overline{\mathcal{M}}_{g_2,|N_2|+1}$ under the natural map that identifies the two 'extra' points and labels the $|N_i|$ remaining points on C_i with the labels from N_i. We choose to denote this boundary component in the case $n>0$ by Δ_{g_i,N_i}, where N_i is the subset of $\underline{n}$ containing 1. In the case $n=0$ the N_i are empty and we may drop them in the notation; note that $\Delta_{g_1}=\Delta_{g_2}$ and that this component is usually denoted by $\Delta_{\min(g_1,g_2)}$.

Although this plays no role in the sequel, we point out that the classes in the rational Picard group introduced so far are independent whenever

$g \geq 3$. For $g = 2$, there is one relation, arising from the fact that κ_1 on $\overline{\mathcal{M}}_2$ comes from the boundary. For $g = 1$, both κ_1 and the ψ_i come from the boundary; the boundary components are independent. For $g = 0$, the boundary components generate, but are not independent; the relations arise from the various projections to $\overline{\mathcal{M}}_{0,4}$ and the equivalence of its 3 boundary cycles. For proofs of these statements, see [Ke] and [AC 3].

There is one other divisor class which is most useful: λ_1, the first Chern class of the Hodge bundle. The Hodge bundle is the locally free rank g sheaf (on the moduli functor) whose fiber at a curve C is $H^0(C, \omega_C)$. So it is 0 in genus 0, while it is a pullback from $\overline{\mathcal{M}}_{1,1}$ or $\overline{\mathcal{M}}_g$ in case $g = 1$, respectively $g \geq 2$.

2 The idea of the algorithm

Suppose given a monomial of degree $3g-3+n$ in the divisor classes $\kappa_1, \psi_1, \ldots, \psi_n, \delta_{\mathrm{irr}}$ and the δ_{g_i,N_i} on $\overline{\mathcal{M}}_{g,n}$; we want to compute the corresponding intersection number. We interpret the divisor classes as classes on the moduli functor. In the case of a boundary divisor, this means that we divide the usual fundamental class by the order of the automorphism group of the generic curve parametrized by the divisor. We denote these divisor classes by $\delta_{\ldots}$, to distinguish them from the actual boundary divisors $\Delta_{\ldots}$.

The case in which the monomial involves only the ψ_i is of course covered by the Witten conjecture [Wi], proved by Kontsevich [Ko]. As explained in [AC 2], this also allows us to compute intersection numbers involving both κ_1 and the ψ_i.

It remains to compute the intersection numbers involving a boundary class. We think of such a number as the intersection of the remaining classes on the corresponding boundary component. A problem with this approach appears to be that most boundary components have singularities that are not quotient singularities, which means that one cannot properly do intersection theory on them. This problem is easily solved: we have seen that each boundary component is the image under a finite map of a moduli space of stable pointed curves or a product of two such spaces. (The map almost always has degree 1; the only exceptions are the degree 2 map from $\overline{\mathcal{M}}_{g-1,n+2}$ to Δ_{irr} and, in the case g even, the degree 2 map from $\overline{\mathcal{M}}_{g/2,1} \times \overline{\mathcal{M}}_{g/2,1}$ to $\Delta_{g/2}$.) So we wish to pull the remaining divisor classes back by means of this map. If we can express the pullbacks in terms of the basic classes on the new moduli space (or product of moduli spaces), we will be done, by induction on the dimension of the moduli space.

So the point is to understand the pullbacks of the basic divisor classes from $\overline{\mathcal{M}}_{g,n}$ to the moduli spaces occurring in its boundary components.

It is clear that the ψ_i pull back to ψ_i on the new moduli space(s): to the first n on $\overline{\mathcal{M}}_{g-1,n+2}$, and to the $|N_1|$ classes ψ_i on $\overline{\mathcal{M}}_{g_1,|N_1|+1}$ and the $|N_2|$ classes ψ_i on $\overline{\mathcal{M}}_{g_2,|N_2|+1}$ that correspond to the points not identified in the map to Δ_{g_i,N_i}.

As explained in [AC 2], the class κ_1 pulls back to κ_1 on $\overline{\mathcal{M}}_{g-1,n+2}$, respectively to the sum of the pullbacks of the κ_1 from the two factors on the product $\overline{\mathcal{M}}_{g_1,|N_1|+1} \times \overline{\mathcal{M}}_{g_2,|N_2|+1}$.

Pulling back a boundary divisor other than the one under consideration to the new moduli space(s) is not difficult. The main point is that two distinct boundary divisors intersect transversally in the universal deformation space (see [DM]). It remains to identify the boundary divisors on the new moduli space(s) that arise as the inverse image of the intersection of the boundary divisor under consideration with a distinct one.

For example, in the case Δ_{irr}, the pullback of the class $\delta_{h,M}$ to $\overline{\mathcal{M}}_{g-1,n+2}$ is the sum

$$\delta_{h-1,M\cup\{n+1,n+2\}} + \delta_{h,M},$$

with some exceptions: when $n = 0$ and $2h = g$, the two classes in the sum are equal and the pullback consists of that class just once; when $n = 0$ otherwise, $\delta_{h,\emptyset}$ was denoted $\delta_{g-1-h,\{1,2\}}$ above; when $h = 0$ or $h = g$, the first, respectively the second, summand is not defined, and should be omitted.

We now consider the pullbacks of boundary divisors to a product of the form $\overline{\mathcal{M}}_{g_1,|N_1|+1} \times \overline{\mathcal{M}}_{g_2,|N_2|+1}$. The pullback of δ_{irr} is the sum of the δ_{irr} on the two factors. It remains to find the pullbacks of the boundary divisors parametrizing reducible curves of type other than that under consideration.

We start with the case $n = 0$. Hence $N_1 = N_2 = \emptyset$. We may assume that $g_1 \leq g_2$, and we want to pull back a class δ_h, with $h \leq g - h$ and $h \neq g_1$. The general point in a component of the intersection of the two boundary divisors parametrizes a chain of 3 irreducible curves. There are a priori 4 possibilities for the genera of the 3 components:

(1) $[h, g_1 - h, g_2]$, occurring when $g_1 > h$;

(2) $[h, g_2 - h, g_1]$, occurring when $g_2 > h$;

(3) $[g_1, h - g_1, g - h]$, occurring when $h > g_1$;

(4) $[g_2, h - g_2, g - h]$, occurring when $h > g_2$.

Here the second entry refers to the genus of the middle component; note that $[a, b, c]$ and $[c, b, a]$ describe the same type of curves.

Note that $h \leq g/2 \leq g_2$. In fact $h < g_2$, since equality implies $h = g_1$, which we have excluded. So (4) never occurs, while (2) always occurs. We conclude that, with one exception, the pullback of δ_h equals

$$\begin{cases} \mathrm{pr}_1^* \delta_{g_1-h,\{1\}} + \mathrm{pr}_2^* \delta_{g_2-h,\{1\}} & \text{if } g_1 > h; \\ \mathrm{pr}_2^* \delta_{g_2-h,\{1\}} + \mathrm{pr}_2^* \delta_{h-g_1,\{1\}} & \text{if } h > g_1, \end{cases}$$

where pr_1, pr_2 are the projections to the two factors of $\overline{\mathcal{M}}_{g_1,|N_1|+1} \times \overline{\mathcal{M}}_{g_2,|N_2|+1}$. The exception is when $h = g - h$ (implying $h > g_1$), and the two summands in the second line are equal; then the actual pullback consists of that class just once.

Now for the case $n > 0$. We may assume that $1 \in N_1$ and we want to pull back a class $\delta_{h,M}$ other than δ_{g_1,N_1}. So $1 \in M$ and $(h, M) \neq (g_1, N_1)$, but we no longer have $g_1 \leq g_2$ or $h \leq g - h$. Again there are *a priori* 4 possibilities for the genera of the 3 components, where as before the second entry refers to the genus of the middle component:

(1) $[h, g_1 - h, g_2]$, occurring when $g_1 \geq h$ and $M \subset N_1$;

(2) $[h, g_2 - h, g_1]$, occurring when $g_2 \geq h$ and $M \subset N_2$;

(3) $[g_1, h - g_1, g - h]$, occurring when $h \geq g_1$ and $N_1 \subset M$;

(4) $[g_2, h - g_2, g - h]$, occurring when $h \geq g_2$ and $N_2 \subset M$.

Observe that (2) never occurs, since $1 \in M$ and $1 \notin N_2$. Note also that the other possibilities indeed yield stable curves in all cases: in (1) and (3) the necessary condition $M \neq N_1$ when $h = g_1$ is fulfilled, and in (4) the equality $M = N_2$ never occurs. Finally, note that types (1), (3) and (4) are mutually exclusive.

This means that the pullback of $\delta_{h,M}$ consists of the sum of 0, 1 or 2 of the classes from the following list, depending on which conditions hold:

(1) $\mathrm{pr}_1^* \delta_{h,M}$, pullback from $\overline{\mathcal{M}}_{g_1,N_1\cup\{*\}}$ when $g_1 \geq h$ and $M \subset N_1$;

(3) $\mathrm{pr}_2^* \delta_{h-g_1,M-N_1\cup\{1\}}$, pullback from $\overline{\mathcal{M}}_{g_2,N_2\cup\{1\}}$ when $h \geq g_1$ and $N_1 \subset M$;

(4) $\mathrm{pr}_1^* \delta_{h-g_2,M-N_2\cup\{*\}}$, pullback from $\overline{\mathcal{M}}_{g_1,N_1\cup\{*\}}$ when $h \geq g_2$ and $N_2 \subset M$.

Here we have identified the factors of the product of moduli spaces by means of sets of marked points instead of just their number of elements. In light of our convention to label a divisor parametrizing reducible curves by the genus of the component containing 1 and by the set of marked points on that component, it is natural to give the 'extra' point in case (3) the label 1, rather than $*$.

Finally, we have to deal with self-intersections of boundary divisors: we need to pull the class of a boundary divisor back to the corresponding moduli space (or product of moduli spaces). It is not difficult to deal with this directly, but it is easier to use the fundamental identity on $\overline{\mathcal{M}}_{g,n}$

$$\kappa_1 = 12\lambda_1 - \delta + \psi,$$

where δ is the sum of the functorial classes of the boundary divisors and ψ is the sum of the n classes ψ_i (see [Co]). Namely, every divisor class on $\overline{\mathcal{M}}_{g,n}$ discussed so far occurs in this identity. So a given boundary divisor class can be expressed as a linear combination of other divisor classes, and if we know how to pull back the other classes, we will also know how to deal with self-intersections. We have discussed the pullbacks of κ_1, the ψ_i and the other boundary divisor classes above, so we only need to determine the pullback of λ_1. Note that the pullback of the Hodge bundle to $\overline{\mathcal{M}}_{g-1,n+2}$ is an extension of a trivial line bundle by the Hodge bundle in genus $g-1$, whereas the pullback of the Hodge bundle to $\overline{\mathcal{M}}_{g_1,|N_1|+1} \times \overline{\mathcal{M}}_{g_2,|N_2|+1}$ is the direct sum of the Hodge bundle in genus g_1 and the Hodge bundle in genus g_2 (see [Kn 2] or [Co].) Hence we find that the pullback of λ_1 to $\overline{\mathcal{M}}_{g-1,n+2}$ equals λ_1, whereas its pullback to $\overline{\mathcal{M}}_{g_1,|N_1|+1} \times \overline{\mathcal{M}}_{g_2,|N_2|+1}$ equals $\operatorname{pr}_1^* \lambda_1 + \operatorname{pr}_2^* \lambda_1$.

This determines the pullback of a boundary divisor class to its corresponding (product of) moduli space(s). We find that for $n = 0$, the pullback of δ_{irr} to $\overline{\mathcal{M}}_{g-1,2}$ equals

$$-\psi_1 - \psi_2 + \delta_{\mathrm{irr}} + \sum_{h=1}^{g-2} \delta_{h,\{1\}},$$

whereas in the case $n > 0$ the pullback of δ_{irr} to $\overline{\mathcal{M}}_{g-1,n+2}$ equals

$$-\psi_{n+1} - \psi_{n+2} + \delta_{\mathrm{irr}} + \sum_{\substack{0 \le h \le g-1, 1 \in M \subset \underline{n} \\ M \ne \underline{n} \text{ when } h = g-1}} (\delta_{h,M\cup\{n+1\}} + \delta_{h,M\cup\{n+2\}}).$$

In the case of a boundary divisor parametrizing reducible curves, an actual self-intersection is much rarer. We find that the pullback of δ_{g_1,N_1} to $\overline{\mathcal{M}}_{g_1,N_1\cup\{*\}} \times \overline{\mathcal{M}}_{g_2,N_2\cup\{1\}}$ equals

$$\begin{cases} -\operatorname{pr}_1^* \psi_{\{*\}} - \operatorname{pr}_2^* \psi_1 + \operatorname{pr}_2^* \delta_{g_2-g_1,\{1\}} & \text{if } n = 0 \text{ and } g_1 < g_2; \\ -\operatorname{pr}_1^* \psi_{\{*\}} - \operatorname{pr}_2^* \psi_1 + \operatorname{pr}_1^* \delta_{g_1-g_2,\underline{n}\cup\{*\}} & \text{if } n > 0, N_1 = \underline{n} \text{ and } g_1 \ge g_2 > 0; \\ -\operatorname{pr}_1^* \psi_{\{*\}} - \operatorname{pr}_2^* \psi_1 & \text{otherwise.} \end{cases}$$

This finishes the theoretical description of the algorithm.

3 Implementation and results

We have implemented the algorithm outlined in §2 in Maple [Ma]. This first requires an implementation of Witten's algorithm [Wi] to compute the intersection numbers of the ψ_i on $\overline{\mathcal{M}}_{g,n}$. For this, we gratefully use the results of Chris Zaal [Za] who, using such an implementation, computed a table containing all the intersection numbers

$$\langle \tau_{d_1} \tau_{d_2} \cdots \tau_{d_n} \rangle = \psi_1^{d_1} \psi_2^{d_2} \cdots \psi_n^{d_n}$$

on $\overline{\mathcal{M}}_{g,n}$ with $g \leq 9$ for which all $d_i \geq 2$ (hence $n \leq 24$ since $\sum_{i=1}^n (d_i - 1) = 3g-3$). The intersection numbers for which a d_i equals 0 or 1 can be computed from these by means of the string and dilaton equations [Wi]:

$$\begin{cases} \langle \tau_0 \tau_{d_1} \tau_{d_2} \cdots \tau_{d_n} \rangle = \sum_{i:d_i>0} \langle \tau_{d_1} \cdots \tau_{d_i - 1} \cdots \tau_{d_n} \rangle & \text{(string equation)}, \\ \langle \tau_1 \tau_{d_1} \tau_{d_2} \cdots \tau_{d_n} \rangle = (2g-2+n) \langle \tau_{d_1} \tau_{d_2} \cdots \tau_{d_n} \rangle & \text{(dilaton equation)}. \end{cases}$$

From the intersection numbers of the ψ_i, one can determine the intersection numbers of the classes κ_i on $\overline{\mathcal{M}}_g$ introduced by Mumford [Mu], as outlined in [Wi]. Arbarello and Cornalba [AC 2] introduced classes κ_i on $\overline{\mathcal{M}}_{g,n}$ generalizing Mumford's classes, and they show that the intersection numbers of the ψ_i determine the intersection numbers of these κ_i as well as the 'mixed' intersection numbers of the divisor classes κ_i and ψ_i. So in particular the intersection numbers of κ_1 and the ψ_i are determined. It is easy to implement the calculation of these numbers from the intersection numbers of the ψ_i (especially so with a formula we learned from Dijkgraaf [Dij]), and we have for instance calculated the numbers κ_1^{3g-3} on $\overline{\mathcal{M}}_g$ for $g \leq 9$.

Naturally the various divisors on $\overline{\mathcal{M}}_{g,n}$ have to be ordered in some consistent way. On $\overline{\mathcal{M}}_g$ we start with κ_1, followed by δ_{irr} and then the 'reducible' boundary divisors, ordered by the minimum of the genera of the 2 components, for a total of $[g/2]+2$ classes. (The class λ_1 was introduced only to deal with self-intersections of boundary divisors and is not actually used in the program.) When $n > 0$, there are $(g+1)2^{n-1}+1$ classes: first $\psi_1, \ldots, \psi_n, \kappa_1$ and δ_{irr}, then the reducible classes, ordered first by the genus of the component that contains the point 1, then by the number of points on that component, and finally by the lexicographic ordering of subsets of $\underline{n}$ (of equal size and containing 1). (Recall that in genus 0 the class δ_{irr} is 0; it is included for convenience.)

In the case of a pullback to a product of moduli spaces, we need to renumber the indices from $N_1 \cup \{*\}$ as well as those from $\{1\} \cup N_2$. For this we just use the natural ordering of the elements of $\underline{n}$, taking $*$ as the $(n+1)$st point.

The implementation of the actual algorithm is now rather straightforward. Given a monomial M in the divisor classes involving at least one boundary divisor, we order the classes as above, and pull M back to the (product of) moduli space(s) corresponding to the last divisor that occurs. Note that whenever the monomial contains several distinct boundary divisors, we have a choice here; while the ordering we choose is probably optimal for $\overline{\mathcal{M}}_g$, it is probably not for $n > 0$: we have recently experimented with another ordering, which indeed appears to be better; the idea is that in the reducible case one should try to pull back to two moduli spaces parametrizing curves of approximately equal genus and number of points.)

In case the last divisor is Δ_{irr}, we find a homogeneous polynomial of degree $3g-4+n$ in the divisor classes on $\overline{\mathcal{M}}_{g-1,n+2}$. After expanding, it is a sum of monomials (with coefficients) in the divisor classes; these are evaluated by means of the (heavily recursive) algorithm.

In the reducible case, we also find a homogeneous polynomial of degree $3g-4+n$, but this time in two sets of variables, the divisors on $\overline{\mathcal{M}}_{g_1,|N_1|+1}$ and those on $\overline{\mathcal{M}}_{g_2,|N_2|+1}$. Many of the monomials will not have the correct degree $3g_1-2+|N_1|$ in the first set of variables and are 0 for trivial reasons. The others automatically have degree $3g_2-2+|N_2|$ in the second set of variables; writing such a monomial as $c \cdot M_1 \cdot M_2$, where c is the coefficient of the monomial and M_i is the monic monomial in the ith set of variables, it contributes $c \cdot a(M_1) \cdot a(M_2)$, where $a(M_i)$ is the result of applying the algorithm to M_i on $\overline{\mathcal{M}}_{g_i,|N_i|+1}$.

Using this implementation of the algorithm, we have computed, for example, the 28 intersection numbers of κ_1, δ_{irr} and δ_1 on $\overline{\mathcal{M}}_3$, confirming the results of [Fa 1]. However, because of its heavily recursive character, the algorithm already becomes impracticable in the computation of certain intersection numbers on $\overline{\mathcal{M}}_4$. Most intersection numbers are still easy to compute, but especially the numbers $\kappa_1^{9-i}\delta_{\mathrm{irr}}^i$ with i large take a long time. It is quite clear why: firstly, a pullback to $\overline{\mathcal{M}}_{g-1,n+2}$ is the 'worst case', since the changes in genus and in dimension of the moduli space are minimal, while the number of points increases by 2, so that the number of divisors increases by a factor of almost 4; secondly, as we saw in §2, the pullback of δ_{irr} to $\overline{\mathcal{M}}_{g-1,n+2}$ involves by far the highest number of terms.

Observe however that the class δ_{irr} is a pullback from $\overline{\mathcal{M}}_g$, respectively $\overline{\mathcal{M}}_{1,1}$. This first of all implies

$$\delta_{\mathrm{irr}}^{m+1} = 0,$$

where $m = \max(g, 3g-3)$, for all $g \geq 0$. Using this identity systematically already saves a considerable amount of time. Moreover, any product involving only κ_1, the ψ_i and δ_{irr} can be pushed down to $\overline{\mathcal{M}}_g$, respectively $\overline{\mathcal{M}}_{1,1}$, by using the projection formula and the formulas in [AC 2]. This leads to intersection numbers on those spaces of monomials in δ_{irr} and the higher κ_i mentioned before. As observed by Arbarello and Cornalba, the κ_i behave very well under pullback, and it is clear that all intersection numbers involving divisors as well as the κ_i can be computed by means of an algorithm almost identical to that in §2. Even the implementation is easy to adapt. The point is that this greatly simplifies the calculation of the numbers involving only κ_1, the ψ_i and $\delta_{\rm irr}$: in the naive implementation, the complexity of a calculation increases at least exponentially with the number of points, and one would have to calculate certain numbers on $\overline{\mathcal{M}}_{0,2g}$ to get all numbers on $\overline{\mathcal{M}}_g$; but now many of the hardest numbers can be computed using at most 2-pointed curves at all stages of the computation.

With these simple changes implemented, the calculation of many more numbers becomes practical. We have calculated all intersection numbers of divisors on $\overline{\mathcal{M}}_g$ for $g \leq 6$ as well as on $\overline{\mathcal{M}}_{3,1}$ and $\overline{\mathcal{M}}_{4,1}$. To obtain the 2 numbers $\kappa_1\delta_{\rm irr}^{14}$ and $\delta_{\rm irr}^{15}$ on $\overline{\mathcal{M}}_6$, we used the relations $\lambda_1^{13}\delta_{\rm irr}\kappa_1 = 0$ and $\lambda_1^{13}\delta_{\rm irr}^2 = 0$, consequences of the relation $\lambda_1^{13}\delta_{\rm irr} = 0$, which is geometrically obvious.

This section of the paper would hardly be complete without some actual intersection *numbers*. Here are a few:

$$\text{On } \overline{\mathcal{M}}_4: \qquad \delta_{\rm irr}^9 = \frac{-251987683}{4320} \quad \text{and} \quad \lambda_1^9 = \frac{1}{113400}\,;$$

$$\text{On } \overline{\mathcal{M}}_5: \qquad \delta_{\rm irr}^{12} = \frac{-1766321028967}{6048} \quad \text{and} \quad \lambda_1^{12} = \frac{31}{680400}\,;$$

$$\text{On } \overline{\mathcal{M}}_6: \qquad \delta_{\rm irr}^{15} = \frac{-324679884372720659 77}{7257600} \quad \text{and} \quad \lambda_1^{15} = \frac{431}{481140}\,.$$

We computed the number λ_1^9 on $\overline{\mathcal{M}}_4$ in [Fa 2] by a completely different ad hoc method. Calculating it with the algorithm amounts to calculating all 220 intersection numbers of divisors on $\overline{\mathcal{M}}_4$, which provides a nice check of the implementation.

4 The class of the locus of Jacobians

We used the calculation of λ_1^9 on $\overline{\mathcal{M}}_4$ in [Fa 2] to obtain the well-known result that the class of the locus $\mathcal{J}_4$ of Jacobians of curves of genus 4 in the moduli space $\mathcal{A}_4$ of principally polarized abelian varieties of dimension 4 equals $8\lambda_1$. Using the computation of λ_1^{12} on $\overline{\mathcal{M}}_5$ as well as some computations in the tautological ring of $\mathcal{M}_5$ as in [Fa 3], we could determine the class of $\mathcal{J}_5$ in $\mathcal{A}_5$, as we explain in a moment. Recently we realized that Mumford's formula [Mu] for the Chern character of the Hodge bundle on $\overline{\mathcal{M}}_g$, together with an algorithm similar to that discussed in §2, enables one to compute the class of $\mathcal{J}_g$ in $\mathcal{A}_g$ for *all* g, at least in principle. In practice, we have carried this out for $g \leq 7$. We also discuss these results here.

We first recall the set-up, and explain our restricted interpretation of the "class of the locus of Jacobians" in the moduli space of principally polarized abelian varieties: by this, we mean a class in the *tautological ring* of $\widetilde{\mathcal{A}}_g$, the $\mathbb{Q}$-subalgebra of the cohomology ring of a toroidal compactification $\widetilde{\mathcal{A}}_g$ of $\mathcal{A}_g$ generated by the Chern classes λ_i of the Hodge bundle $\mathbb{E}$ on $\widetilde{\mathcal{A}}_g$. From [Mu], §5 we know the relation

$$\big(1-\lambda_1+\lambda_2-\lambda_3+\cdots+(-1)^g\lambda_g\big)\big(1+\lambda_1+\lambda_2+\lambda_3+\cdots+\lambda_g\big)=1;$$

equivalently, $\mathrm{ch}_{2k}(\mathbb{E})=0$ for all $k\geq 1$. The tautological ring is in fact the quotient of $\mathbb{Q}[\lambda_1,\ldots,\lambda_g]$ by the ideal generated by the homogeneous components of the relation above and is thus a complete intersection ring. A detailed description of the tautological ring may be found in [vdG]. In particular, the relation with the cohomology ring of the *compact dual* of the Siegel upper half space via Hirzebruch's proportionality principle is explained there. This includes the fundamental identity

$$\prod_{i=1}^{g}\lambda_i=\prod_{i=1}^{g}\frac{|B_{2i}|}{4i}$$

that enables one to compute intersection numbers in the tautological ring of $\widetilde{\mathcal{A}}_g$: the monomial of top degree on the left, interpreted as an intersection number, equals the number on the right. Here B_m are the Bernoulli numbers, defined as in [BS] via $t/(e^t-1)=1+\sum_{m=1}^{\infty}(B_m/m!)t^m$. So for $g=1,2,3$ respectively,

$$\lambda_1=\frac{1}{24},\quad \lambda_1\lambda_2=\frac{1}{5760},\quad \lambda_1\lambda_2\lambda_3=\frac{1}{2903040}.$$

Denoting by $t\colon\overline{\mathcal{M}}_g\to\widetilde{\mathcal{A}}_g$ the extended Torelli morphism and its image by $\widetilde{\mathcal{J}}_g$, we are after the functorial class $\big[\widetilde{\mathcal{J}}_g\big]_Q$ of the locus of (generalized) Jacobians, which is one half its usual fundamental class. In other words, we wish to determine $\frac{1}{2}t_*1$, since a generic curve of genus at least 3 has no nontrivial automorphisms, while the generic p.p.a.v. and the generic Jacobian of dimension at least 3 have two automorphisms.

It is important to point out that this is *not* what we actually compute. We do not know whether modulo boundary classes the class of the locus of Jacobians lies in the tautological ring; our feeling is that this is probably false for g large enough, but we cannot even envisage a method to decide this. Instead, we compute the *projection* $\big[\widetilde{\mathcal{J}}_g\big]_T$ of this class in the tautological ring; this is well defined by the perfect pairing in the tautological ring and the cohomology ring of $\widetilde{\mathcal{A}}_g$. In other words, we compute the class $\big[\widetilde{\mathcal{J}}_g\big]_Q$ modulo a class X that has zero pairing with all tautological classes of the complementary dimension.

The method to compute $[\widetilde{\mathcal{J}}_g]_T$ is the following. It is a class in the tautological ring of $\widetilde{\mathcal{A}}_g$ of dimension $3g-3$, hence of codimension $c = \binom{g-2}{2}$. Write it as a linear combination with unknown coefficients a_i of the elements s_i of a basis of the degree c part of the tautological ring:

$$[\widetilde{\mathcal{J}}_g]_T = a_1 s_1 + a_2 s_2 + \cdots + a_k s_k\,, \quad [\widetilde{\mathcal{J}}_g]_Q = [\widetilde{\mathcal{J}}_g]_T + X.$$

(A natural basis is for instance the collection of square free monomials of degree c in $\lambda_1, \ldots, \lambda_g$.) To compute the coefficients a_i, we have to evaluate the k monomials Λ_i of a basis of the degree $3g-3$ part of the tautological ring on this class. The evaluation of expressions $\Lambda_i s_j$ in the tautological ring of $\widetilde{\mathcal{A}}_g$ uses the relations between the λ_i and the proportionality relation stated above. So the expressions $\Lambda_i(a_1 s_1 + \cdots + a_k s_k)$ yield k linearly independent rational linear combinations of the unknowns a_i. The values of these expressions can be determined as $\frac{1}{2}t_*(t^*\Lambda_i) = (\frac{1}{2}t_* 1)\cdot\Lambda_i$, provided we know how to evaluate $t^*\Lambda_i$ on $\overline{\mathcal{M}}_g$.

The simplest nontrivial example is $g = 4$. Here $c = 1$, a basis in codimension 1 is λ_1, a basis in codimension 9 is λ_1^9 (or any nonzero monomial in the λ_i), so to compute the class of $\widetilde{\mathcal{J}}_4$ in $\widetilde{\mathcal{A}}_4$ we only need λ_1^9 on $\overline{\mathcal{M}}_4$. We have seen above that this can be evaluated, for example, by means of the implementation of the algorithm; we find the well-known result that $[\widetilde{\mathcal{J}}_4]_T = 8\lambda_1$.

The situation for $g = 5$ is more interesting. Here $c = 3$ and a basis in codimension 3 is given by $\lambda_1\lambda_2 (= \frac{1}{2}\lambda_1^3)$ and λ_3. We need to evaluate 2 independent monomials of degree 12 in the λ_i on $\overline{\mathcal{M}}_5$. The algorithm will naturally yield only the number λ_1^{12} (whose value we gave at the end of §3). However, we can use the simple observation that the class $\lambda_g\lambda_{g-1}$ vanishes on the boundary $\overline{\mathcal{M}}_g - \mathcal{M}_g$ (see [Fa 3] or [Fa 4]). As a corollary, the numbers $\lambda_1\lambda_2\lambda_4\lambda_5$ and $\lambda_3\lambda_4\lambda_5$ on $\overline{\mathcal{M}}_5$ satisfy the same relation as the classes $\lambda_1\lambda_2$ and λ_3 in the degree 3 part of the *tautological ring* $R^*(\mathcal{M}_5)$ of $\mathcal{M}_5$, which is 1-dimensional. This relation was worked out in [Fa 3]: $10\lambda_3 = 3\lambda_1\lambda_2$. A quick calculation in the tautological ring of $\widetilde{\mathcal{A}}_5$ shows that this implies that the class of the locus of Jacobians satisfies

$$[\widetilde{\mathcal{J}}_5]_T = a(3\lambda_1\lambda_2 - 2\lambda_3).$$

Another such calculation, using $\lambda_1^{12} = \frac{31}{680400}$ on $\overline{\mathcal{M}}_5$, then shows that $a = 24$, hence

$$[\widetilde{\mathcal{J}}_5]_T = 72\lambda_1\lambda_2 - 48\lambda_3.$$

For higher genus, this method only provides some of the coefficients, not all of them. We start with a general formula:

Conjecture 1 *In the basis of monic square free monomials in $\lambda_1, \ldots, \lambda_g$ of degree $\binom{g-2}{2}$, the coefficient $C_{1,2,\ldots,g-3}$ of $\lambda_1\lambda_2\cdots\lambda_{g-3}$ in the (projected) class $[\widetilde{\mathcal{J}}_g]_T$ equals*

$$\frac{1}{2g-2}\prod_{i=1}^{g-2}\frac{2}{(2i+1)|B_{2i}|}\,.$$

Thus for $g=3,\ldots,7$ it equals $1, 8, 72, 384, 768$ respectively, while it is not an integer for any larger value of g.

The conjecture holds for $g \leq 15$: it is a consequence of a conjectural formula in [Fa 3] for the number λ_{g-1}^3 on $\overline{\mathcal{M}}_g$ that is proved for $g \leq 15$. To derive it from that formula, note simply that $\lambda_{g-1}^3 = 2\lambda_{g-2}\lambda_{g-1}\lambda_g$ and that $\lambda_1\lambda_2\cdots\lambda_{g-3}$ is the unique monomial in the basis that has nonzero pairing with $\lambda_{g-2}\lambda_{g-1}\lambda_g$.

Note that we could have used this formula instead of that for λ_1^{12} to compute the class of $[\widetilde{\mathcal{J}}_5]_T$.

For $g=6$ we find the coefficient of $\lambda_2\lambda_4$ using the relation between $\lambda_1\lambda_3$ and λ_4 in $R^*(\mathcal{M}_6)$:

$$C_{2,4} = -3C_{1,2,3} = -1152.$$

The knowledge of λ_1^{15} (see §3) gives a nontrivial relation between the remaining coefficients of $\lambda_1\lambda_5$ and λ_6:

$$C_6 + 16C_{1,5} = \frac{7336704}{691}\,.$$

A new ingredient will be required to solve for these coefficients. It is provided by Mumford's formula [Mu] for the Chern character of the Hodge bundle on $\overline{\mathcal{M}}_g$:

$$\mathrm{ch}(\mathbb{E}) = g + \sum_{i=1}^{\infty}\frac{B_{2i}}{(2i)!}\left[\kappa_{2i-1} + \frac{1}{2}\sum_{h=0}^{g-1} i_{h,*}\left(K_1^{2i-2} - K_1^{2i-3}K_2 + \cdots + K_2^{2i-2}\right)\right].$$

(Note that Mumford's convention for the Bernoulli numbers differs from the one we use: $B_2 = \frac{1}{6}$, $B_4 = \frac{-1}{30}$ etc.) Here $i_0\colon \overline{\mathcal{M}}_{g-1,2} \to \Delta_{\mathrm{irr}} \subset \overline{\mathcal{M}}_g$ and $i_h\colon \overline{\mathcal{M}}_{h,1}\times\overline{\mathcal{M}}_{g-h,1} \to \Delta_h \subset \overline{\mathcal{M}}_g$ are the natural maps, and K_i is the first Chern class of the relative cotangent line bundle at the ith point.

The formula is ideally suited for a recursive computation of the intersection numbers of the κ_i and the $\mathrm{ch}_{2j-1}(\mathbb{E})$. Namely, suppose given a monomial in those classes of degree $3g-3+n$ on $\overline{\mathcal{M}}_{g,n}$. If only κ_i occur, we can proceed as explained in §2, using the Witten conjecture (Kontsevich's theorem). In any case, as the Hodge bundle is a pullback from $\overline{\mathcal{M}}_g$ or $\overline{\mathcal{M}}_{1,1}$, we can push down the expression to $\overline{\mathcal{M}}_g$, respectively $\overline{\mathcal{M}}_{1,1}$, and obtain a sum of similar expressions. In genus 1, we only need the well-known equalities

$$\mathrm{ch}_1(\mathbb{E}) = \lambda_1 = \kappa_1 = \psi_1 = \langle \tau_1 \rangle = \frac{1}{24}\,.$$

So assume the genus is at least 2 and the monomial on $\overline{\mathcal{M}}_g$ contains at least one $\mathrm{ch}_{2j-1}(\mathbb{E})$. Take the highest odd Chern character component that occurs, and expand it using Mumford's formula. In the first term, that ch_{2k-1} is replaced by a κ_{2k-1} (up to a factor), so it is determined inductively. The other terms involve expressions in the K_i pushed forward via the maps i_h. The point is that these can be written as pushforwards from $\overline{\mathcal{M}}_{g-1,2}$ or $\overline{\mathcal{M}}_{h,1} \times \overline{\mathcal{M}}_{g-h,1}$ of intersection numbers of the classes K_i, κ_j and Chern character components of the Hodge bundles in genus $g-1$, respectively genus h and $g-h$. This is clear from the Arbarello–Cornalba formulas for $i_h^*\kappa_j$ and the fact that $i_0^*(\mathbb{E})$ is the extension of a trivial line bundle by the Hodge bundle in genus $g-1$, while for h positive $i_h^*(\mathbb{E})$ is the direct sum of the Hodge bundles in genera h and $g-h$.

After expanding and omitting the terms that are 0 for dimension reasons, we find an expression in intersection numbers of the classes just mentioned on spaces of 1- or 2-pointed curves. These can be pushed down again to intersection numbers of κ_i and $\mathrm{ch}_{2j-1}(\mathbb{E})$ on $\overline{\mathcal{M}}_h$ (with $h < g$) or $\overline{\mathcal{M}}_{1,1}$. By induction on the genus, these numbers are known.

Having discussed the implementation of the divisor algorithm in some detail, we content ourselves with saying that the implementation of the new algorithm proceeds along similar lines and is considerably easier.

Because the λ_i can be expressed in the $\mathrm{ch}_j(\mathbb{E})$, this means that all the intersection numbers of the λ_i on $\overline{\mathcal{M}}_g$ can be computed recursively. By the discussion above, it follows that for *all* g the projection in the tautological ring of $\widetilde{\mathcal{A}}_g$ of the class of the locus of Jacobians can be computed, at least in principle.

Currently, we have carried this out for $g \le 7$. For $g = 6$, only one more relation was required. Either one of the two relations following from

$$\lambda_2\lambda_3\lambda_4\lambda_6 = \frac{1697}{2988969984000} \quad \text{and} \quad \lambda_1\lambda_2\lambda_3\lambda_4\lambda_5 = \frac{150719}{15692092416000}$$

suffices to solve for C_6 and $C_{1,5}$ (the first relation involves $C_{1,5}$ only). The result is

$$\begin{aligned}[\widetilde{\mathcal{J}}_6]_T &= 384\lambda_1\lambda_2\lambda_3 - 1152\lambda_2\lambda_4 + \frac{474048}{691}\lambda_1\lambda_5 - \frac{248064}{691}\lambda_6 \\ &= 2^7 3\left(\lambda_1\lambda_2\lambda_3 - 3\lambda_2\lambda_4 + \frac{2469}{1382}\lambda_1\lambda_5 - \frac{646}{691}\lambda_6\right).\end{aligned}$$

It may be worthwhile to point out the relation $15C_6 + 28C_{1,5} = 2^9 3^3$.

In genus 7 the result is:

$$[\widetilde{\mathcal{J}}_7]_T = 768\lambda_1\lambda_2\lambda_3\lambda_4 - 6912\lambda_2\lambda_3\lambda_5 + \frac{2209152}{691}\lambda_1\lambda_4\lambda_5 + \frac{7522176}{691}\lambda_1\lambda_3\lambda_6 - \frac{8842752}{691}\lambda_4\lambda_6 + \frac{968832}{691}\lambda_3\lambda_7 - \frac{3276672}{691}\lambda_1\lambda_2\lambda_7.$$

As stated, this result is not very pretty; perhaps the class looks better in a different basis. We would like to point out though that the class can be computed with any choice of 7 independent monomials of degree 18 in the λ_i; once this is accomplished, the values of all other such monomials can be determined by means of an easy calculation in the tautological ring of $\widetilde{\mathcal{A}}_7$. In particular, one can choose 'easy' monomials to compute the class, and get the 'hard' ones for free. In this way we computed for instance λ_1^{18} on $\overline{\mathcal{M}}_7$:

$$\lambda_1^{18} = \frac{32017001}{638512875}.$$

Finally, we have another general formula:

Conjecture 2 *In the basis of monic square free monomials in $\lambda_1, \ldots, \lambda_g$ of degree $\binom{g-2}{2}$, the coefficient $C_{2,3,\ldots,g-4,g-2}$ of $\lambda_2\lambda_3\cdots\lambda_{g-4}\lambda_{g-2}$ in the (projected) class $\left[\widetilde{\mathcal{J}}_g\right]_T$ equals*

$$\left(\frac{g(2g-2)}{12} - 2^{g-3}\right) C_{1,2,\ldots,g-3} = \frac{g}{12}\prod_{i=1}^{g-2}\frac{2}{(2i+1)|B_{2i}|} - \frac{1}{4g-4}\prod_{i=1}^{g-2}\frac{4}{(2i+1)|B_{2i}|}.$$

Thus for $g = 5, 6, 7$ it equals $-48, -1152, -6912$ respectively, while it is not an integer for any larger value of g.

The conjecture again holds for $g \leq 15$; it is a consequence of Conjecture 1 and of the conjectural formula $\kappa_1\lambda_{g-3} = g(2g-2)\lambda_{g-2}$ in $R^{g-2}(\mathcal{M}_g)$ that can be proved for $g \leq 15$ using the results of [Fa 3] (note that the only 2 monomials in the basis that have nonzero pairing with $\lambda_1\lambda_{g-3}\lambda_{g-1}\lambda_g$ are $\lambda_2\lambda_3\cdots\lambda_{g-4}\lambda_{g-2}$ and $\lambda_1\lambda_2\ldots\lambda_{g-3}$).

5 Acknowledgments and final remarks

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Finally, some remarks about possible uses of the implemented algorithms. The algorithm for computing intersection numbers involving Chern classes of the Hodge bundle (described in §4) has already found applications in the recent work of Graber and Pandharipande [GP] and Kontsevich and Manin [KM]. For moduli spaces $\overline{\mathcal{M}}_{g,n}$ of small dimension and with not too many divisors, one can use the divisor algorithm to determine the part of the cohomology ring generated by divisors. Recently, Getzler [Ge] did this for $\overline{\mathcal{M}}_{2,2}$; he shows in fact that the divisors generate the cohomology ring. As in that paper, such calculations may have applications to computing Gromov–Witten invariants. Also, the algorithms involving the κ_i and the ch_{2j-1} allow one to do calculations that include those classes; this covers, for example, the Chow ring of $\overline{\mathcal{M}}_3$. It is even possible to include arbitrary boundary strata as *module* generators, by pulling back all other classes to the corresponding product of moduli spaces via a sequence of maps, each identifying a single pair of points. Writing algorithms that can handle *intersections* of arbitrary boundary strata will be considerably more difficult, however.

The various algorithms, and some tables of intersection numbers computed with it, are available from the author by e-mail, although only the initial version of the divisor algorithm is currently available in a user friendly format.

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