Kähler hyperbolicity and variations of Hodge structures

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There is a well-known duality, pervasive throughout the theory of Hermitian symmetric spaces, sending a symmetric space of compact type to its noncompact dual (see, for example, [9], Chap. VIII, §4 and Chap. XI, p. 354); whereas the former is a projective variety, the latter can be realized as a bounded domain in $\mathbb{C}^n$. For example, the Grassmannian $\mathrm{Grass}(p, p+q) = \mathrm{SU}(p+q)/\mathrm{S}\big(\mathrm{U}(p)\times\mathrm{U}(q)\big)$ has as its dual the quotient $\mathrm{SU}(p,q)/\mathrm{S}(\mathrm{U}(p)\times\mathrm{U}(q))$, which is the bounded symmetric domain $\{Z \in \mathbb{C}^{p,q} \mid {}^t\overline{Z}Z - I_q < 0\}$. A geometric statement concerning a Hermitian symmetric space of compact type often has an associated 'dualized' statement about its noncompact dual. I formulate such a dual pair of problems and discuss some aspects of the version on a Hermitian symmetric space of noncompact type.

Let Ω be a Hermitian symmetric space of noncompact type that has a compact dual (that is, a bounded symmetric domain). Let A_Ω be the evenly graded commutative real algebra generated by the Chern–Weil forms of homogenous holomorphic Hermitian vector bundles on Ω; then A_Ω is isomorphic to the real cohomology algebra of the compact dual of Ω. Setting $\deg c_i = 2i$ defines a grading on the polynomial algebra $B_\Omega = A_\Omega[c_1, \dots, c_i, \dots]$.

Let $T = (M, \rho, i)$ be a triple consisting of a connected n-dimensional complex compact manifold M, a representation ρ of $\pi_1(M)$ in the isometry group $\mathrm{Aut}\,\Omega$ and a ρ-equivariant holomorphic immersion $i\colon \widetilde{M} \hookrightarrow \Omega$ from the universal covering space $\widetilde{M}$ of M into Ω. Typical examples arise from complex submanifolds of Hermitian locally symmetric manifolds uniformized by Ω. Since every homogenous vector bundle V_Ω on Ω descends to a vector bundle V_M on M, there is a canonical morphism $i^*\colon B_\Omega \to H^*(M, \mathbb{R})$ defined by $i^*c_i(V_\Omega) = c_i(V_M), i^*c_i = c_i(M)$. Evaluation on the fundamental class of M defines a linear form $e_T\colon B_\Omega^{2n} \to \mathbb{R}$. I call such a triple an n-dimensional *positive cycle* of Ω.

Define the closed convex cone of universal Chern numbers inequalities to be $K_\Omega^n = \{\alpha \in B_\Omega^{2n}; e_T(\alpha) \geq 0 \text{ for all } T\}$. Consider the following problem:

Problem Describe completely the closed convex cone K_Ω^n. Construct optimal universal inequalities, and describe the cases of equality. A specialization of the latter question is of interest: describe the optimal inequalities whose cases of equality are the totally geodesic submanifolds of Ω/Γ for Γ a discrete cocompact subgroup of $\operatorname{Aut}\Omega$.

The dual problem is easy to describe. To the compact dual $\widehat{\Omega}$, we associate the algebra $A_{\hat{\Omega}}$ generated by the Chern–Weil forms of homogenous vector bundles on $\widehat{\Omega}$ and we set $B_{\hat{\Omega}} = A_{\hat{\Omega}}[c_1, \dots, c_n, \dots]$, with $\deg c_i = 2i$. Then B_Ω and $B_{\hat{\Omega}}$ are isomorphic. We define the n-dimensional *positive cycles* as the n-dimensional projective algebraic submanifolds of $\widehat{\Omega}$. The cone $K^n_{\hat{\Omega}}$ is defined as before. The structure of $K^n_{\hat{\Omega}} \cap A^n_{\hat{\Omega}}$ is known when $\widehat{\Omega}$ is a Grassmannian: this is the content of the Fulton–Lazarsfeld theorem on universal inequalities for ample vector bundles [5].

However, there are also very significant differences between the problems of describing universal Chern numbers inequalities for Ω and $\widehat{\Omega}$. In the latter case, every projective algebraic manifold M (of sufficiently small dimension) may occur as the first term of a cycle (M, ρ, i). In the case of interest, the projective manifolds M that occur are of general type and have large fundamental groups.

I cannot formulate any reasonable conjecture describing K^n_Ω, and I only explain how to construct some interesting elements. There are various ways of constructing elements of K^n_Ω. For instance, because Chern–Weil theory enables us to compute Chern numbers as integrals of differential forms, starting from curvature tensors (whose explicit form is well known, see for instance [14]), explicit computations can be performed. The computational complexity of the problem grows so quickly with n that more subtle approaches are necessary.

In the situation of interest, Ω carries many homogeneous flat complex vector bundles which in addition underly a complex variation of Hodge structure. This structure pulls back to the universal covering $\widetilde{M}$ of M and then descends to M. The universal inequalities we discuss in this article are deduced from the study of these complex variations of Hodge structures. In the case that $\widetilde{M}$ is a totally geodesic complex submanifold of Ω, we call the resulting variation of Hodge structure a *locally homogeneous variation* and its period map a *homogeneous period map.*

More generally, we can consider complex variations of Hodge structures on M with their period map $\widetilde{M} \to D$, where D is the associated Griffiths period domain. See [3] for this construction and [7] for a thorough study of the homogenous domain D.

The manifold $\widetilde{M}$ is almost never compact. Fortunately, if the period map does not contract curves, the metric properties of the period map $\widetilde{M} \to D$ implied by the horizontality of the period map ensure that $\widetilde{M}$ admits a smooth exact Kähler form which is Lipschitz equivalent to the pullback of any given

Kähler form on M. Such a Kählerian compact manifold M is called *weakly Kähler hyperbolic.*

This structure makes it possible to apply Gromov's methods from [8] to the L^2 cohomology with values in the variation of Hodge structure on the universal cover and the main result from [4] follows.

Main Theorem *Let M be a compact weakly Kähler hyperbolic manifold, and suppose that $(M, \mathbb{V}, F^\bullet, S)$ is a complex polarized variation of Hodge structures (VHS for short). Denote by K_P^r the holomorphic vector bundle on M given by $K_P^r = F^{P-r}/F^{P-r+1} \otimes \Omega^r$ and by $K_P^\bullet$ the complex*

$$K_P^0 \xrightarrow{\nabla'} K_P^1 \xrightarrow{\nabla'} \cdots \longrightarrow K_P^{\dim_{\mathbb{C}} M}.$$

Then $(-1)^{\dim_{\mathbb{C}}(M)}\chi(M, K_P^\bullet) \geq 0$.

Via the Riemann–Roch theorem, these give inequalities for the Chern classes, which I call *Arakelov type inequalities*. Going back to homogeneous variations over a bounded domain $D = \Omega$, one obtains nontrivial elements of K_Ω^n. In the general case, one has to work with A_D, the algebra generated by the Chern forms of homogeneous Hermitian vector bundles, and with $B_D = A_D[c_1, \ldots]$. I define a *positive n-dimensional cycle* of D to be a triple $T = (M, \rho, i)$ with M a connected n-dimensional complex compact manifold, $\rho\colon \pi_1(M) \to \mathrm{Aut}(D)$ a representation and $i\colon \widetilde{M} \to D$ a ρ-equivariant holomorphic horizontal immersion; the cone K_D^n is defined as the dual cone of the convex cone generated by irreducible positive cycles. The Arakelov type inequalities define nontrivial elements in this cone. An interesting feature is that, for a general Griffiths domain, these Arakelov type inequalities cannot be deduced from the explicit formulas for the curvature, even when $n = 1$. Another interesting phenomenon is that one can construct cases of equality for some of them that are locally homogenous. I conjecture that in fact every case of equality is locally homogenous (I call the corresponding positive cycles *geodesic* cycles). I can only prove this when the geodesic cycles satisfy some strong rigidity properties.

We describe the organization of this article. Section 1 reviews Gromov's notion of Kähler hyperbolicity, and vanishing theorems for L^2 cohomology. Section 2 recalls the definition of a VHS and sketches the proof of the Main Theorem; for more details, see [4], on which my talk at the Warwick Euro-Conference was based. The rest of the article is devoted to a systematic investigation of the cases of equality in the main theorem, which was lacking in [4]. Section 3 describes locally homogenous VHS on irreducible symmetric domains along the lines of Zucker's article [17] and gives a criterion derived from a formula of Kostant that identifies the cases of equality in the main theorem among them (Proposition 3.5.2). This criterion leads in principle to an algorithm producing the list of every locally homogenous VHS that gives rise to a case of equality in the main theorem. In this spirit, Appendix A.2 gives an algorithm to determine every such locally homogenous VHS uniformized by a classical domain. Section 4 establishes a part of the conjecture that every nontrivial case of equality of the universal inequalities among Chern numbers of Theorem 2.2.1 is actually a locally homogenous VHS, and describes the strong rigidity properties satisfied by certain of these examples (a list is given in Appendix A.3), which gives some more evidence for the gap phenomenon conjectured by N. Mok and the author (see [12], §4 and the introduction to [13]).

I thank the editors and referees for help in improving the exposition.

1 Harmonic bundles and Kähler hyperbolicity

1.1 Kähler hyperbolicity

Recall the following definition of Gromov [8]: let (M, ω) be a compact Kähler manifold and $\widetilde{M} \to M$ its universal covering. (M, ω) is *Kähler hyperbolic* if the Kähler form ω pulled back to $\widetilde{M}$ is the de Rham coboundary of a bounded smooth 1-form. This notion depends on the chosen Kähler form. The basic examples are Kähler manifolds admitting a Riemannian metric of strictly negative sectional curvature (for any Kähler form) and submanifolds of Hermitian locally symmetric manifolds of noncompact type (for the restriction of the symmetric Kähler form). I need the following variant: M is said to be *weakly Kähler hyperbolic* if $\widetilde{M}$ carries a smooth real 1-form α whose de Rham coboundary $\mathrm{d}\alpha$ is a Kähler form Lipschitz equivalent to the pullback of some (and hence every) Kähler form on M.

1.2 The Lefschetz–Gromov vanishing theorem

Let (X, g) be a complete Riemannian manifold and V a flat vector bundle endowed with a flat connection D and a Hermitian metric h, not necessarily flat. Let $L^2\,\mathrm{dR}^p(X, g, V, h)$ be the Hilbert space of square integrable p-forms on X with values in V whose distributional de Rham coboundary is still square integrable. Define the L^2 de Rham complex as the following complex of topological vector spaces:

$$L^2\,\mathrm{dR}^0(X,g,V,h) \xrightarrow{\mathrm{d}} L^2\,\mathrm{dR}^1(X,g,V,h) \xrightarrow{\mathrm{d}} \cdots \xrightarrow{\mathrm{d}} L^2\,\mathrm{dR}^{\dim X}(X,g,V,h).$$

The cohomology of this complex is called the L^2 cohomology of (X, g) with values in (V, h), and denoted by $H^\bullet_{(2)}(X, g, V, h)$. It is obviously a bi-

Lipschitz invariant. Gromov [8], 1.2.B, p. 273, asserts that the L^2 cohomology of the universal covering of a compact Kähler hyperbolic manifold (M,ω) with values in the trivial flat vector bundle (or any unitary flat vector bundle) vanishes in degree $\neq \dim_{\mathbb{C}} M$.

As observed in [4], Théorème 1, Gromov's proof also extends to VHS (see below). In fact, the same proof also holds for Higgs bundles equipped with a harmonic metric. Recall ([16]) that a Higgs bundle on a complex manifold M is a holomorphic vector bundle V together with a holomorphic map $\theta\colon V \to V \otimes \Omega^1_M$ with $\theta \wedge \theta = 0$ in $\mathrm{End}(V) \otimes \Omega^2_M$. Not all such bundles admit harmonic metrics; one needs that the underlying flat bundle is semisimple. Such Higgs bundles are called *harmonic Higgs bundles.* Variations of Hodge structures provide examples, as we see below.

Proposition 1.2.1 *Let M be a compact weakly Kähler hyperbolic manifold. Then the L^2 cohomology of its universal covering with coefficients in a harmonic Higgs bundle endowed with its harmonic metric (cf. [16]) vanishes in degrees* $\neq \dim_{\mathbb{C}} M$.

Note that on $\widetilde{M}$ the harmonic metric is Lipschitz equivalent to a Hermitian metric constructed by parallel transport of a Hermitian metric on some fibre if and only if the (real) monodromy group is compact, which is rarely the case.

2 Variations of Hodge structures

2.1 Definition

Definition 2.1.1 Let M be a complex manifold. A quadruple $(M, \mathbb{V}, F^\bullet, S)$ is called a *complex polarized variation of Hodge structure* if $\mathbb{V}$ is a flat bundle of finite dimensional complex vector spaces with a flat connection D, $F^\bullet$ a decreasing filtration of $\mathbb{V} \otimes \mathcal{O}_M$ by holomorphic subbundles indexed by the integers, and S a flat nondegenerate sesquilinear pairing such that

1. The C^∞ vector bundle V associated to $\mathbb{V}$ decomposes as a direct sum $V = \bigoplus_p H^p$ with $F^P = \bigoplus_{p \geq P} H^p$;
2. $S(H^p, H^r) = 0$ for $p \neq r$, and $(-1)^p S$ is positive definite on H^p;
3. $D^{1,0} F^p \subset F^{p-1} \otimes \Omega^{1,0}_M$.

The numerical vector $\{\dim_{\mathbb{C}} H^r\}_{r \in \mathbb{Z}}$ is called the *Hodge vector* of the VHS.

The subbundle H^p can be given a holomorphic structure by the isomorphism $H^p \to F^p/F^{p+1}$. Write d''_p for the corresponding Dolbeault operator, and set $\mathrm{d}'' = \bigoplus_p \mathrm{d}''_p$. Then $D^{1,0}$ induces a C^∞-linear map $\nabla'_p\colon H^p \to$

$H^{p-1} \otimes \Omega^1$ called the *Gauss–Manin connection*; we set $\nabla' = \bigoplus_p \nabla'_p$. The Hermitian metric $H = \bigoplus_p (-1)^p S_{H^p}$ is called the *Hodge metric*. The pair $(\mathbb{V} \otimes \mathcal{O}_M, \nabla')$ is a Higgs bundle and H is a harmonic metric on the associated flat bundle $\mathbb{V}$.

For our purposes, the weight of a VHS is not relevant; instead, we are interested in the *length* of a VHS, the number $p_{\max} - p_{\min}$, where $p_{\max}$ and $p_{\min}$ are the maximum and minimum of the (finite) set of integers p with $\dim H^p \neq 0$.

The Kähler hyperbolic assumption in Proposition 1.2.1 is well adapted to nondegenerate VHS by [4], Proposition 4.6.1:

Proposition 2.1.2 *Let M be a compact Kähler manifold and $(M, \mathbb{V}, F^\bullet, S)$ a VHS on M. Assume that for any connected smooth curve C mapping to M with 1-dimensional image, the induced VHS on C has a nonzero Gauss–Manin connection (equivalently, no curve on M is contracted by the Griffiths period map). Then M is a projective weakly Kähler hyperbolic manifold. Thus the L^2 cohomology $H^\bullet_{(2)}(\widetilde{M}, \mathbb{V})$ vanishes in degree $\neq \dim_{\mathbb{C}} M$.*

2.2 Arakelov type inequalities

Set $E^{P,Q} = \bigoplus_{p+r=P, s-p=Q} H^p \otimes \Omega^{rs}$ and $D'' = \mathrm{d}'' + \nabla'$. It follows that $D'' E^{P,Q} \subset E^{P,Q+1}$. Furthermore, Deligne proved (cf. [4], Proposition 2.2.1) that given any Kähler metric, if we take formal adjoints of differential operators with respect to this Kähler metric and the Hodge metric on $\mathbb{V}$, the usual Kähler identities hold, for instance $2\big(D''(D'')^* + (D'')^* D''\big) = DD^* + D^*D$. It follows that the L^2 cohomology on the universal covering $\widetilde{M}$ with coefficients in $\mathbb{V}$ has a Hodge decomposition.

Theorem 2.2.1 *Let M be a compact weakly Kähler hyperbolic manifold and $(M, \mathbb{V}, F^\bullet, S)$ a VHS. Denote by K^r_P the holomorphic vector bundle on M given by $K^r_P = H^{P-r} \otimes \Omega^r$ and by $K^\bullet_P$ the complex*

$$0 \to K^0_P \xrightarrow{\nabla'} K^1_P \xrightarrow{\nabla'} \cdots \to K_P^{\dim_{\mathbb{C}} M} \to 0.$$

Then $(-1)^{\dim_{\mathbb{C}}(M)} \chi(M, K^\bullet_P) \geq 0$. In terms of Chern numbers, this reads:

$$(-1)^{\dim_{\mathbb{C}}(M)} \sum_r (-1)^r \operatorname{ch}(H^{P-r}) \operatorname{ch}(\Omega_M^{p-r}) \operatorname{Todd}(T_M)[M] \geq 0.$$

Sketch of Proof Let G be a countable group, endowed with the counting measure, and write l or $r \colon G \to \mathrm{U}(L^2(G))$ for its left (respectively right) regular representation. The commutant of l is a von Neumann algebra $W^*_r(G)$.

Every element a of $W_r^*(G)$ can be written uniquely as $a = \sum_{g \in G} a_g r(g)$, with $a_g \in \mathbb{C}$. Set $\tau(a) = a_e$. The linear form τ is a finite trace on $W_r(G)$, because

$$\tau(ab) = \tau(ba) \text{ with } a \neq 0 \implies +\infty > \tau(aa^*) > 0.$$

The orthogonal projection p onto a closed G-invariant subspace E of $L^2(G)$ belongs to $W_r(G)$ and we may try to define $\dim_G(E) = \tau(p)$. In fact, this definition can be extended to any Hilbert space with a left G-action, provided it can be realized as a G-stable closed linear subspace of $L^2(G)^{\oplus N}$. This provides a dimension theory for a certain class of unitary representations of G, similar to the dimension theory of linear algebra over $\mathbb{C}$ (the particular case $G = \{1\}$). For instance, $E = 0 \iff \dim_G E = 0$. The differences that have to be stressed in this article are that, when the group is infinite, the dimension function takes values in $\mathbb{R}^+$ and that an admissible G-module of nonzero G-dimension is infinite dimensional in the usual sense.

We can now explain Atiyah's L^2 index theorem (for details, see Atiyah [1]). Let D be an elliptic differential operator on the compact connected C^∞-manifold X, and $\operatorname{Ind} D$ its index. Then D can be lifted to a differential operator $\widetilde{D}$ on the universal covering space $\widetilde{X}$ which is invariant under the natural action of $\pi_1(X)$. Then the $\pi_1(X)$-module $\ker_{L^2}(\widetilde{D}) = \{u \in L^2(\widetilde{X}); \widetilde{D}u = 0\}$ has a well-defined $\pi_1(X)$-dimension and

$$\dim_{\pi_1(X)} \ker_{L^2}(\widetilde{D}) - \dim_{\pi_1(X)} \ker_{L^2}(\widetilde{D}^*) = \operatorname{Ind} D.$$

This theorem is the analogue for infinite coverings of the multiplicative property of the usual index under finite coverings.

$\chi(M, K_P^\bullet)$ is the usual index of the elliptic complex $D''\colon E^{P,*} \to E^{P,*+1}$. By Atiyah's L^2 index theorem, it can be computed as the L^2 index of the lifted elliptic complex on $\widetilde{M}$ where the vanishing theorem Proposition 2.1.2 is available. □

3 Locally homogenous variations of Hodge structures

3.1 Hermitian symmetric spaces of noncompact type

Let Ω be an irreducible Hermitian symmetric space of noncompact type. Its group of isometries is an almost simple adjoint Lie group $\operatorname{Aut}\Omega$, and we set $\mathfrak{g} = \operatorname{Lie}(\operatorname{Aut}\Omega)$. Let $G_{\mathbb{C}}$ be the simply connected complex semisimple algebraic group associated to $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$. Then $G_{\mathbb{C}}$ has a connected real form G with Lie algebra $\mathfrak{g}$, and $\operatorname{Aut}\Omega$ is the adjoint group G^{ad} of G. Let K be a maximal compact subgroup of G and observe that $\Omega = G/K$. Since Ω

is Hermitian symmetric, $\mathfrak{l} = \mathrm{Lie}\, K$ has a 1-dimensional center $z(\mathfrak{l})$ and splits as $\mathfrak{l} = z(\mathfrak{l}) \oplus [\mathfrak{l}, \mathfrak{l}]$. Thus $Z = \exp(z(\mathfrak{l}))$ is isomorphic to $\mathrm{U}(1)$. We write μ for the degree of the covering map $Z \to Z^{\mathrm{ad}}$.

Now suppose that $z_0 \in Z^{\mathrm{ad}}$ is an element of order 4. Then $z_0^2 = \theta$ is a Cartan involution. Write $\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{p}$ for the corresponding Cartan decomposition. Then z_0 induces on $\mathfrak{p} = T_{eK}G/K$ the tensor $\pm J$, where J is the almost complex structure of Ω. We may thus choose z_0 inducing J on $\mathfrak{p}$. Write accordingly $\mathfrak{p} \otimes \mathbb{C} = \mathfrak{p}^+ \oplus \mathfrak{p}^-$, where $\mathfrak{p}^\pm = \ker(\mathrm{ad}\, z_0 - \pm\sqrt{-1})$.

We can choose a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ contained in $\mathfrak{l}$. Let $\Delta_\mathfrak{g} \subset \mathfrak{h}_\mathbb{C}^*$ be the root system for $\mathfrak{g}_\mathbb{C}$. These roots take purely imaginary values on $\mathfrak{h}$. There exists a partition $\Delta_\mathfrak{g} = \Delta_\mathfrak{l} \cup \Delta_\mathfrak{p}$ for which $\mathfrak{p}_\mathbb{C} = \bigoplus_{\alpha \in \Delta_\mathfrak{p}} \mathfrak{g}^\alpha$ and $\mathfrak{l}_\mathbb{C} = \bigoplus_{\alpha \in \Delta_\mathfrak{l}} \mathfrak{g}^\alpha$. One can choose an ordering on $\sqrt{-1}\mathfrak{h}^*$ such that $\mathfrak{p}^+ = \bigoplus_{\alpha \in \Delta_\mathfrak{p}^+} \mathfrak{g}^\alpha$, where $\Delta_\mathfrak{p}^+ = \Delta^+ \cap \Delta_\mathfrak{p}$ (respectively, $\Delta_\mathfrak{l}^+ = \Delta^+ \cap \Delta_\mathfrak{l}$).

3.2 Homogenous holomorphic vector bundles

We say that a holomorphic vector bundle $V \to \Omega$ of finite rank is *homogenous* if the natural action of G on Ω lifts to an action on V which preserves the complex structure. Homogenous holomorphic vector bundles correspond one-to-one to finite dimensional K-modules. Under this correspondence the holomorphic tangent (or cotangent) bundle of Ω corresponds to $\mathfrak{p}^+$ (respectively $\mathfrak{p}^-$).

3.3 Homogenous VHS

There is another natural correspondence between flat homogenous vector bundles on Ω of finite rank and finite dimensional complex linear representations of G. As observed in [17], an irreducible flat homogenous complex vector bundle $\mathbb{V}$ on Ω can be given a structure of a homogenous VHS.

Let us describe Zucker's construction. $V = \mathbb{V}_{eK}$ is the representation space of the associated representation ρ of G. Let $\widehat{Z} = \mathrm{Hom}(Z, \mathrm{U}(1)) = \chi_0^\mathbb{Z}$ with χ_0^μ the character of Z acting on $\mathfrak{p}^+$. Decompose V according to the action of Z: $V = \bigoplus_{\chi \in \widehat{Z}} V(\chi)$. Then $V(\chi)$ is K-invariant. Observe that $\rho(\mathfrak{p}^\pm V(\chi)) \subset V(\chi \cdot \chi_0^{\pm\mu})$. Let χ_V be the highest nontrivial character (with the convention that $\chi_0 > 0$). It follows from the irreducibility of ρ that there exists an integer N_ρ such that $V = \bigoplus_{l=0}^{N_\rho} V(\chi_V {\chi_0}^{-l\mu})$, and every summand is nonzero.

Define a decreasing filtration of V by setting $F^pV = \bigoplus_{l=p}^{N_\rho} V(\chi_V {\chi_0}^{-l\mu})$. The Lie subalgebra $\mathfrak{g}_c = \mathfrak{l} \oplus \sqrt{-1}\mathfrak{p} \subset \mathfrak{g}_\mathbb{C}$ is compactly embedded, and preserves a positive definite Hermitian sesquilinear pairing H (unique up to scalar multiple). The decomposition $V = \bigoplus_{l=0}^{N_\rho} V(\chi_V {\chi_0}^{-l\mu})$ is orthogonal and preserved by $\mathfrak{l}$.

Since $\rho(\mathfrak{p}) \bigoplus_{l \text{ odd}} V(\chi_V {\chi_0}^{-l\mu}) = \bigoplus_{l \text{ odd}} V(\chi_V {\chi_0}^{-l\mu})$, it follows that $\mathfrak{g}$ preserves the Hermitian nondegenerate pairing $S = \bigoplus_l (-1)^l H|_{V(\chi_V {\chi_0}^{-l\mu})}$ which polarizes the filtration $F^\bullet$.

Let $x = gK \in \Omega$, and set $F^p\mathbb{V}_x = \rho(g) \cdot F^p V \subset \mathbb{V}_x$ $S_x = S \circ \rho(g)^{-1}$. The $F_x^\bullet$ and S_x glue together to yield a holomorphic filtration on $\mathbb{V}$ satisfying Griffiths transversality (because $\rho(\mathfrak{p}^+)F^p \subset F^{p-1}$) and a polarization S of this filtration. I henceforth refer to this VHS as the irreducible homogenous VHS associated to ρ. The number N_ρ is the length of the VHS (see Definition 2.1).

For the reader's convenience, Appendix A.1 describes all fundamental VHS on classical domains.

A simple modification of this construction works for reducible Hermitian symmetric spaces and yields:

Proposition 3.3.1 *Let Ω be a Hermitian locally symmetric space of noncompact type, $\mathfrak{g}$ the Lie algebra of its automorphism group and $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$ a complex linear representation.*

There exists a $\mathfrak{g}$-invariant Hermitian form S_ρ that is nondegenerate (but indefinite unless ρ is trivial). Moreover, the constant local system $V \times \Omega \to \Omega$ underlies a VHS polarized by S_ρ which is invariant under a finite covering group G of the automorphism group of Ω, G acting on $\Omega \times V$ by $g \cdot (o, v) = (g \cdot o, g \cdot v)$.

For every $p \in \Omega$, we let $\varphi(p) = \{F^\bullet(V)\}$ be the corresponding Hodge flag. The group $\mathrm{U}(S_\rho)$ acts transitively on Hodge flags and if we fix some origin $o \in \Omega$ and let U be the stabilizer of the flag defined by o, the set of Hodge flags can be described as a homogeneous domain $D_\rho = D(\mathrm{U}(S_\rho), \varphi(o)) = \mathrm{U}(S_\rho)/\,\mathrm{U}$. This domain is called the *Griffiths domain*. The Griffiths period map corresponding to the VHS constructed above is the holomorphic horizontal map $f\colon \Omega \to D_\rho$ described by $f(p) := \varphi(p)$.

From a *homogenous* VHS, that is, a VHS on a bounded symmetric domain Ω invariant under some (finite) covering group G of the automorphism group of Ω, one associates a representation ρ of G in the obvious way, so that there is a one-to-one correspondence between homogenous VHS on a bounded symmetric domain and complex linear representations of the Lie algebra of its automorphism group.

3.4 Locally homogenous VHS

Let $M = \Omega/\Gamma$ be a Hermitian locally symmetric manifold of noncompact type with universal covering space the Hermitian symmetric space Ω. Say that a

VHS on M is *locally homogenous* if its lift to Ω is a homogenous VHS. It is proved in [4] that the partial converse to Theorem 2.2.1 holds:

Theorem 3.4.1 *Let M be a compact Hermitian locally symmetric manifold of noncompact type and $(M, \mathbb{V}, F^\bullet, S)$ a locally homogenous VHS. Recall that K_P^r is the holomorphic vector bundle on M given by $K_P^r = H^{P-r} \otimes \Omega_M^r$ and $K_P^\bullet$ the complex $0 \to K_P^0 \xrightarrow{\nabla'} K_P^1 \xrightarrow{\nabla'} \cdots \to K_P^{\dim_{\mathbb{C}} M} \to 0$. Then*

$$(-1)^{\dim_{\mathbb{C}}(M)} \chi(M, K_P^\bullet) \geq 0,$$

with equality if and only if $K_P^\bullet$ is acyclic.

Matsushima and Murakami ([11], Theorem 12.3, p. 33) prove that, if ρ is a irreducible complex representation of G associated to a *nonsingular* highest weight, $H^i(\pi_1(M), \rho) = H^i(M, \mathbb{V}_\rho) = 0$ unless $i = \dim_{\mathbb{C}}(M)$. This vanishing theorem is clearly stronger than the L^2 vanishing theorem applied to an Hermitian locally symmetric space. Furthermore, Theorem 3.4.1 can be deduced, for a nonsingular weight, from their work, and should be viewed as a weak version of their theorem holding for singular weights.

3.5 A formula of Kostant

Let $\rho \colon G \to \mathrm{GL}(V_\rho)$ be an irreducible complex representation. The complex $K_P^\bullet$ associated to the homogenous VHS it defines on Ω is obviously the complex of homogenous vector bundles associated to the following complex of K-modules:

$$K_\chi^\bullet(V_\rho) \colon \ \cdots \to \Lambda^r \mathfrak{p}^- \otimes V(\chi\chi_0^{r\mu}) \xrightarrow{\nabla'_\rho} \Lambda^{r+1}\mathfrak{p}^- \otimes V(\chi\chi_0^{(r+1)\mu}) \to \cdots,$$

where $\chi = \chi_V \chi_0^{-P\mu}$ and $\nabla'_\rho = \sum_{\alpha \in \Delta_{\mathfrak{p}}^+} \rho(e_\alpha) \otimes e_{-\alpha} \wedge$. This complex computes the part of the K-module $H^\bullet(\mathfrak{p}^+, V_\rho)$ on which the central subgroup Z acts with character χ.

Write $W_{\mathfrak{g}}$ and $W_{\mathfrak{l}} = W_{[\mathfrak{l},\mathfrak{l}]}$ for the Weyl groups of $\mathfrak{g}$, respectively $\mathfrak{l}$, and $\Pi_{\mathfrak{g}} \in \mathfrak{h}$ and $\Pi_{\mathfrak{l}} \in \mathfrak{h}$ for the weight lattice of $\mathfrak{g}$, respectively $\mathfrak{l}$ in $\sqrt{-1}\mathfrak{h}^*$. The Grothendieck ring of the category of representations of $\mathfrak{g}$ and $\mathfrak{l}$ is isomorphic to the ring $\mathbb{Z}[\Pi_{\mathfrak{g}}]^{W_{\mathfrak{g}}}$, respectively $\mathbb{Z}[\Pi_{\mathfrak{l}}]^{W_{\mathfrak{l}}}$. The fact that the Weyl group is simply transitive on the Weyl chambers yields:

Lemma 3.5.1 ([10]) *Denote by $W_{\mathfrak{g},\mathfrak{l}}$ the right coset space $W_{\mathfrak{l}} \backslash W_{\mathfrak{g}}$. Then $W_0 = \{w \in W_{\mathfrak{g}} : w\Delta_{\mathfrak{g}}^- \cap \Delta_{\mathfrak{g}}^+ \subset \Delta_{\mathfrak{p}}^+\} \to W_{\mathfrak{g},\mathfrak{l}}$ is a set of representatives.*

Set $W_0 = \bigcup_q W_0(q)$ where $W_0(q) = \{w \in W_0, \ |w\Delta_{\mathfrak{g}}^- \cap \Delta_{\mathfrak{p}}^+| = q\}$.

Let $C_{\mathfrak{g}}^+$ and $C_{\mathfrak{l}}^+$ be positive Weyl chambers for $\mathfrak{g}$ and $\mathfrak{l}$. For a dominant weight $\lambda \in \Pi_{\mathfrak{g}} \cap \overline{C}_{\mathfrak{g}}^+$ for $\mathfrak{g}$, write $E_\lambda^{\mathfrak{g}}$ for the irreducible $\mathfrak{g}$-module with highest weight λ; similarly, for a dominant weight $\mu \in \Pi_{\mathfrak{l}} \cap \overline{C}_{\mathfrak{l}}^{+}$ for $\mathfrak{l}$, write $E_{\mu}^{\mathfrak{l}}$ for the irreducible $\mathfrak{l}$-module with highest weight μ.

Let $\rho_{\mathfrak{g}}$, $\rho_{\mathfrak{l}}$ and $\rho_{\mathfrak{p}}$ be half the sum of the positive roots in $\Delta_{\mathfrak{g}}^{+}$ (respectively, $\Delta_{\mathfrak{l}}^{+}$ and $\Delta_{\mathfrak{p}}^{+}$).

The following is due to Kostant:

Proposition 3.5.2 (see [17], [10]) *Let $\lambda \in \Pi_{\mathfrak{g}} \cap C_{\mathfrak{g}}^{+}$. Then*

$$H^{q}(\mathfrak{p}^{+}, E_{\lambda}^{\mathfrak{g}}) = \sum_{w_0 \in W_0(q)} E_{w_0(\lambda+\rho_{\mathfrak{g}})-\rho_{\mathfrak{g}}}^{\mathfrak{l}}$$

We need to decompose this formula further according to the action of Z. First of all, observe that:

Lemma 3.5.3 $w\rho_{\mathfrak{p}} = \rho_{\mathfrak{p}}$ *for all* $w \in W_{\mathfrak{l}}$.

Proof Let ζ be a generator of $z(\mathfrak{l})$. Clearly $(\rho_{\mathfrak{p}}, \zeta) = c|\Delta_{\mathfrak{p}}^{+}| \neq 0$. Observe that $w \cdot \Delta_{\mathfrak{p}} = \Delta_{\mathfrak{p}}$. It follows that

$$(w\rho_{\mathfrak{p}}, \zeta) = c(|w\Delta_{\mathfrak{p}}^{+} \cap \Delta_{\mathfrak{p}}^{+}| - |w\Delta_{\mathfrak{p}}^{+} \cap \Delta_{\mathfrak{p}}^{-}|).$$

The Weyl group $W_{\mathfrak{l}}$ acts orthogonally on $\mathfrak{h}$ (w.r.t. the Killing form of $\mathfrak{g}$), leaving $\mathfrak{h} \cap [\mathfrak{l}, \mathfrak{l}]$ invariant. Thus, $z(\mathfrak{l}) = [\mathfrak{l}, \mathfrak{l}]^{\perp}$ is also $W_{\mathfrak{l}}$-invariant. Since $\varepsilon(w)$, equal to -1 to the power the number of reflections in any decomposition of w as a product of reflections, does not depend on whether we view w as an element of $W_{\mathfrak{l}}$ or of $W_{\mathfrak{g}}$ and is the determinant of the linear transformation it induces on $\mathfrak{h}$ and on $[\mathfrak{l}, \mathfrak{l}]$, $z(\mathfrak{l})$ is fixed by $W_{\mathfrak{l}}$. It follows that $(w(\rho_{\mathfrak{p}}), \zeta) = (\rho_{\mathfrak{p}}, w\zeta) = (\rho_{\mathfrak{p}}, \zeta)$, and thus $w\rho_{\mathfrak{p}} = \rho_{\mathfrak{p}}$. $\square$

The latter argument implies also that, by means of the identification of $\mathfrak{h}$ and its dual space via the Killing form, $\rho_{\mathfrak{p}}$ is identified with a positive generator of $z(\mathfrak{l})$, since $[\mathfrak{l}, \mathfrak{l}]^{W_{\mathfrak{l}}} = \{0\}$. Set $\overline{\omega} = \rho_{\mathfrak{p}}/\mu(\rho_{\mathfrak{p}}, \rho_{\mathfrak{p}}) \dim_{\mathbb{C}}(\mathfrak{p}^{+})$. Let $\Pi_{\mathfrak{g}} \subset \Pi_{\mathfrak{l}}^{K} \subset \Pi_{\mathfrak{l}}$ be the lattice consisting of those weights of $\mathfrak{l}$ arising from K-modules, and let $\lambda \in \Pi_{\mathfrak{l}}^{K}$ be a dominant weight. Observe that Z acts by the character χ_0^{l} on $E_{\lambda}^{\mathfrak{l}}$ if and only if $(\lambda, \overline{\omega}) = l$.

This implies the following:

Corollary 3.5.4 *Let $\lambda \in \Pi_{\mathfrak{g}}$ be a dominant weight. Then:*

$$H^{q}(K_{\chi_0^{l}}^{\bullet}(E_{\lambda}^{\mathfrak{g}})) = \sum\nolimits_{\substack{w_0 \in W_0(q) \text{ such that} \\ (w_0(\lambda+\rho_{\mathfrak{g}})-\rho_{\mathfrak{g}}, \overline{\omega}) = l}} E_{w_0(\lambda+\rho_{\mathfrak{g}})-\rho_{\mathfrak{g}}}^{\mathfrak{l}}.$$

We apply this to $\rho = 0$, where $\nabla'_{\rho} = 0$ and $H^{\bullet}(K_{\chi_0^{l}}^{\bullet}(E_0^{\mathfrak{g}})) = K_{\chi_0^{l}}^{\bullet}(E_0^{\mathfrak{g}}) = \Lambda^{-l/\mu}\mathfrak{p}^{-}$ $(l \leq 0)$, and call $c_{\mathfrak{p}}$ the (unique) element of W_0 that corresponds to $\Lambda^{\dim_{\mathbb{C}} \mathfrak{p}^{+}}\mathfrak{p}^{-}$.

In particular $W_0(0) = \{\mathrm{id}\}$, $W_0(\dim \Omega) = \{c_{\mathfrak{p}}\}$. For a possibly reducible Hermitian symmetric space, elements of $W_0(1)$ are in one-to-one correspondence with its irreducible factors.

It follows from Corollary 3.5.4 that for any $\lambda \in \Pi_{\mathfrak{g}} \cap \overline{C}^+_{\mathfrak{g}}$ that $H^0(\mathfrak{p}^+, E^{\mathfrak{g}}_\lambda)$ and $H^{\dim \Omega}(\mathfrak{p}^+, E^{\mathfrak{g}}_\lambda)$ are irreducible K-modules; these modules are easily identified. They correspond to the two complexes $K^\bullet_\chi(\lambda)$ with a single term; namely $K^\bullet_{\chi_V}(\lambda) = V(\chi_V)$, and

$$K^\bullet_{\chi_V \chi_0^{-(N_\lambda + \dim(\mathfrak{p}^+))\mu}}(\lambda) = V(\chi_V \chi_0^{-N_\lambda \mu}) \otimes \Lambda^{\dim(\mathfrak{p}^+)}\mathfrak{p}^-$$

correspond to the elements $w = e \in W_0$ respectively $w = c_{\mathfrak{p}}$. Therefore:

Lemma 3.5.5 *The length of the VHS associated to $\lambda \in \Pi_{\mathfrak{g}} \cap C^+_{\mathfrak{g}}$ is*

$$N_\lambda = (\lambda, (\mathrm{id} - c_{\mathfrak{p}}^{-1})\overline{\omega}).$$

We say that the character $\chi \in \widehat{Z}$ is *critical* with respect to the dominant weight $\lambda \in \Pi_{\mathfrak{g}}$ if some $K^q_{\chi_0^l}(E^{\mathfrak{g}}_\lambda)$ is not 0 but $K^\bullet_{\chi_0^l}(E^{\mathfrak{g}}_\lambda)$ is acyclic. In view of Theorem 3.4.1, we get the following criterion:

Proposition 3.5.6 *χ_0^l is critical with respect to λ if and only if the following hold:*

1. *$l \equiv (\lambda, \overline{\omega}) \mod \mu$;*
2. *$l \notin \{(w_0(\lambda + \rho_{\mathfrak{g}}) - \rho_{\mathfrak{g}}, \overline{\omega})\}_{w_0 \in W_0}$;*
3. *$(c_{\mathfrak{p}}(\lambda + \rho_{\mathfrak{g}}) - \rho_{\mathfrak{g}}, \overline{\omega}) \leq l \leq (\lambda, \overline{\omega})$.*

This provides us with many nontrivial examples. For example:

Corollary 3.5.7 *Let Ω be an Hermitian symmetric space of noncompact type. Set $l_c(\Omega) = |W_0| - \dim_{\mathbb{C}} \Omega$. Every dominant weight whose associated VHS has length $\geq l_c(\Omega)$ has critical characters. Moreover, every dominant weight in the series $(n\rho_{\mathfrak{g}})_{n\geq 1}$ or $(n\rho_{\mathfrak{g}} + (n+1)\lambda)_{n\geq 1, \lambda \in \Pi_{\mathfrak{g}} \cap \overline{C}^+{}_{fg}}$ has critical characters.*

Since the length of the VHS associated to the dominant root λ is given by the positive linear form $l(E_\lambda) = (\lambda, (\mathrm{id} - c_{\mathfrak{p}}^{-1})\overline{\omega})$, the corollary shows that, for each bounded symmetric domain Ω, only a finite number of weights have no critical character (I refer to these as the *special* weights of Ω).

Although I did not find a simple condition for a weight of a classical domain of type I, II or III to be special, the proposition gives in principle an algorithm to determine every special weight. See Appendix A.2 for details and specific examples.

We consolidate our gains in the following result.

Theorem 3.5.8 *Let Ω be an n-dimensional Hermitian symmetric space of noncompact type and λ a nonspecial weight with critical character $\chi_V{\chi_0}^{-P\mu}$. Let $E_\lambda^{\mathfrak{g}}$ be the highest weight module attached to λ. Let $\mathbb{G} = (\Omega, \mathbb{V}, F^\bullet, S)$ be the homogenous VHS attached to $E_\lambda^{\mathfrak{g}}$ (see Proposition 3.3.1).*

Then equality occurs in the universal inequality

$$(-1)^n \big(\mathrm{ch}(K_P^\bullet)\,\mathrm{Todd}(T_M)\big)_{n,n}[M] \geq 0$$

(which, in virtue of Theorem 2.2.1, is valid for every VHS with the same Hodge vector as $\mathbb{G}$ having an immersive period map) if the VHS is a locally homogenous VHS uniformized by $\mathbb{G}$.

4 Hyperrigid locally homogenous variations of Hodge structure

4.1 Scalar curvature for a Griffiths period map

Let $D = W/U$ be a Griffiths domain. The Lie algebra of W has a real Hodge structure of weight 0, polarized by the Killing form.

On D there is an invariant closed symplectic form ω which is not a Kähler form with respect to its complex structure. In fact, $h(X) = \omega(X, JX)$ is an indefinite nondegenerate form. However ω is positive definite on horizontal directions.

Let $T = (M, \rho, i)$ be an irreducible n-dimensional positive cycle. The form ω restricts to a Kähler metric ω_h on $\widetilde{M}$ which descends to M. In fact, it may also be defined as the curvature of the pullback of a certain equivariant holomorphic line bundle on D. Assume that the Griffiths period map i sends $x \in \widetilde{M}$ to $eU \in D(W, \varphi) = W/U$. Then $i_* T_x \widetilde{M}$ can be identified with a complex Abelian subspace of $\mathrm{Lie}(W)^{-1,1}$ which we denote by this subspace. Let $(e_i)_i$ be a unitary basis of this subspace. Using the Gauss equation and the formulas in [7] (for quick reference, see [14], Corollary 2.2), one easily computes:

Lemma 4.1.1 *The scalar curvature of M at x is:*

$$\mathrm{Scal}(M, x) = -\|\sum_i [e_i, \bar{e}_i]\|^2 - \Sigma^2 = S(\mathcal{A}) - \Sigma^2.$$

Viewing $T\widetilde{M}$ as a subbundle of the bundle associated to $\mathrm{Lie}(W)^{-1,1}$ (this is the horizontal tangent bundle) identifies the term Σ^2 with the square of the norm of the metric second fundamental form; thus it vanishes identically on M if and only if T is a geodesic cycle.

If ρ is a linear representation of W, one constructs the associated VHS $(M, \mathbb{V}_\rho, F^\bullet, S)$, and for every integer P the complex $K_P^\bullet$:

$$K_P^\bullet\colon 0 \to F^P/F^{P-1} \xrightarrow{\nabla'} K^{P-1}/F^{P-2} \otimes \Omega^1_M \xrightarrow{\nabla'} \\ \cdots \xrightarrow{\nabla'} K^{P-n}/F^{P-n-1} \otimes \Omega^n_M \to 0.$$

Lemma 4.1.2 *There exists a constant C_D^P and a function S_D^P on the space E_{ab} of n-dimensional complex Abelian subspaces of* $\mathrm{Lie}(W)^{-1,1}$ *such that, if $T = (M, \rho, i)$ denotes an n-dimensional positive cycle, then:*

$$c_1(K_P^\bullet)\omega_h^{n-1} = C_D^P(\mathrm{Scal}(M, x) + S_D^P(T_x M))\omega_h^n \quad \textit{at every } x \in M.$$

4.2 Hyperrigidity

4.2.1 Definitions

Definition 4.2.1 Let $D = W/U$ be a Griffiths domain and $\Omega = G/K$ a Hermitian symmetric space of noncompact type; we say that the homogeneous period map $\Omega \to D$ is 0- (respectively 1)-hyperrigid if it satisfies Condition 1 (respectively, Conditions 1–2) below:

1. $S(T\Omega)$ is the maximal value of the scalar curvature in the space on n-dimensional Abelian subspaces of $\mathrm{Lie}(W)^{-1,1}$.

2. There is a linear representation ρ of W and an integer P such that for the induced homogenous VHS on Ω, the complex $K_P^\bullet$ is nontrivial but acyclic, satisfies $C_{\mathcal{D}}^P \neq 0$ and S_D^P is maximal at $T\Omega$. (This condition follows automatically if $\mathrm{Lie}(W)^{-1,1}$ happens to be an irreducible U-module.)

4.2.2 Rigidity phenomena

The following lemma is elementary:

Lemma 4.2.2 *Assume that there is some 0-hyperrigid period map $\Omega \to D$ with scalar curvature S, and $\dim \Omega = n$. Then, for each n-dimensional positive cycle (M, ρ, i) of D, one has $(S\omega_h^n - c_1(M)\omega_h^{n-1})[M] \geq 0$ and every case of equality is a geodesic cycle uniformized by some 0-hyperrigid locally homogenous period map.*

Now, assume that there is some 1-hyperrigid homogenous period map $\Omega \to D$. Denote by $F^\bullet$ the Hodge filtration on the representation space of

ρ. For each Abelian n-dimensional complex subspace $\mathcal{A} \subset \mathrm{Lie}(W)^{-1,1}$ one defines the complex:

$$K_P^\bullet(\mathcal{A})\colon 0 \to F^P/F^{P-1} \xrightarrow{\nabla'} F^{P-1}/F^{P-2} \otimes \mathcal{A}^* \xrightarrow{\nabla'} \cdots \xrightarrow{\nabla'} F^{P-n}/F^{P-n+1} \otimes \Lambda^n \mathcal{A}^* \to 0$$

When $\mathcal{A}$ arises as the tangent space to a Griffiths period map, this complex is identified with the already defined complex $K_P^\bullet$ of the associated VHS. Let $U \subset E_{ab}$ be the (Zariski open) subset of the space of all Abelian n-dimensional complex subspaces $\mathcal{A}$ of $\mathrm{Lie}(W)^{-1,1}$ such that $K_P^\bullet(\mathcal{A})$ is acyclic.

One may build holomorphic homogenous fibre bundles $\mathbf{U} \to D$ and $\mathbf{E}_{ab} \to D$ with fibres at the origin respectively U and E_{ab}. The Gauss map of a positive cycle $T = (M, \rho, i)$ is the map $\gamma\colon M \to i^*\mathbf{E}_{ab}/\pi_1(M)$ sending each point to its embedded tangent space.

The following theorem follows almost immediately from the definition of 1-hyperrigidity:

Theorem 4.2.3 *Let Ω be an n-dimensional Hermitian symmetric space of noncompact type. Suppose that there is some 1-hyperrigid homogenous period map $\Omega \to D$. Let $T = (M, \rho, i)$ be a positive n-dimensional cycle of D. The Gauss map of T takes values in $i^*\mathbf{U}/\pi_1(M)$ if and only if T is a 0-hyperrigid geodesic cycle.*

Proof Theorem 3.4.1 tells us that for any cocompact fixed point free discrete subgroup Γ of $\mathrm{Aut}\,\Omega$, one can construct a positive cycle $T = (\Omega/\Gamma, \rho|_\Gamma, i)$ such that $\mathrm{ch}(K_P^\bullet) = 0$. In particular, $\int_{\Omega/\Gamma} c_1(K_P^\bullet)\omega_h^{n-1} = 0$. The integrand from Lemma 4.1.2 is constant along Ω/Γ. So the maximal value of S_D^P is exactly minus the scalar curvature of Ω. In particular, due to Definition 4.2.1, for any nongeodesic n-dimensional cycle $T = (M, \rho, i)$ (and also for any geodesic cycle of nonmaximal scalar curvature) $\int_M c_1(K_P^\bullet)\omega_h^{n-1} \neq 0$.

But, tautologically, to say that the Gauss map γ of the cycle T should take values in $i^*\mathbf{U}/\pi_1(M)$ means that $K_P^\bullet$ is an acyclic complex. Thus $\int_M c_1(K_P^\bullet)\omega_h^{n-1} = 0$, and T is a geodesic cycle. $\square$

On the other hand, the following theorem of [4] is more difficult to prove, and uses in a crucial way an improvement, due to Gromov [8], of Atiyah's L^2 index theorem.

Assume that there is some 1-hyperrigid period map $\Omega \to D = W/U$. Set $n = \dim \Omega$. Let ρ be the representation of W and P the integer provided by Definition 4.2.1. For a positive n-dimensional cycle of D, $T = (M, \rho, i)$, we write $(M, \mathbb{V}, F^\bullet, S)$ for the VHS on M associated to the representation ρ.

Theorem 4.2.4 *Assume that (M, ω_h) is a Kähler hyperbolic manifold in the strong sense. (This condition is automatic when D is Hermitian symmetric.) Then $\chi(M, K_P^\bullet) = 0$ if and only if T is a 0-hyperrigid geodesic cycle.*

Sketch of proof Let L be the homogenous holomorphic vector bundle on D such that $c_1(L,h) = \omega_h$. Since $i^*\omega_h$ is closed on $\widetilde{M}$, it follows that $i^*L \to \widetilde{M}$ is topologically trivial. Thus, one can construct, for each $\varepsilon \in \mathbb{R}$, a Hermitian holomorphic line bundle L^ε on $\widetilde{M}$ such that $c_1(L^\varepsilon) = \varepsilon\omega_h$. Furthermore, there is an extension Γ_ε of $\pi_1(M)$ by $\mathrm{U}(1)$ whose action on $\widetilde{M}$ lifts to L^ε. One can construct a variant of Atiyah's L^2 index theory for Γ_ε-invariant elliptic operators.

Set $E_\varepsilon^{P,Q} = \bigoplus_{p+r=P, s-p=Q} H^p \otimes \Omega^{rs} \otimes L^\varepsilon$ and $D''_\varepsilon = \mathrm{d}''_\varepsilon + \nabla'_\varepsilon$. In view of the higher index theory mentioned above, the L^2 D''_ε-complex with its Γ_ε-action has index given by the polynomial

$$P(\varepsilon) = \int_M \exp(\varepsilon\omega_h)\,\mathrm{ch}(K_P^\bullet)\,\mathrm{Todd}(T_M).$$

Now by Proposition 1.2.1, $\chi(X, K_P^\bullet) = 0$ implies that the L^2 D''-complex is acyclic for $\varepsilon = 0$.

Due to the boundedness of the postulated primitive of ω_h, the L^2 D''_ε-complex is also acyclic for small values of ε. In particular:

$$\chi(M, K_P^\bullet) = 0 \implies a_k = \omega_h^{n-k}\big(\mathrm{ch}(K_p^\bullet)\,\mathrm{Todd}(T_M)\big)_{k,k}[M] = 0 \quad \text{for all } k \le n.$$

Now, for $k = n-1$, $a_k = \int_M c_1(K_P^\bullet)\omega_h^{n-1}$ (note that $a_n = 0$ since there is some n-dimensional geodesic cycle M' which satisfies $\chi(M', K_P^\bullet) = 0$). We are thus reduced to the situation of Theorem 4.2.3. $\square$

4.3 Constructing examples

The conditions in Definition 4.2.1 are so strong (and so artificial) that one might wonder if 1-hyperrigid homogenous period maps can exist at all. Since I have not found any general principle to classify 1-hyperrigid homogenous period maps, I only explain here how to construct all the nontrivial examples I know (see Appendix A.3 for a list of examples).

Lemma 4.3.1 *Let $\Omega = G/K$ be an irreducible Hermitian symmetric space of noncompact type and $\lambda \in \Pi_\mathfrak{g} \cap \overline{C}_\mathfrak{g}^+$ a weight whose associated VHS has length 1 (or 2 and is defined over the real numbers with Hodge vector (a,b,a)). Then:*

(1) The Griffiths period map associated with the direct sum of k-copies of $E_\lambda^\mathfrak{g}$, $\Omega \to D^{\mathrm{I}}_{kp,kq}$ (respectively,

$$\Omega \to D_\mathbb{R}(ka, kb, ka) = \mathrm{SO}(2ka + kb)/\mathrm{S}\big(\mathrm{U}(ka) \times \mathrm{O}(kb)\big)$$

is 0*-hyperrigid.*

(2) *Assume moreover that* λ *has a critical character, or that some multilinear functor (for example, a symmetric or an exterior power) of* $E_\lambda^{\mathfrak{g}}$ *is an irreducible* $\mathfrak{g}$ *module whose highest weight has a critical character. If* $C_D^P \neq 0$*, then the homogenous period map is also* 1*-hyperrigid. In the case of a length* 1 *VHS with a critical character, this is always satisfied. In the length* 2 *case,*

$$C_D^P = 0 \iff \dim\Omega = 2P \quad \textit{and} \quad b/a = 2\dim\Omega/(\dim\Omega + 2).$$

Proof I give the proof of (1) only for VHS of length 1, the case of length 2 being similar. Recall the formula of Lemma 4.1.1 for the scalar curvature of a Griffiths period map. Observe that, for any locally homogenous VHS, $\sum_i [e_i, \bar{e}_i] \in \sqrt{-1}\mathfrak{s}(\mathfrak{u}(kp) \times \mathfrak{u}(kq))$ is the image under the representation of $\rho_{\mathfrak{p}}$. Thus it is a central element, that is, a matrix of the form (aI_{kp}, bI_{kq}), with $pa + bq = 0$.

For any n-dimensional subspace of $D^{\mathrm{I}}_{kp,kq}$, the element

$$\sum_i [e_i, \bar{e}_i] \in \sqrt{-1}\mathfrak{s}(\mathfrak{u}(kp) \times \mathfrak{u}(kq))$$

is a pair (A, B) of symmetric matrices, with A positive definite, B negative definite, $\operatorname{Tr} A = n$ and $\operatorname{Tr} A + \operatorname{Tr} B = 0$. Then $\| \sum_i [e_i, \overline{e}_i] \|^2 = \operatorname{Tr}(A^2) + \operatorname{Tr}(B^2)$ takes its minimal value at (aI_{kp}, bI_{kq}) with $pa + bq = 0$. This gives the maximality assumption on the scalar curvature.

For (2), the irreducibility of T_h^1 under the action of the isotropy group of the origin of the Griffiths domain D enables to conclude that S_D^P is constant on the space of Abelian subspaces. The main point to be verified for the statement on 1-hyperrigidity is thus $C_D^P \neq 0$. This follows from simple calculations. □

A Tables

The tables in this appendix were completed using the tables in Bourbaki [2], p. 251–276 and Helgason [9], p. 518.

A.1 Length of the fundamental representations

In Table 1, ω_i denotes the weight of the ith fundamental representation using the notation of [2].

The exceptional isomorphisms between classical domains are: $D_2^{\mathrm{II}} = \Delta$, $D_3^{\mathrm{II}} = B^3$, $D_4^{\mathrm{II}} = D_6^{\mathrm{IV}}$, $D_2^{\mathrm{III}} = D_3^{\mathrm{IV}}$, $D_{2,2}^{\mathrm{I}} = D_4^{\mathrm{IV}}$.

Ω	$\mathfrak{g}$ and $\mathfrak{l}$	$\dim_{\mathbb{C}}(\Omega)$	$\lvert W_0\rvert$	$l(E_{\omega_i})$
B^n	$\mathfrak{su}(n,1)$, $\mathfrak{s}(\mathfrak{u}(n)\times\mathfrak{u}(1))$	n	$n+1$	$l(E_{\omega_i})=1, 1\le i\le n$
$D^{\mathrm{I}}_{p,q}$, for $q\ge p\ge 2$	$\mathfrak{su}(p,q)$, $\mathfrak{s}(\mathfrak{u}(p)\times\mathfrak{u}(q))$	pq	$\binom{p+q}{p}$	$l(E_{\omega_i})=i,\ 1\le i\le p$ $l(E_{\omega_i})=p,\ p\le i\le q$ $l(E_{\omega_i})=p+q-i$ $q\le i\le p+q-1$
D^{II}_n, for $n\ge 4$	$\mathfrak{so}^*(2n),\mathfrak{u}(n)$	$\binom{n}{2}$	2^{n-1}	$l(E_{\omega_i})=i, 1\le i\le n-2$ $l(E_{\omega_{n-1}})=[(n-1)/2]$ $l(E_{\omega_n})=[n/2]$
D^{III}_n, for $n\ge 3$	$\mathfrak{sp}(n,\mathbb{R}),\mathfrak{u}(n)$	$\binom{n+1}{2}$	2^n	$l(E_{\omega_i})=i, 1\le i\le n$
D^{IV}_{2n}, for $n\ge 2$	$\mathfrak{so}(2n,2)$, $\mathfrak{so}(2n)\times\mathfrak{so}(2)$	$2n$	$2n+2$	$l(E_{\omega_i})=2, 1\le i\le n-1$ $l(E_{\omega_n})=1, l(E_{\omega_{n+1}})=1$
D^{IV}_{2n+1}, for $n\ge 2$	$\mathfrak{so}(2n+1,2)$, $\mathfrak{so}(2n+1)\times\mathfrak{so}(2)$	$2n$	$2n+1$	$l(E_{\omega_i})=2, 1\le i\le n$ $l(E_{\omega_{n+1}})=1$
D^{V}	$\mathfrak{e}_{6(-14)}$, $\mathfrak{so}(10)\times\mathfrak{so}(2)$	16	27	
D^{VI}	$\mathfrak{e}_{7(-25)},\mathfrak{e}_6\times\mathfrak{so}(2)$	27	56	

Table 1: Lengths of fundamental representations

In the case of type I domains, the weight ω_1 corresponds to the fundamental representation $\rho_1\colon \mathfrak{su}(p,q)\to\mathfrak{gl}(p+q,\mathbb{C})$. $E_k^{\mathfrak{su}(p,q)}=\Lambda^k\rho_1$.

In the case of type III domains, the weight ω_1 corresponds to the fundamental representation $\rho_1\colon \mathfrak{sp}(n,\mathbb{C})\to\mathfrak{gl}(2n,\mathbb{C})$. $E_k^{\mathfrak{sp}(n,\mathbb{R})}$ is the primitive part of $\Lambda^k\rho_1$ (and can be seen as the primitive part of $R^k\pi_*\mathbb{C}$ where $\pi\colon A_n^{\mathrm{III}}\to D_n^{\mathrm{III}}$ is the universal family parametrizing marked principally polarized n-dimensional Abelian varieties).

In the case of type II_n and IV_{2n-2} domains, E_{ω_1} is the fundamental representation $\rho_1\colon\mathfrak{so}(2n)\to\mathfrak{gl}(2n,\mathbb{C})$. $E_{\omega_i}=\Lambda^i\rho_i$ for $i\le n-2$, and $E_{\omega_{n-1}}$ and E_{ω_n} are the two halfspin representations.

In the case of type IV_{2n+1} domains, E_{ω_1} is the fundamental representation $\rho_1\colon\mathfrak{so}(2n+1,2)\to\mathfrak{gl}(2n+3,\mathbb{C})$. $E_{\omega_i}=\Lambda^i\rho_i$ for $i\le n$, and $E_{\omega_{n+1}}$ is the spin representation.

Lacking expertise on the two exceptional symmetric domains, I did not perform all the relevant computations. However, observing that length one homogenous VHS can be used to classify totally geodesic holomorphic embeddings on symmetric domains into classical ones (compare our table with the list of embeddings of a classical domain into another one given by [15], p. 188), the fact that D^{V} and D^{VI} have no such embedding ([15], p. 187) is equivalent to the fact that they do not carry any length one homogenous VHS.

A.2 Special weights for classical domains

For each classical domain, the combinatorial recipe for testing whether a dominant weight is special is given in Table 2:

Ω	$\lambda = \sum_i n_i\omega_i$ is special if and only if:
B^n	$\lambda = 0$
$D^{\mathrm{I}}_{p,q}$, for $q \geq p \geq 2$	$\forall k \in \{0, \dots, pq + \sum_{i=1}^{p} i(n_i + n_{p+q-i}) + p\sum_{p+1}^{p+q-1} n_i\}$, $\exists A \subset \{1, \dots, p+q\}$ with $\lvert A\rvert = p$ such that $\sum_{j\in A}(1+n_j)(\min(j,p) - \lvert A \cap \{1,\dots,j\}\rvert) = k$
D^{II}_n, for $n \geq 4$	$\forall k \in \{0, \dots, \binom{n}{2} + \sum_{i=1}^{n-2} il_i + [\frac{n-1}{2}]l_{n-1} + [\frac{n}{2}]l_n\}$, $\exists A \subset \{1, \dots, n\}$ with $\lvert A\rvert \equiv 0 \mod 2$ such that $\delta_{n\in A}\frac{l_n - l_{n-1}}{2} + \sum_{\substack{i\in A\\ i<n}}(n - i + \frac{l_{n-1}+l_n}{2} + \sum_{j=i}^{n-2} l_j) = k$
D^{III}_n, for $n \geq 3$	$\forall k \in \{0, \dots, \frac{n(n+1)}{2} + \sum_{i=1}^{n} in_i\}, \exists A \subset \{1, \dots, n\}$ such that $\sum_{i\in A}(n - i + 1 + \sum_{l=i}^{n} n_l) = k$
D^{IV}_{2n}, for $n \geq 2$	$\lambda \in \{0, \omega_n, \omega_{n+1}\}$, where E_{ω_n},$E_{\omega_{n+1}}$ are the half spin representations of $\mathfrak{so}(2n+2)$
D^{IV}_{2n+1}, for $n \geq 2$	$\lambda = 0$

Table 2: Special weights

Except in the particular cases of type IV domains and of the complex ball or of some other small domains, I cannot give the whole list of special weights, but can give specific examples. In the following list, the weight λ is supposed to be the sum $\lambda = \sum_i l_i\omega_i$, with $l_i \in \mathbb{N}$:

Case $D^{\mathrm{I}}_{p,q}$ for $q \geq p \geq 2$

- If $l_p \neq 0$ (or $l_q \neq 0$) then $\lambda = \sum l_i \omega_i$ is not a special weight.
- If $l_{p+1} \neq 0$ and $l_{p-1} \neq 0$ (or $l_{q\pm 1} \neq 0$) then $\lambda = \sum l_i \omega_i$ is not a special weight.
- $r\omega_{p+q-1}$ (respectively $r\omega_1$) is special if and only if $r < 1 + (p-1)(q-1)$.

Case D^{III}_n for $n \geq 2$

- If $l_n \neq 0$ or $l_{n-1} \neq 0$ then λ is not special.
- $p\omega_1 + q\omega_2$ is special if and only if $p \leq \frac{(n-1)(n-2)}{2}$ and $q \leq \frac{(n-2)(n-3)}{2}$.
- $p\omega_1 + q\omega_2 + r\omega_3$ is special if and only if $r \leq \frac{(n-4)(n-3)}{2}$, $q \leq \frac{(n-2)(n-3)}{2}$ and $|n - 3 + r - p| \leq 1 + \frac{(n-3)(n-2)}{2}$.

In particular, for D^{III}_5, $7\omega_1 + 3\omega_2$ is not special whereas $7\omega_1 + 3\omega_2 + \omega_3$ is special; this shows that being nonspecial is not a monotonous property, that is, λ nonspecial does not necessarily implies that $\lambda + \mu$ nonspecial.

Case D^{II}_n for $n \geq 4$

- If $l_n \neq 0$, λ is not special. When $n \equiv 1 \mod 2$, a special weight also has $l_{n-1} = 0$ and $l_n \leq 1$.
- The weight $\sum_{i=1}^{n-2} l_i \omega_i^{\mathrm{II}}$ is a special weight for D^{II}_n if and only if $\sum_{i=1}^{n-2} l_i \omega_i^{\mathrm{III}}$ is a special weight for D^{III}_{n-1}.

A.3 Examples of 1-hyperrigid period maps

The following Table 3 gives the examples of hyperrigid homogenous period maps $\Omega \to D$ constructed thanks to Lemma 4.3.1.

Ω	D	Nonspecial isotypic VHS
B^n	$D^{\mathrm{I}}_{kC_n^{r-1},kC_n^r}, 1 \leq r \leq n, k \in \mathbb{N}$	$E_{\omega_r}^{\oplus k}$
D^{IV}_{2n+1}	$D^{\mathrm{I}}_{k2^n,k2^n}, k \in \mathbb{N}$	$E_{\omega_n}^{\oplus k}$
$D^{\mathrm{II}}_n, n \geq 4$	$D^{\mathrm{I}}_{n,n}$	$\Lambda^{n-2}E^I_{\omega_1} = E^I_{\omega_{n-2}}$
$D^{\mathrm{III}}_n, n \geq 3$ $n \equiv 0, 1[4]$	$D^{\mathrm{I}}_{n,n}$	$S^{1+d}E^I_{\omega_1} = E^I_{(1+d)\omega_1}$, $d = \binom{n-1}{2}$
B^n	$D_{\mathbb{R}}(ka_i, kb_i, ka_i)$	$E^I_{\omega_i+\omega_{n-i}}, i \leq \frac{n}{2}$
D^{IV}_n	$D_{\mathbb{R}}(kC_n^{r-1}, k(C_n^r + C_n^{r-2}), kC_n^{r-1})$	$(E^{\mathrm{II}}_{\omega_i})^{\oplus k} = (\Lambda^r E^{\mathrm{II}}_{\omega_1})^{\oplus k}, r \leq n$

Table 3: Hypperrigid homogeneous period maps

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