Algebraic Gromov–Witten invariants

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Abstract

We present an introduction to the algebraic theory of Gromov–Witten invariants, as developed in collaboration with Yu. Manin and B. Fantechi in [4], [3] and [2]. We try to make these three articles more accessible. Proofs are generally omitted and there is little new material.

Contents

1 Introduction

Gromov–Witten invariants are the basic enumerative geometry invariants of a (nonsingular projective) algebraic variety W. Given a family $\Gamma_1, \ldots, \Gamma_n$ of algebraic cycles on W, one asks how many curves of fixed genus and degree (or homology class) pass through $\Gamma_1, \ldots, \Gamma_n$. The answer is given by the associated Gromov–Witten invariant. (If there is an infinite number of such curves, the Gromov–Witten invariant is a cycle in the moduli space of marked curves, rather than a number.) Isolating the properties satisfied by these invariants (formulated here as Axioms I–VIII) has had tremendous impact on enumerative geometry in recent years. Moreover, Gromov–Witten invariants tell us the *correct* way of counting curves. In simple cases (for example, $W = \mathbb{P}^n$) the Gromov–Witten invariant simply gives the actual number of curves through $\Gamma_1, \ldots, \Gamma_n$ if $\Gamma_1, \ldots, \Gamma_n$ are moved into general position. But such a naive interpretation of Gromov–Witten invariants is impossible in general, and so one should think of Gromov–Witten invariants as the *ideal* number of curves through $\Gamma_1, \ldots, \Gamma_n$.

Gromov–Witten invariants are defined as certain integrals over moduli spaces of maps from curves to W. Integrating over the usual fundamental class of the moduli space is problematic and can give the wrong result, because the moduli space might be of higher dimension than expected. This necessitates the construction of a so-called virtual fundamental class. This is the key step in the definition of Gromov–Witten invariants. Before the virtual fundamental classes were understood, Gromov–Witten invariants had only been constructed in special cases.

It turns out that the Gromov–Witten invariants of W (over $\mathbb{C}$) only depend on the underlying symplectic structure of W. (The only aspect one does not see from the symplectic point of view is the motivic nature of Gromov–Witten invariants.) The history of Gromov–Witten invariants in symplectic geometry is actually much older than in algebraic geometry. Classically, one perturbed the almost complex structure on W, instead of constructing a virtual fundamental class. For an exposition of this theory and its development, see the article by Siebert in this volume, which also explains the fact that the invariants constructed in symplectic geometry equal the algebraic ones.

The necessity of virtual fundamental classes for the definition of Gromov–Witten invariants in algebraic geometry was felt from the very beginning (see the seminal papers [10] and [11]). Before the general construction, several special cases had been studied in detail, usually in genus zero, or for W a homogeneous space. For more information on the results obtained and the history of this part of the subject, see the survey [7].

The theory of virtual fundamental classes explained in this article is due to B. Fantechi and the author (see [3]). Our work was inspired by a talk of J. Li at the Santa Cruz conference on algebraic geometry in the summer of 1995. In his talk, Li reported on work in progress with G. Tian on the subject of virtual fundamental classes. At the time, that approach relied on analytic methods, for example, the existence of the Kuranishi map. Our work [3] grew out of an attempt to understand Li and Tian's work, to construct virtual fundamental classes in an algebraic context, and, most of all, to give the construction as intrinsically as possible. But, of course, our construction owes its existence to theirs. See [12] for the approach of Li and Tian.

Full details of the theory explained here can be found in the series of papers [4], [3] and [2]. In this article, we stress the geometric meaning of Gromov–Witten invariants and skip most proofs.

Our approach uses graphs to keep track of the moduli spaces involved. The graph theory we use here is much simpler than that of [4], for two reasons. Firstly, we restrict to 'absolutely stable' graphs (in the terminology of [4]). We lose a lot of invariants this way, but we gain a high degree of simplification of the formalism. However, even this simplified formalism contains all invariants $I^V_{g,n}(\beta)$ envisioned in [11]. The other aspect we do not go into here is that graphs form a category. Using the full power of the categorical (or 'operad') approach, the number of axioms for Gromov–Witten invariants can be distilled down to two (from the eight we need here), but only at the cost of a lot of formalities.

Introducing graphs here has two purposes. Firstly, we believe that graphs (as presented here) actually *simplify* the theory of Gromov–Witten invariants, and make their properties more transparent. For example, the use of graphs splits the famous 'splitting axiom' into three much simpler axioms. We also hope that presenting a simplified graph theoretical approach here will make [4] and [2] more accessible.

Our approach also relies heavily on the use of stacks. Again, stacks are introduced to simplify the theory; still, a few remarks seem in order. There are two ways in which stacks appear here, and two different kinds of stacks that play a role.

First of all, the moduli stacks involved are *Deligne–Mumford* stacks, which are algebraic geometry analogues of orbifolds. Thus, if one works over $\mathbb{C}$ and uses the analytic topology, such stacks are locally given as the quotient of an analytic space by the action of a finite group (except that the stack 'remembers' these group actions in a certain sense). A good way to think of a Deligne–Mumford stack is as a space (of points) together with a finite group attached to each point. (So if the stack is the quotient of a space by a finite group, the points of the stack are the orbits, and the group attached to an orbit is the isotropy group of any element of the orbit.) If the stack is a moduli stack, its points correspond to isomorphism classes of the objects the stack classifies, and the group attached to such an isomorphism class is the automorphism group of any object in the isomorphism class. The space of isomorphism classes is called the *underlying coarse moduli space*.

Deligne–Mumford stacks behave in many respects just like schemes. For example, their cohomological and intersection theoretic properties are identical to those of schemes, at least if one uses rational coefficients. Care is needed a just one place, when integrating a cohomology class over a Deligne–Mumford stack (which is not a scheme). Then fractions may appear (even if one integrates integral cohomology classes). More generally, one has to use fractions when doing *proper pushforwards* of homology or Chow cycles, if the morphism one pushes forwards along is not representable (i.e., has fibers which are stacks, not schemes).

For example, if our Deligne–Mumford stack X has one point, with finite group G attached to it (notation $X = BG$; we can view it as the quotient of a point by the action of G), then the Euler characteristic of X (i.e., the integral of the top Chern class of the tangent bundle, in this case the integral of $1 \in H^*(X)$) is $\chi(X) = \int_X 1 = 1/\#G$.

To calculate such an integral $\int_X \omega$ over a Deligne–Mumford stack X, one has to find a proper scheme X' together with a generically finite morphism $f\colon X' \to X$, and then one has $\int_X \omega = (1/\deg f)\int_{X'} f^*\omega$. In the above example $X = BG$, we may take X' to the one-point variety and then $X' \to X$ has degree $\#G$ and so $\int_{BG} 1 = (1/\#G)\int_{\mathrm{pt}} 1 = 1/\#G$.

When explaining the general theory, it is not necessary to calculate a nonrepresentable proper pushforward explicitly, and so for this purpose we may as well pretend that all moduli stacks are spaces (i.e., schemes). We do this often, so that even if it says moduli space somewhere, it is implicitly understood that moduli stack is meant.

One reason that we must work with moduli stacks to do things properly, is that the corresponding coarse moduli spaces do not have universal families over them, whereas the construction of Gromov–Witten invariants uses universal families in an essential way.

The second way in which stacks appear is in the construction of virtual fundamental classes. Of course, one could construct the virtual fundamental class without using stacks, but we believe that stacks is the natural language for formulating the construction. The stacks used in this theory are so-called *cone stacks*, which are Artin stacks of a particular type. Artin stacks are more general than Deligne–Mumford stacks in that the groups attached to the points of the stack can be arbitrary algebraic groups, not just finite groups. These groups are too big to sweep under the carpet so easily, so that it is better not to pretend that Artin stacks are spaces, and we therefore include a 'heuristic' definition of cone stacks. (Cone stacks are special, since their groups are always vector groups.) The most important cone stack is the 'intrinsic normal cone'. It is an invariant of any Deligne–Mumford stack, and even for schemes it is an interesting object, which is nontrivial as a stack.

2 What are Gromov–Witten invariants?

Let k be a field[1] and W a smooth projective variety over k. We shall define the Gromov–Witten invariants of W, taking values in the cohomology of moduli spaces of curves.

2.1 Cohomology theories

So before we can begin, we have to choose a cohomology theory,

$$\begin{array}{rcl} H^*\colon (\text{smooth proper DM stacks}/k) & \longrightarrow & (\text{vector spaces over } \Lambda) \\ X & \longmapsto & H^*(X) \end{array}$$

This needs to be a 'graded generalized cohomology theory with coefficients in a field Λ of characteristic zero, with cycle map, such that $\mathbb{P}^1$ satisfies epu'. It should be defined on the category of smooth and proper Deligne–Mumford stacks over k. The precise definition can be found in [8].

Remark (for pedants) In [8], the cohomology theory is of course defined on the category of smooth and proper varieties, but the generalization of the definitions in [8] to Deligne–Mumford stacks is not difficult. The only point is that, strictly speaking, the category of (smooth, proper) Deligne–Mumford stacks is a 2-category, and so the cohomology theory is a functor from a 2- to a 1-category (i.e., a usual category). This means that it factors through the associated 1-category of the 2-category of Deligne–Mumford stacks, i.e., the category in which one passes to isomorphism classes of morphisms. In other words, one pretends that the category of Deligne–Mumford stacks is a usual category.

Rather than recalling the precise definition of a generalized cohomology theory with the required properties, we give a few examples.

1. If the ground field k is $\mathbb{C}$ and the coefficient field Λ is $\mathbb{Q}$, then let

$$H^*(X) = H^*_{\mathrm{B}}(X) = \text{Betti cohomology of } X.$$

This can be defined in several ways.

[1]Because the theory is somewhat limited in positive characteristic (see footnote 4) the most important case is char $k = 0$.

The easiest case is when X has a moduli space $\widetilde{X}$. Then we can simply set

$$H^*_{\mathrm{B}}(X) = H^*_{\mathrm{sing}}(\widetilde{X}(\mathbb{C}), \mathbb{Q}),$$

the usual (singular) cohomology of the underlying topological space with the analytic topology. All the X that we consider have moduli spaces.[2]

More generally, one can consider $[X(\mathbb{C})]$, the set of isomorphism classes of the groupoid $X(\mathbb{C})$, in other words, the set of isomorphism classes of the objects the stack classifies. It comes with a natural topology, because the quotient of any groupoid exists in the category of topological spaces. The space $[X(\mathbb{C})]$ is thus the quotient of the topological groupoid associated to any presentation of X (with the analytic topology). Then we have

$$H^*_{\mathrm{B}}(X) = H^*_{\mathrm{sing}}([X(\mathbb{C})], \mathbb{Q}).$$

The canonical definition is the following. To the algebraic $\mathbb{C}$-stack X we associate a topological stack X^{top} (a stack on the category of topological spaces with the usual Grothendieck topology). This has an associated site (or topos) of sheaves $X^{\mathrm{top}}_{\mathrm{et}}$. (By abuse of notation we denote the usual topology by the subscript ét.) The Betti cohomology of X is then the cohomology of this topos

$$H^*_{\mathrm{B}}(X) = H^*(X^{\mathrm{top}}_{\mathrm{et}}, \mathbb{Q}).$$

This can also be defined in terms of geometric realizations.

2. Let ℓ be a prime not equal to the characteristic of k and consider the coefficient field $\Lambda = \mathbb{Q}_\ell$. Then we may take the ℓ-adic cohomology of X:

$$H^*(X) = H^*_\ell(X) = H^*(\overline{X}_{\mathrm{et}}, \mathbb{Q}_\ell) = \varprojlim_n H^*(\overline{X}_{\mathrm{et}}, \mathbb{Z}/\ell^n).$$

Here $\overline{X} = X \times_k \overline{k}$ and $\overline{X}_{\mathrm{et}}$ denotes the étale site of $\overline{X}$.

3. In the case $\operatorname{char} k = 0$, we may take $\Lambda = k$ and consider algebraic de Rham cohomology

$$H^*(X) = H^*_{\mathrm{DR}}(X) = \mathbb{H}^*(X_{\mathrm{et}}, \Omega^\bullet_X).$$

[2]Note, however, that the existence of the moduli spaces is a nontrivial, *additional* fact, that we never need.

4. We may also take Chow cohomology

$$H^*(X) = A^{*/2}(X),$$

where the coefficient field is $\Lambda = \mathbb{Q}$. The Chow rings one needs for this definition were constructed by Vistoli [13].

2.2 Moduli stacks of curves

Gromov–Witten invariants take values in the cohomology of moduli stacks of curves. To list the axioms of Gromov–Witten invariants efficiently, we need slightly more general moduli spaces than the well-known $\overline{M}_{g,n}$. These are indexed by modular graphs.

Definition 2.1 A *graph* τ is a quadruple $(F_\tau, V_\tau, j_\tau, \partial_\tau)$ where F_τ is a finite set, the set of *flags*, V_τ is another finite set, the set of *vertices*, $\partial\colon F_\tau \to V_\tau$ is a map and $j_\tau\colon F_\tau \to F_\tau$ an involution. We use the notation:

$$S_\tau = \{f \in F_\tau \mid jf = f\}, \quad \text{the set of } \textit{tails} \text{ of } \tau.$$
$$E_\tau = \{\{f_1, f_2\} \subset F_\tau \mid f_2 = jf_1, f_1 \neq f_2\}, \quad \text{the set of } \textit{edges}.$$

For every vertex $v \in V$, the set $F_\tau(v) = \partial_\tau^{-1}(v)$ is the set of flags of v and $\# F_\tau(v)$ the *valency* of v.

We draw graphs by representing a vertex as a dot, and edge as a curve connecting vertices, and a tail as a half open curve, connected only to a vertex at its closed end. (The map ∂ specifies which vertex a flag is connected to.) Drawing graphs in this manner suggests an obvious notion of *geometric realization* of a graph. This is the topological space obtained in the way just indicated. The geometric realization of a graph τ is denoted by $|\tau|$.

Definition 2.2 A *modular graph* is a pair (τ, g), where τ is a graph and $g\colon V_\tau \to \mathbb{Z}_{\geq 0}$ a map. We use the terminology:

$g(v)$ is the *genus* of the vertex v.

$\chi(\tau) = \chi(|\tau|) - \sum_{v \in V_\tau} g(v)$ is the *Euler characteristic* of the graph τ.

If the geometric realization $|\tau|$ of τ is nonempty and connected then

$$g(\tau) = \sum_{v \in V_\tau} g(v) + \dim H^1(|\tau|, \mathbb{Q}) = 1 - \chi(\tau)$$

is the *genus* of τ. A graph of genus zero is a *tree*, and a possibly disconnected graph all of whose connected components are trees a *forests*. A nonempty connected graph without edges is called a *star*. Note that a star has exactly one vertex.

The Deligne–Mumford moduli stacks we are interested in are indexed by modular graphs. But we do not associate a moduli stack to every modular graph, only to the *stable* modular graphs.

Definition 2.3 The modular graph τ is *stable* if each vertex is stable, i.e., if

$$2g(v) + \# F_\tau(v) \geq 3 \quad \text{for all } v \in V_\tau.$$

We are now ready to define the moduli stacks of curves. First, for a nonnegative integer g and a finite set S such that $2g + \# S \geq 3$, we define the stack $\overline{M}_{g,S}$ to be the moduli stack of stable curves of genus g with marked points indexed by S.

Thus each point of $\overline{M}_{g,S}$ corresponds to a pair (C, x), where C is a nodal curve of arithmetic genus g (i.e., a connected but possibly reducible curve with at worst nodes as singularities) and x is an injective map $x\colon S \to C$, which avoids all nodes. The pair (C, x) is moreover required to be *stable*, meaning that for every irreducible component C' of C we have

$$2g(C') + \#\{\text{special points of } C'\} \geq 3.$$

Here $g(C')$ is the geometric genus of C', and a point is *special* if it is in the image of x or is a node of C. If both branches of a node belong to C', then this node counts as *two* special points.

If we choose an identification $S = \{1, \ldots, n\}$, then we get an induced identification

$$\overline{M}_{g,S} = \overline{M}_{g,n},$$

where the $\overline{M}_{g,n}$ are the moduli stacks of stable marked curves introduced by Mumford and Knudsen [9], and for $n = 0$ the stacks of stable curves defined by Deligne and Mumford [5].

Definition 2.4 The moduli stack associated to a stable modular graph τ is now simply defined to be

$$\overline{M}_\tau = \prod_{v \in V_\tau} \overline{M}_{g(v), F_\tau(v)}.$$

It may seem surprising that the involution j_τ does not enter here. The usefulness of this definition will become clear later.

Let (C, x) be a stable marked curve. We obtain its associated modular graph by associating

- to each irreducible component C' of C a vertex of genus $g(C')$ (geometric genus, that is, the genus of its normalization),

- to each node of C an edge connecting the vertices corresponding to the two branches of the node,
- to each marked point of C a tail attached to the vertex corresponding to the component containing the marked point.

If τ is the modular graph associated to (C,x), we say that (C,x) is of *degeneration type* τ. Note that τ is connected and $g(\tau) = g(C)$ (arithmetic genus).

If τ is a stable modular graph which is nonempty and connected, there exists a morphism

$$\overline{M}_\tau \longrightarrow \overline{M}_{g(\tau),S_\tau},$$

defined by associating to a V_τ-tuple of stable marked curves $(C_v,(x_i)_{i\in F_v})_{v\in V_\tau}$ the single curve $(C,(x_i)_{i\in S_\tau})$ obtained by identifying any two marks x_i that correspond to an edge of τ. This morphism is finite and its image is the stack of curves *of degeneration type τ or worse.* It is of generic degree $\#\operatorname{Aut}'(\tau)$ onto the image; here $\operatorname{Aut}'(\tau)$ is the group of automorphisms of τ fixing the tails.

If one fixes g and n and considers all connected stable modular graphs τ such that $g(\tau) = g$ and $S_\tau = \{1,\ldots,n\}$, then one gets in this way the stratification of $\overline{M}_{g,n}$ by degeneration type.

2.3 Systems of Gromov–Witten invariants

Fix a smooth projective variety W over k. We use the notation

$$H_2(W)^+ = \left\{\varphi \in \operatorname{Hom}(\operatorname{Pic} W,\mathbb{Z}) \;\middle|\; \begin{array}{l}\varphi(L)\geq 0 \text{ for all ample}\\ \text{invertible sheaves } L \text{ on } W\end{array}\right\}$$

Of course, if $k=\mathbb{C}$, then $H_2(W)^+$ contains the semigroup of effective cycle classes in $H_2(W,\mathbb{Z})$ (or, in general, the semigroup of effective cycle classes in $A_1(W)$) and we would not lose anything by restricting to this subsemigroup.

Definition 2.5 A *system of Gromov–Witten invariants* for W is a collection of (multi-)linear maps[3]

$$I_\tau(\beta)\colon H^*(W)^{\otimes S_\tau} \longrightarrow H^*(\overline{M}_\tau), \tag{1}$$

[3]To take Tate twists into account, one has to twist in a certain way, explained below, in the context of the grading axiom. So what we say here is only true up to Tate twists. Of course in the most important case, the Betti case, this is of no concern.

for every stable modular graph τ and every[4] $H_2(W)^+$ marking $\beta\colon V_\tau \to H_2(W)^+$ of τ, satisfying the list of eight axioms below.

Before listing the axioms, we say a few words about the geometric interpretation of Gromov–Witten invariants. For this, assume that we are over $\mathbb{C}$ and use singular cohomology. For purposes of intuition, it is better to dualize. So using Poincaré duality we identify H^* with H_* and get

$$\begin{aligned} I_\tau^\vee(\beta)\colon H_*(W)^{\otimes S_\tau} &\longrightarrow H_*(\overline{M}_\tau) \\ \gamma_1 \otimes \cdots \otimes \gamma_n &\longmapsto I_\tau^\vee(\beta)(\gamma_1, \ldots, \gamma_n). \end{aligned}$$

Note that, as the notation suggests, we are thinking of the $I_\tau^\vee(\beta)$ as multilinear maps (and we have chosen an identification $S_\tau = \{1, \ldots, n\}$).

To explain what $I_\tau^\vee(\beta)(\gamma_1, \ldots, \gamma_n)$ should be, choose cycles $\Gamma_1, \ldots, \Gamma_n \subset W$ in sufficiently general position representing the homology classes $\gamma_1, \ldots, \gamma_n$.

Consider all triples (C, x, f), where

- $C = (C_v)_{v \in V_\tau}$ is a family of connected curves.
- $x = (x_i)_{i \in F_\tau}$ is a family of 'marks', i.e., for each $i \in F_\tau$ the mark x_i is a point on the curve $C_{\partial(i)}$. We also demand that (C, x) be a family of stable marked curves.
- $f = (f_v)_{v \in V_\tau}$ is a family of maps $f_v\colon C_v \to W$ such that
  1. for each edge $\{i_1, i_2\}$ of τ we have $f_{\partial(x_1)}(x_{i_1}) = f_{\partial(i_2)}(x_{i_2})$,
  2. for all $v \in V_\tau$ we have $f_*[C_v] = \beta(v)$,
  3. for all $i \in S_\tau$ we have that $f_{\partial(i)}(x_i) \in \Gamma_i$.

Let T be the 'space' of all such triples up to isomorphism. (An isomorphism from a triple (C, x, f) to a triple (D, y, g) is a V_τ-tuple $\varphi = (\varphi_v)_{v \in V_\tau}$ of isomorphisms of curves $\varphi_v\colon C_v \to D_v$ such that $\varphi_{\partial(i)}(x_i) = y_i$ for all $i \in F_\tau$ and $g_v \circ \varphi_v = f_v$ for all $v \in V_\tau$.)

We have a morphism $\varphi\colon T \to \overline{M}_\tau$, which simply maps a triple (C, x, f) to the first two components (C, x). The 'naive' definition of $I_\tau(\beta)$ is then

$$I_\tau^\vee(\beta)(\gamma_1, \ldots, \gamma_n) = \varphi_*[T].$$

Remark For simplicity, assume that τ is connected. To a triple (C, x, f) we may associate, as above, a single marked curve $(\widetilde{C}, \widetilde{x})$ by identifying the

[4] If $\operatorname{char} k > 0$, then choose a very ample invertible sheaf L on W and consider only $\beta \in H_2(W)^+$ such that $\beta(L) < \operatorname{char} k$. This ensures that all maps considered are separable, which is needed for all the arguments (as stated here) to go through. One only does not get 'as many' Gromov–Witten invariants as in characteristic zero.

two marks corresponding to each edge of τ, obtaining a stable marked curve of degeneration type τ or worse. The V_τ-tuple of maps f induces a map $\widetilde{f}\colon \widetilde{C} \to W$.

Let $\widetilde{T}$ be the space of triples (D, y, g), where (D, y) is a stable marked curve of degeneration type τ and $g\colon D \to W$ a morphism such that $g(y_i) \in \Gamma_i$ for all $i = 1, \ldots, n$, and $g_*[D_v] = \beta(v)$ for all $v \in V_\tau$. (Here D_v is the component of D corresponding to v.) Then we have a rational map $T \to \widetilde{T}$ of degree $\#\operatorname{Aut}'(\tau)$. (It is not defined everywhere, as we do not allow worse degeneration types than τ in $\widetilde{T}$.)[5]

So a slightly more naive but less abstract definition of $I_\tau(\beta)$ would be

$$I_\tau^\vee(\beta)(\gamma_1, \ldots, \gamma_n) = \#\operatorname{Aut}'(\tau)\varphi_*[\widetilde{T}].$$

Note that in the most important case, where τ is a star, the factor $\#\operatorname{Aut}'(\tau)$ is equal to 1.

For example, assume that T is finite (usually the case of interest). Then

$$\# T = I_\tau^\vee(\beta)(\gamma_1, \ldots, \gamma_n) \in \mathbb{Q}$$

is the 'ideal' number of solutions to an enumerative geometry problem.[6]

More precisely, passing to $(\widetilde{C}, \widetilde{x}, \widetilde{f})$ as before and then to $\widetilde{f}(\widetilde{C})$, we get a curve in W passing through $\Gamma_1, \ldots, \Gamma_n$. If $\Gamma_1, \ldots, \Gamma_n$ are sufficiently generic, then one would hope that this process sets up a bijection between points of $\widetilde{T}$ and the curves of degeneration type τ (or worse) through $\Gamma_1, \ldots, \Gamma_n$. Thus (if the hope is justified) $\frac{1}{\#\operatorname{Aut}'(\tau)} I_\tau^\vee(\beta)(\gamma_1, \ldots, \gamma_n)$ is the number of such curves intersecting $\Gamma_1, \ldots, \Gamma_n$.

For example, let $W = \mathbb{P}^2$ be the projective plane. Then $H_2(W)^+ = \mathbb{Z}_{\geq 0}$, and one writes $d = \beta$. Assume $d \geq 2$ and let $n = 3d - 1$. Let τ be the star of genus zero with n tails: $S_\tau = F_\tau = \{1, \ldots, n\}$. So we have

$$I_\tau^\vee(\beta) = I_{0,n}^\vee(d)\colon H^*(\mathbb{P}^2)^{\otimes n} \longrightarrow H^*(\overline{M}_{0,n}).$$

If we consider the homology class of a point in $\mathbb{P}^2$, call it γ, and consider $I_{0,n}^\vee(d)(\gamma^{\times n})$, where $\gamma^{\times n}$ stands for the n-tuple $(\gamma, \ldots, \gamma)$, then the corresponding 'space' T is a discrete set of points (if the n points $\Gamma_1, \ldots, \Gamma_n$ representing γ are in sufficiently general position).

[5]Allowing more degenerate curves in $\widetilde{T}$ would not make sense, because D_v would no longer be well defined.

[6]The word 'ideal' is essential here. In many cases, the Gromov–Witten invariant differs from the actual curve count.

One sees easily, that T corresponds one-to-one to the rational curves of degree d through $\Gamma_1, \ldots, \Gamma_n$. Thus

$$\begin{aligned} I_{0,n}(d)(\gamma^{\times n}) &= I^{\vee}_{0,n}(d)(\gamma^{\times n}) \\ &= \# \left\{ \begin{matrix} \text{rational curves of degree } d \text{ through} \\ n \text{ points in general position} \end{matrix} \right\}. \end{aligned}$$

For example, the number of conics through 5 points is $I_{0,5}(2)(\gamma^{\times 5}) = 1$, and the number of rational cubics through 8 points is $I_{0,8}(3)(\gamma^{\times 8}) = 12$.

In view of this 'intuitive definition' the following eight axioms that we require of Gromov–Witten invariants are all very natural. Note, however, that there are two problems with this definition. First of all, T must be compactified in order for φ_* in homology to make sense. This can be dealt with using stable maps (see below). A more serious problem is that in general it is not possible to put the Γ_i into sufficiently general position to assure that T is smooth and of the 'correct' dimension. This necessitates the construction of a 'virtual fundamental class' in T, which is a homology class in the correct degree, whose image in $\overline{M}_\tau$ is taken to be the Gromov–Witten invariant.

The reasons for using this axiomatic approach are largely historical. Kontsevich and Manin [11] introduced these axioms before Gromov–Witten invariants were rigorously defined. Today there are several natural constructions of invariants satisfying the axioms. We present one of these later.

One should note that the axioms do not determine the invariants uniquely. For example, one can set all the I equal to zero (except for those forced to be nonzero by the axiom concerning mapping to a point). Certain re-scalings are also possible.

The axioms do comprise all properties used to construct quantum cohomology out of the Gromov–Witten invariants, and certainly imply all characteristic properties of Gromov–Witten invariants that do not involve change of the variety W.

2.4 Axioms for Gromov–Witten invariants

I. The grading axiom

This says that

$$I_\tau(\beta) \colon H^*(V)^{\otimes S_\tau}[2\chi(\tau)\dim W] \longrightarrow H^*(\overline{M}_\tau)[2\beta(\tau)(\omega_W)]$$

respects the natural grading on both vector spaces.[7] Here $[\,\cdot\,]$ denotes shifts of grading: if $H^* = \bigoplus H^k$ is a graded vector space then $H^*[m]$ is the graded

[7] If one is concerned about Tate twists, one needs to also twist by $(\chi(\tau)\dim W)$ on the left and $(\beta(\tau)(\omega_W))$ on the right.

vector space such that $(H^*[m])^k = H^{k+m}$. In other words, $I_\tau(\beta)$ raises degrees by $2(\beta(\tau)(\omega_W) - \chi(\tau) \dim W)$. We use the notation

$$\beta(\tau) = \sum_{v \in V_\tau} \beta(v)$$

and ω_W is the canonical line bundle on W.

The idea behind this axiom is that the moduli 'space' T of triples, alluded to in the section on geometric intuition, has an expected dimension. It is computed using deformation theory (assuming that there are no obstructions). Even if there are obstructions, one still requires that $I_\tau(\beta)$ changes the grading by this expected dimension (minus $\sum_i \deg \gamma_i$), which is then called 'virtual dimension'.

The reasoning behind this is that one wants Gromov–Witten invariants to be invariant under continuous (or better, algebraic) deformations of the whole situation, like all good enumerative geometry numbers are. So one supposes that one could deform the situation into sufficiently general position for the obstructions to vanish and the space T to actually attain the expected dimension.

Note however, that in general it is not possible to deform the variety W algebraically to make it sufficiently generic in this sense.

For the computation of the expected dimension see Section 4.7. See also Remark 3.3

II. Isomorphisms

Let $\varphi\colon \sigma \to \tau$ be an isomorphism of $H_2(W)^+$-marked stable modular graphs. Then we get induced isomorphisms $V^{S_\sigma} \to V^{S_\tau}$ and $\overline{M}_\sigma \to \overline{M}_\tau$ and the isomorphism axiom requires the diagram

$$\begin{array}{ccc} H^*(V)^{\otimes S_\tau} & \xrightarrow{I_\tau(\beta)} & H^*(\overline{M}_\tau) \\ \downarrow & & \downarrow \\ H^*(V)^{\otimes S_\sigma} & \xrightarrow{I_\sigma(\beta)} & H^*(\overline{M}_\sigma) \end{array}$$

to commute.

This axiom leads to a covariant behavior of the $I_{g,n}(\beta)$ with respect to the action of the symmetric group on n letters. It is motivated by the expectation that the ideal number of curves through the cycles $\Gamma_1, \ldots, \Gamma_n$ should not depend on the labelling of the cycles.

III. Contractions

Let $\varphi\colon \sigma \to \tau$ be a contraction of stable modular graphs. This means that there exists an edge $\{f, \overline{f}\}$ of σ which, on the level of geometric realizations, is literally contracted to a vertex by φ. It also implies a certain compatibility between the genera of the vertices involved. There are two cases to distinguish.

Case (a) The edge $\{f, \overline{f}\}$ is a loop with vertex v. Then if v' is the corresponding vertex of τ to which our edge is contracted, we have $g(v') = g(v) + 1$.

Case (b) The edge $\{f, \overline{f}\}$ has two different vertices v_1 and v_2. In this case we let v be the vertex of τ obtained by merging the vertices v_1 and v_2 via the contraction φ. The requirement is that $g(v) = g(v_1) + g(v_2)$.

All other vertices of τ have the genus of the corresponding vertex of σ.

In either case we get an induced morphism $\Phi\colon \overline{M}_\sigma \to \overline{M}_\tau$, defined as follows.

Case (a)

$$\begin{aligned} \Phi\colon \overline{M}_\sigma &\longrightarrow \overline{M}_\tau \\ (C, \dots) &\longmapsto (C/x_f = x_{\overline{f}}, \dots) \end{aligned}$$

Here C stands for the component of the V_σ-tuple of stable marked curves corresponding to the vertex v. This curve has two marked points, indexed by f and $\overline{f}$. The morphism Φ identifies them with each other, creating a node in the curve C and losing two marked points in the process. This curve obtained from C by creating an additional node we call C' and then C' is the component of the V_τ-tuple of stable marked curves corresponding to the index v'.

Case (b)

$$\begin{aligned} \Phi\colon \overline{M}_\sigma &\longrightarrow \overline{M}_\tau \\ (C_1, C_2, \dots) &\longmapsto (C_1 \amalg C_2/x_f = x_{\overline{f}}, \dots) \end{aligned}$$

Here C_1 and C_2 are the components of the V_σ-tuple of stable marked curves corresponding to the vertices v_1 and v_2, respectively. On C_1 there is a marked point indexed by f, and on C_2 there is a marked point indexed by $\overline{f}$, and $C_1 \amalg C_2/x_f = x_{\overline{f}}$ refers to the curve obtained by identifying these two points in the disjoint union of these two curves. In the process one loses two marked points, which is OK, because the graph also lost two flags.

In either case, the image of Φ is a 'boundary' divisor in $\overline{M}_\tau$. Usually, Φ is a closed immersion. Only if exchanging the two flags f and $\overline{f}$ can be extended to an automorphism of σ inducing the identity on τ (i.e., always in Case (a), almost never in Case (b)) is Φ a degree two cover followed by an immersion. In Case (a) the image of Φ can also intersect itself.

The axiom now demands that for each $H_2(W)^+$ marking β on τ, the diagram

$$\begin{array}{ccc} H^*(W)^{\otimes S_\tau} & \xrightarrow{I_\tau(\beta)} & H^*(\overline{M}_\tau) \\ \downarrow & & \downarrow \Phi^* \\ H^*(W)^{\otimes S_\sigma} & \xrightarrow{\sum_{\beta'} I_\sigma(\beta')} & H^*(\overline{M}_\sigma) \end{array}$$

commutes. Here the vertical map on the left is the canonical isomorphism coming from the fact that the contraction φ does not affect the tails of the graphs involved.

The sum in the lower horizontal map is taken over all maps

$$\beta' \colon V_\sigma \longrightarrow H_2(W)^+$$

that are compatible with β. This means

- in Case (a), that $\beta'(w) = \beta(w)$ for all $w \in V_\sigma$. In particular that $\beta'(v) = \beta(v')$,
- in Case (b), that $\beta'(w) = \beta(w)$ for all $w \neq v_1, v_2$, and $\beta'(v_1) + \beta'(v_2) = \beta(v)$.

Note that in Case (a) there is only one summand and in Case (b) there is a finite number of summands.

The meaning of this axiom is very simple. For example, in Case (b) it says that the number of curves in class β that have two components is the sum over all pairs (β_1, β_2) such that $\beta_1 + \beta_2 = \beta$ of the number of curves that have two components whose first component is of class β_1 and whose second component is of class β_2. (The invariant $I_\tau^\vee(\beta)$ might be a 1-cycle in $\overline{M}_\tau$ and Φ^* would intersect it with the boundary divisor $\overline{M}_\sigma$ and so count the number of curves in the family $I_\tau^\vee(\beta)$ that have two components, where the generic member has one.) In case Φ is generically two to one, Φ^* involves a multiplication by a factor of two, which reflects the ambiguity in marking the two points lying over the node.

IV. Glueing tails

Let τ be a stable modular graph and $\{f, \overline{f}\}$ an edge of τ. Let σ be the modular graph obtained from τ by 'cutting the edge' $\{f, \overline{f}\}$. This means that all the data describing σ is the same as the data describing τ, except for the involution j. In the case of τ, the set $\{f, \overline{f}\}$ is an orbit of j_τ and in the case of σ it is the union of two orbits of j_σ.

In this situation we have a 'morphism of stable modular graphs of type cutting edges' $\tau \to \sigma$ and an 'extended isogeny of type glueing tails' $\sigma \to \tau$. For the definitions of these terms see [4]. The direction of the arrow joining σ and τ depends on the context. Part I of [4] takes the 'morphism of stable modular graphs' approach to describe the morphisms between moduli spaces. In Part II, where Gromov–Witten invariants are given a graph theoretic treatment, the 'extended isogeny' viewpoint is needed.

Anyway, to state our axiom, the direction of the arrow joining σ and τ is not relevant. What is important to note is that σ has two tails more than τ, and therefore we have

$$H^*(W)^{\otimes S_\sigma} = H^*(W)^{\otimes S_\tau} \otimes H^*(W \times W).$$

The axiom now requires the diagram

$$\begin{array}{ccccc} H^*(W) \otimes H^*(W)^{\otimes S_\tau} & \overset{p^*}{\longleftarrow} & H^*(W)^{\otimes S_\tau} & \overset{I_\tau(\beta)}{\longrightarrow} & H^*(\overline{M}_\tau) \\ \Delta_* \downarrow & & & & \downarrow \cong \\ H^*(W \times W) \otimes H^*(W)^{\otimes S_\tau} & = & H^*(W)^{\otimes S_\sigma} & \overset{I_\sigma(\beta)}{\longrightarrow} & H^*(\overline{M}_\sigma) \end{array}$$

to commute. Here $p\colon W \times W^{S_\tau} \to W^{S_\tau}$ is the projection onto the second factor, and $\Delta\colon W \to W \times W$ is the diagonal. This diagram is required to commute for any $H_2(W)^+$ marking β one can put on $V_\tau = V_\sigma$.

Note that the image of $\Gamma_1 \times \cdots \times \Gamma_n$ under $\Delta_* \circ p^*$ is $\Delta \times \Gamma_1 \times \cdots \times \Gamma_n$, where Δ takes up the two first components in W^{S_τ}. So $I_\sigma(\beta) \circ \Delta_* \circ p^*$ should count the number of marked curves whose first two marks map to the same point in W. These are exactly the curves that $I_\tau(\beta)$ should count.

V. Products

Let τ and τ' be two stable modular graphs and σ the stable modular graph whose geometric realization is the disjoint union of the those of τ and τ'. We write $\sigma = \tau \times \tau'$ (and not $\tau \amalg \tau'$). For $H_2(W)^+$ markings β on τ and β' on τ' we denote by $\beta \times \beta'$ the induced $H_2(W)^+$ marking on σ. The product axiom requires that under such conditions the diagram

$$\begin{array}{ccc} H^*(W)^{\otimes S_\tau} \otimes H^*(W)^{S_{\tau'}} & \overset{I_\tau(\beta) \otimes I_{\tau'}(\beta')}{\longrightarrow} & H^*(\overline{M}_\tau) \otimes H^*(\overline{M}_{\tau'}) \\ \downarrow & & \downarrow \\ H^*(W)^{\otimes S_\sigma} & \overset{I_\sigma(\beta \times \beta')}{\longrightarrow} & H^*(\overline{M}_\sigma) \end{array}$$

always commutes. Here the vertical maps are the isomorphisms induced by the isomorphisms $W^{S_\sigma} = W^{S_\tau} \times W^{S_{\tau'}}$ and $\overline{M}_\sigma = \overline{M}_\tau \times \overline{M}_{\tau'}$.

This axiom expresses the expectation that the number of solutions to the enumerative geometry problem (τ, β) multiplied by the number of solutions of the problem (τ', β') is the number of *pairs* solving the 'composite enumerative geometry problem'.

VI. Fundamental class

Let σ be a stable modular graph and $f \in S_\sigma$ a tail of σ. Let τ be the modular graph obtained by simply omitting f. We assume that τ is still stable.

Remark Since there is a canonical way of associating a stable modular graph to any modular graph (called its 'stabilization'), one might wonder if there is also an axiom that applies in the case that τ is not stable. The answer is that such an axiom would follow from the others and is therefore not necessary. To see this, assume that the stabilization of τ is not empty. Then the process of removing the tail from σ and stabilizing the graph thus obtained can also be described (albeit not uniquely) as an edge contraction followed by a tail omission that does not lead to an unstable graph.[8]

In this situation we get a morphism

$$\Phi\colon \overline{M}_\sigma \longrightarrow \overline{M}_\tau,$$

defined in the following way: take the curve corresponding to the vertex of the tail f, which has a marked point on it indexed by f. Omit this point x_f and stabilize the marked curve thus obtained. To stabilize means to contract (blow down) the component on which x_f lies, if it becomes unstable by omitting x_f. (This can only happen in case this component is rational.) It is proved in [9] that $\overline{M}_\sigma \to \overline{M}_\tau$ is the universal curve corresponding to the vertex of f. More on stabilization in the next section.

Our axiom requires that the diagram

$$\begin{array}{ccc} H^*(W)^{\otimes S_\tau} & \xrightarrow{I_\tau(\beta)} & H^*(\overline{M}_\tau) \\ {\scriptstyle p^*}\downarrow & & \downarrow{\scriptstyle \Phi^*} \\ H^*(W)^{\otimes S_\sigma} & \xrightarrow{I_\sigma(\beta)} & H^*(\overline{M}_\sigma) \end{array}$$

commutes for every $H_2(W)^+$ marking β one can put on $V_\sigma = V_\tau$. Note that σ has exactly one tail more than τ and that therefore we can identify $W^{S_\sigma} = W \times W^{S_\tau}$ and p is the projection onto the second factor.

The geometric meaning of this axiom is that if one of the homology classes $\gamma_1, \ldots, \gamma_n$, say γ_1, is $[W]$, then the space T_1 obtained for $\gamma_1, \ldots, \gamma_n$ is a curve over the corresponding space T for $\gamma_2, \ldots, \gamma_n$. This is because for x_1 to be in W is no condition, so it can move anywhere on C leading to $T_1 \to T$ being the universal curve.

[8] It is precisely for this reason that the notion of 'isogeny' of stable graphs is introduced in Part II of [4]. If one were to use only the morphisms defined in Part I, one would not be able to decompose a tail omission that necessitates stabilization in this way.

VII. Divisor

The setup is the same as in Axiom VI for the fundamental class. The divisor axiom says that for every line bundle $L \in \operatorname{Pic}(W)$ (and every β) the diagram

$$\begin{array}{ccc} H^*(W)^{\otimes S_\tau} & \xrightarrow{\beta(L) I_\tau(\beta)} & H^*(\overline{M}_\tau) \\ {\scriptstyle c_1(L)} \downarrow & & \uparrow {\scriptstyle \Phi_*} \\ H^*(W)^{\otimes S_\sigma} & \xrightarrow{I_\sigma(\beta)} & H^*(\overline{M}_\sigma) \end{array}$$

commutes. Here the vertical map on the left is

$$\begin{array}{rcl} H^*(W)^{\otimes S_\tau} & \longrightarrow & H^*(W) \otimes H^*(W)^{\otimes S_\tau} \\ \gamma & \longmapsto & c_1(L) \otimes \gamma. \end{array}$$

This axiom expresses the expectation that modifying an enumerative problem by adding a divisor D (such that $L = \mathcal{O}(D)$) to the list $\Gamma_1, \ldots, \Gamma_n$ multiplies the number of solutions by $\beta(L)$, because for a curve C of class β to intersect D is no condition, and in fact the additional marking on C can be any of the $\beta(L)$ points of intersection of C with D.

VIII. Mapping to point

This axiom deals with the case that $\beta = 0$. Let τ be a nonempty connected stable modular graph. Over the moduli space $\overline{M}_\tau$ there are universal curves, one for each vertex of τ, obtained by pulling back the universal curves from the factors of $\overline{M}_\tau$. If $v \in V_\tau$ is a vertex of τ then the associated universal curve C_v has sections (x_f), one for each flag $f \in F_\tau(v)$. Now define a new curve $\widetilde{C}$ over $\overline{M}_\tau$ by identifying x_f with $x_{\overline{f}}$ for each edge $\{f, \overline{f}\}$ of τ. We call $\widetilde{C}$ the *universal curve* over $\overline{M}_\tau$. It has connected fibers since the geometric realization of τ is connected. Denote the structure morphism by $\pi\colon \widetilde{C} \to \overline{M}_\tau$.

Consider the direct product of $\overline{M}_\tau$ and W, with projections labelled as in the diagram

$$\begin{array}{ccc} \overline{M}_\tau \times W & \xrightarrow{p} & W \\ {\scriptstyle q} \downarrow & & \\ \overline{M}_\tau & & \end{array}$$

We get an induced homomorphism

$$\begin{array}{rcl} \rho\colon H^*(W) & \longrightarrow & H^*(\overline{M}_\tau) \\ \gamma & \longmapsto & q_*(p^*(\gamma) \cup c_{\mathrm{top}}(R^1\pi_*\mathcal{O}_{\widetilde{C}} \boxtimes T_W)). \end{array}$$

Here T_W stands for the tangent bundle of W and c_{top} for the highest Chern class, which in this case will be of degree $g(\tau) \dim W$.

The mapping to point axiom now states that

$$I_\tau(0)\colon H^*(W)^{\otimes S_\tau} \longrightarrow H^*(\overline{M}_\tau)$$

is given by

$$I_\tau(0)(\gamma_1,\ldots,\gamma_n) = \rho(\gamma_1 \cup \cdots \cup \gamma_n).$$

This axiom expresses the fact that in this case

$$T = \overline{M}_\tau \times \Gamma_1 \cap \cdots \cap \Gamma_n,$$

since a constant map to W has to map to $\Gamma_1 \cap \cdots \cap \Gamma_n$. Note that for $g(\tau) \geq 1$ the factor $c_{\mathrm{top}}(R^1\pi_*\mathcal{O}_{\widetilde{C}} \boxtimes T_W)$ is put in to satisfy the grading axiom. It is a sort of excess intersection term coming from the fact that there are obstructions in this case. More on this later (see Section 4.6).

Remark Axioms I–VIII imply the axioms listed in [11], except for the motivic axiom. This follows from the construction we give below. Axioms III, IV and V (contractions, glueing tails and products) imply the splitting and genus reduction axioms.

3 Construction of Gromov–Witten invariants

3.1 Stable maps

Gromov–Witten invariants are constructed as integrals over moduli spaces, namely, moduli spaces of stable maps. The notion of stable map is a natural generalization, due to Kontsevich, of stable curve (Deligne and Mumford [5]) and stable marked curve (Knudsen and Mumford [9]). Let us recall the definition.

Definition 3.1 Fix a smooth projective k-variety W. A *stable map* (to W) over a k-scheme T, of genus $g \in \mathbb{Z}_{\geq 0}$, class $\beta \in H_2(W)^+$ and marked by a finite indexing set S is:

1. a proper flat curve $C \to T$ such that every geometric fiber C_t is connected, one dimensional, has only ordinary double points (i.e., nodes) as singularities and arithmetic genus $1 - \chi(\mathcal{O}_{C_t}) = g$;

2. a family $(x_i)_{i \in S}$ of sections $x_i\colon T \to C$ such that for every geometric point $t \in T$, the points $(x_i(t))_{i \in S}$ are distinct, and not equal to a node;

3. a morphism $f: C \to W$ such that for every geometric point $t \in T$, denoting the restriction of f to the fiber C_t by $f_t: C_t \to W$, we have $\beta(L) = \deg f_t^*(L)$ for all $L \in \operatorname{Pic} W$ (or written more suggestively, $f_{t*}[C_t] = \beta$);

such that, for every geometric point $t \in T$, the normalization C' of every irreducible component of C_t satisfies:

$$f_t(C') \text{ is a point} \implies 2g(C') + \#\{\text{special points of } C'\} \geq 3;$$

here a *special* point is one that that lies over a mark x_i or a node of C_t.

A *morphism* of stable maps $\varphi: (C, x, f) \to (C', x', f')$ over T is a T-isomorphism $\varphi: C \to C'$ such that $\varphi(x_i) = x_i'$ for all $i \in S$ and $f'(\varphi) = f$.

Let $\overline{M}_{g,S}(W, \beta)$ denote the k-stack of stable maps of type (g, S, β) to W. Just like an algebraic space, or a scheme, or a variety (all over k), a stack is defined by giving its set of T-valued points for every k-scheme T, except that the set of T-valued points is not a set, but a category, in fact a category in which all morphisms are isomorphisms, in other words a *groupoid*. So the moduli stack $\overline{M}_{g,S}(W, \beta)$ is given by

$$\overline{M}_{g,S}(W, \beta)(T) = \text{category of stable maps over } T \text{ of type } (g, S, \beta) \text{ to } W,$$

for every k-scheme T.

The concept of stable maps was invented to make the following theorem true.

Theorem 3.2 (Kontsevich) *The k-stack $\overline{M}_{g,S}(W, \beta)$ is a proper algebraic Deligne–Mumford stack.*[9]

The Deligne–Mumford property signifies that the 'points' of $\overline{M}_{g,S}(W, \beta)$ have finite automorphism groups. The properness says two things. First, that every one dimensional family in $\overline{M}_{g,S}(W, \beta)$ has a 'limit', and secondly that this limit is unique. This translates into two facts about stable maps, namely first of all that every stable map over $T - \{t\}$, where T is one dimensional, extends to a stable map over T. For this to be true one has to allow certain degenerate maps, namely those with singular curves. The amazing fact is that by including exactly the degenerate maps which are *stable*, one picks out exactly one extension to T from all the possible extensions of the stable map over $T - \{t\}$. This makes the 'limit' unique, and hence the stack $\overline{M}_{g,S}(W, \beta)$ proper.

For a proof of this theorem we refer to [7].

[9] At least if $\operatorname{char} k = 0$ or if $\beta(L) < \operatorname{char}(k)$ for some ample invertible sheaf L on W.

Note For each $i \in S$ there is an evaluation morphism

$$\begin{aligned} \mathrm{ev}_i \colon \overline{M}_{g,S}(W,\beta) &\longrightarrow W \\ (C,x,f) &\longmapsto f(x_i). \end{aligned}$$

These moduli stacks are now taken as building blocks to define moduli stacks of stable maps associated to graphs.

So let (τ,β) be a stable modular graph with an $H_2(W)^+$ marking. The associated moduli stack we construct is denoted by $\overline{M}(W,\tau)$, abusing notation by leaving out β. We list three conditions on these moduli stacks that determine them completely.

1. Stars If τ is a star, i.e., a graph with only one vertex v, and set of flags S, which are all tails, then

$$\overline{M}(W,\tau) := \overline{M}_{g(v),S}(W,\beta(v)).$$

2. Products If τ and σ are stable modular graphs with $H_2(W)^+$ markings, and $\sigma \times \tau$ denotes the obvious stable modular graph with $H_2(W)^+$ marking whose geometric realization is the disjoint union of the geometric realizations of σ and τ, then

$$\overline{M}(W,\tau\times\sigma) := \overline{M}(W,\tau) \times \overline{M}(W,\sigma).$$

3. Edges If τ has two tails i_1 and i_2, and σ is obtained from τ by glueing these two tails to an edge (so that conversely, τ is obtained from σ by cutting an edge), then $\overline{M}(W,\sigma)$ is defined to be the fibered product

$$\begin{array}{ccc} \overline{M}(W,\sigma) & \longrightarrow & W \\ \downarrow & & \downarrow \Delta \\ \overline{M}(W,\tau) & \stackrel{\mathrm{ev}_{i_1}\times \mathrm{ev}_{i_2}}{\longrightarrow} & W \times W \end{array}$$

It is not difficult to see that $\overline{M}(W,\tau)$ is well defined by these conditions for every stable modular graph τ with $H_2(W)^+$ marking. Moreover, all $\overline{M}(W,\tau)$ are proper Deligne–Mumford stacks.

Note There exists an evaluation morphism $\mathrm{ev}_i \colon \overline{M}(W,\tau) \to W$ for each tail $i \in S_\tau$, and the product of these is the *evaluation morphism* $\mathrm{ev} \colon \overline{M}(W,\tau) \to W^{S_\tau}$.

For future reference, we now construct the *universal curve* on $\overline{M}(W,\tau)$. Fix a vertex $v \in V_\tau$. By construction, there exists a projection morphism

$$\overline{M}(W,\tau) \longrightarrow \overline{M}_{g(v),F_\tau(v)}(W,\beta(v)),$$

and we can pull back the universal stable map. This gives us a curve C_v over $\overline{M}(W,\tau)$ together with a morphism $f_v\colon C_v \to W$, and sections $x_i\colon \overline{M}(W,\tau) \to C_v$ for all $i \in F_\tau(v)$.

We glue the C_v according to the edges of τ (i.e., identify x_i with $x_{j_\tau(i)}$) to get a curve $C \to \overline{M}(W,\tau)$ called the *universal curve*, even though its fibers are only connected if $|\tau|$ is. There are induced sections $(x_i)_{i\in S_\tau}$ of C and an induced morphism $f\colon C \to W$.

Stabilization

Let $\tau = (\tau,\beta)$ be a stable modular graph with $H_2(W)^+$ structure. There exists a morphism

$$\overline{M}(W,\tau) \longrightarrow \overline{M}_\tau$$

given by 'stabilization'. To define it, it suffices to consider the case that τ is a star. So we are claiming that there exists a morphism

$$\begin{array}{rcl} \overline{M}_{g,S}(W,\beta) & \longrightarrow & \overline{M}_{g,S} \\ (C,x,f) & \longmapsto & (C,x)^{\mathrm{stab}}. \end{array}$$

In other words, we take a stable map (C,x,f) and forget about the map f, retaining only the marked curve (C,x). The problem is that (C,x) might not be stable, so to get a point in $\overline{M}_{g,S}$ we need to associate to (C,x) a stable marked curve, in a natural way.

This is done as follows. Let $\pi\colon C \to T$ be a curve with a family of sections $x\colon T \to C$, $x = (x_i)_{i\in S}$. Then its stabilization is defined to be the curve

$$C' = \operatorname{Proj}_T\left(\bigoplus_{\nu\geq 0} \pi_*(L^{\otimes\nu})\right),$$

where

$$L = \omega_{C/T}\left(\sum_{i\in S} x_i\right).$$

Here $\omega_{C/T}$ is the relative dualizing sheaf twisted by the Cartier divisor given by the images of the sections x_i in C. Note that there is a natural map $C \to C'$ and so one gets induced sections in C'.

One proves that C' together with these induced sections is a stable marked curve, and one calls it $(C,x)^{\mathrm{stab}}$. For details of this construction, see [9].

The morphism $C \to C'$ just contracts (blows down) all the unstable rational components any fiber of $C \to T$ might have.

3.2 The construction: an overview

As usual, let $\tau = (\tau, \beta)$ be a stable modular graph with vertices marked by elements of $H_2(W)^+$.

Consider the diagram

$$\begin{array}{ccc} \overline{M}(W,\tau) & \xrightarrow{\text{ev}} & W^{S_\tau} \\ \downarrow{\scriptstyle \text{stab}} & & \\ \overline{M}_\tau & & \end{array} \tag{2}$$

In what follows, we construct a rational equivalence class

$$[\overline{M}(W,\tau)]^{\text{virt}} \in A_{\dim(W,\tau)}(\overline{M}(W,\tau))$$

called the *virtual fundamental class* of $\overline{M}(W,\tau)$. Here A stands for the Chow group (with rational coefficients) of a separated Deligne–Mumford stack, constructed by Vistoli [13]. This class has degree

$$\dim(W,\tau) = \chi(\tau)(\dim W - 3) - \beta(\tau)(\omega_W) + \# S_\tau - \# E_\tau,$$

which is the 'expected dimension' of the moduli stack $\overline{M}(W,\tau)$. If $\overline{M}(W,\tau)$ happens to be of dimension $\dim(W,\tau)$, then $[\overline{M}(W,\tau)]^{\text{virt}} = [\overline{M}(W,\tau)]$ is just the usual fundamental class.

Then the Gromov–Witten invariant

$$\begin{array}{rcl} I_\tau(\beta)\colon H^*(W)^{\otimes S_\tau} & \longrightarrow & H^*(\overline{M}_\tau) \\ \gamma & \longmapsto & I_\tau(\beta)(\gamma) \end{array}$$

is defined by

$$I_\tau(\beta)(\gamma) \cap [\overline{M}_\tau] = \text{stab}_*(\text{ev}^*(\gamma) \cap [\overline{M}(W,\tau)]^{\text{virt}}). \tag{3}$$

Note that this condition defines $I_\tau(\beta)(\gamma)$ uniquely, because of Poincaré duality on the smooth stack $\overline{M}_\tau$.

Alternatively, consider the morphism

$$\pi\colon \overline{M}(W,\tau) \longrightarrow \overline{M}_\tau \times W^{S_\tau}$$

(induced by Diagram 2) which is proper and so we may consider the push-forward

$$\pi_*[\overline{M}(W,\tau)]^{\text{virt}},$$

which is a correspondence $\overline{M}_\tau \rightsquigarrow W^{S_\tau}$, so we get $I_\tau(\beta)$ through pullback via this correspondence:

$$I_\tau(\beta)(\gamma) = p_{1*}(p_2^*(\gamma) \cup \text{cl}\, \pi_*[\overline{M}(W,\tau)]^{\text{virt}}).$$

Hence this construction implies the 'motivic axiom' of [11].

Now all the axioms required of $I_\tau(\beta)$ reduce to axioms for

$$J(W,\tau) := [\overline{M}(W,\tau)]^{\mathrm{virt}}.$$

Before we list these, some more remarks.

As in the above discussion of the geometric interpretation of Gromov–Witten invariants, consider the situation where $S = \{1,\dots,n\}$, and $\Gamma_1,\dots,\Gamma_n$ are dual cycles to the cohomology classes $\gamma_1,\dots,\gamma_n \in H^*(W)$. For ease of exposition, let us assume that the Γ_i are actually algebraic subvarieties of W. We can now give a more precise definition of the moduli space T mentioned above. It is defined to be the fibered product

$$\begin{array}{ccc} T & \longrightarrow & \Gamma_1 \times \cdots \times \Gamma_n \\ \downarrow & & \downarrow \\ \overline{M}(W,\tau) & \xrightarrow{\mathrm{ev}} & W \times \cdots \times W \end{array}$$

This will in fact assure that T is proper, and thus we have solved the problem of compactifying the earlier T.

Now, if the Γ_i are in general position, then T should be smooth of the expected dimension, which is

$$\dim(W,\tau) - \sum_{i=1}^{n} \operatorname{codim}_W \Gamma_i.$$

In fact, one could use this principle as a *definition* of general position, *defining* $\Gamma_1,\dots,\Gamma_n$ to be in general position if T is smooth[10] of this dimension. As mentioned above, the problem is that one cannot always find Γ_i in general position.

But let us assume that $\Gamma_1,\dots,\Gamma_n$ are in general position. Then

$$[T] = \mathrm{ev}^*[\Gamma_1 \times \cdots \times \Gamma_n]$$

and

$$I_\tau^\vee(\beta)([\Gamma_1],\dots,[\Gamma_n]) = \mathrm{stab}_*[T],$$

because the virtual fundamental class agrees with the usual one in this case.

So defining Gromov–Witten invariants by (3) leads to the situation anticipated by our geometric interpretation detailed above. In particular, for the case that $\overline{M}(W,\tau)$ is of dimension $\dim(W,\tau)$ the above heuristic arguments explaining the motivations of the various axioms give proofs of the axioms.

[10] In fact, purely of the expected dimension would be enough, if one is willing to count components of T with multiplicities given by the scheme (or stack) structure. The difference is the same as that between transversal and proper intersection in intersection theory.

Remark 3.3 If one prefers cohomology, in the case that $\overline{M}(W,\tau)$ of smooth of the expected dimension $\dim(W,\tau)$, one can also think of $I_\tau(\beta)(\gamma_1,\ldots,\gamma_n)$ as obtained by pulling back $\gamma_1,\ldots,\gamma_n$ by the evaluation maps, taking the cup product of these pullbacks and then integrating over the fibers of the morphism stab: $\overline{M}(W,\tau) \to \overline{M}_\tau$. Thus $I_\tau(\beta)$ should lower the grading by twice the dimension of the fibers of this map, which is

$$\dim(W,\tau) - \dim \overline{M}_\tau = \chi(\tau)\dim W - \beta(\tau)(\omega_W).$$

This gives another interpretation of the grading axiom.

3.3 Axioms for $J(W,\tau)$

We now list the properties that the

$$J(W,\tau) = [\overline{M}(W,\tau)]^{\mathrm{virt}}$$

must satisfy in order that the induced Gromov–Witten invariants $I_\tau(\beta)$ should satisfy their respective properties. This amounts to five axioms for $J(W,\tau)$, and we refer to them by the names given in [4].

I. Mapping to point

Assume that $|\tau|$ is nonempty connected and that $\beta(\tau) = 0$, so that in fact $\beta(v) = 0$ for all $v \in V_\tau$.

In this situation the universal map $f\colon C \to W$ factors through the structure map $\pi\colon C \to \overline{M}(W,\tau)$ of the universal curve, since a map of class zero maps to a single point in W. We call the resulting map $\mathrm{ev}\colon \overline{M}(W,\tau) \to W$, since it is also equal to all the evaluation maps. Now the morphism

$$\overline{M}(W,\tau) \stackrel{\mathrm{stab}\times\mathrm{ev}}{\longrightarrow} \overline{M}_\tau \times W$$

is an isomorphism, since giving a stable map to a point in W is the same as giving a stable curve and a point in W.

The axiom is that

$$J(W,\tau) = c_{g(\tau)\dim W}(R^1\pi_*\mathcal{O}_{\widetilde{C}} \boxtimes T_W) \cap [\overline{M}(W,\tau)].$$

For future reference, let us give an alternative description of $R^1\pi_*\mathcal{O}_{\widetilde{C}} \boxtimes T_W$. Consider the vector bundle

$$\begin{aligned} R^1\pi_* f^* T_W &= R^1\pi_*(\pi^*\,\mathrm{ev}^*\,T_W) \\ &= R^1\pi_*\mathcal{O}_C \otimes \mathrm{ev}^*\,T_W. \end{aligned}$$

Note that we have a Cartesian diagram

$$\begin{array}{ccc} C & \xrightarrow{\pi} & \overline{M}(W,\tau) \\ \downarrow & & \downarrow \text{stab} \\ \widetilde{C} & \xrightarrow{\pi} & \overline{M}_\tau \end{array}$$

since in the case of mapping to a point, there is no need to stabilize and thus the pullback of the universal curve over the moduli space of curves is the universal curve over the moduli space of stable maps to a point. Thus we can write the above tensor product as an exterior tensor product:

$$R^1\pi_*f^*T_W = R^1\pi_*\mathcal{O}_{\widetilde{C}} \boxtimes T_W.$$

Note that $\operatorname{rank} R^1\pi_*f^*T_W = g(C)\dim W = g(\tau)\dim W$.

II. Products

Let σ and τ be stable modular graphs with $H_2(W)^+$ marking. Recall that we have

$$\overline{M}(W,\tau\times\sigma) = \overline{M}(W,\tau)\times\overline{M}(W,\sigma).$$

Our axiom is that

$$J(W,\tau\times\sigma) = J(W,\tau)\times J(W,\sigma).$$

III. Glueing tails

Let τ be obtained from σ by cutting an edge. Recall that then we have a Cartesian diagram

$$\begin{array}{ccc} \overline{M}(W,\sigma) & \longrightarrow & W \\ \downarrow & & \downarrow \Delta \\ \overline{M}(W,\tau) & \longrightarrow & W\times W \end{array}$$

Since W is smooth, Δ is a regular immersion, and so the Gysin pullback $\Delta^!\colon A_*(\overline{M}(W,\tau)) \to A_*(\overline{M}(W,\sigma))$ exists. (See [6], Section 6.2 for Gysin pullbacks in the context of schemes, [13] in the context of stacks.) The axiom is that

$$\Delta^! J(W,\tau) = J(W,\sigma).$$

IV. Forgetting tails

Let σ and τ be as in the fundamental class axiom for Gromov–Witten invariants. Endow both σ and τ with the same $H_2(W)^+$ marking β. Then we get an induced morphism of moduli of stable maps

$$\Phi\colon \overline{M}(W,\sigma) \longrightarrow \overline{M}(W,\tau).$$

To construct it, it suffices to consider the case of stars. So let σ be a star with set of tails $F_\sigma = S = \{0,\ldots\}$ and let τ have set of tails $F_\tau = S' = \{\ldots\}$. Then Φ is defined by

$$\begin{array}{rcl} \Phi\colon \overline{M}_{g,S}(W,\beta) & \longrightarrow & \overline{M}_{g,S'}(W,\beta) \\ (C,x_0,(x_i),f) & \longmapsto & (C,(x_i),f)^{\mathrm{stab}}. \end{array}$$

The construction of the stabilization is as before. One chooses a very ample invertible sheaf M on W. Then stabilization replaces the curve $\pi\colon C \to T$ by

$$C' = \operatorname{Proj} \bigoplus_{k\geq 0} \pi_*(L^{\otimes k}),$$

where $L = \omega_{C/T}(\sum x_i) \otimes f^* M^{\otimes 3}$. As before, this amounts to contracting or blowing down any rational components that become unstable on leaving out the section x_0.

Now a slightly nontrivial fact is that

$$\Phi\colon \overline{M}_{g,S}(W,\beta) \longrightarrow \overline{M}_{g,S'}(W,\beta)$$

is isomorphic to the universal curve over $\overline{M}_{g,S'}(W,\beta)$ (see [4], Corollary 4.6).

In the case of general graphs σ and τ this translates into the fact that the morphism x_0 in the diagram

$$\begin{tikzcd} \overline{M}(W,\sigma) \arrow[r, "x_0"] \arrow[dr, "\Phi"'] & C_v \arrow[d] \\ & \overline{M}(W,\tau) \end{tikzcd}$$

is an isomorphism. Here C_v is the universal curve over $\overline{M}(W,\tau)$ corresponding to the vertex $v = \partial(0)$.

In particular, the morphism Φ is flat of constant fiber dimension 1. Therefore the flat pullback homomorphism

$$\Phi^!\colon A_*(\overline{M}(W,\tau)) \longrightarrow A_*(\overline{M}(W,\sigma))$$

exists. Our axiom is that

$$\Phi^! J(W,\tau) = J(W,\sigma).$$

V. Isogenies

This axiom is really four axioms in one. The name of the axiom comes from the fact that it deals with those operations on a graph that do not affect its genus.

So let σ be a stable modular graph and let τ be obtained from σ by contracting an edge or omitting a tail. Assume that τ is also stable. Then choose an $H_2(W)^+$ structure on τ. In each case, we construct a commutative diagram

$$\begin{array}{ccc} \coprod \overline{M}(W,\sigma) & \longrightarrow & \overline{M}(W,\tau) \\ \downarrow & & \downarrow \text{stab} \\ \overline{M}_\sigma & \xrightarrow{\Phi} & \overline{M}_\tau \end{array} \tag{4}$$

where the disjoint sum is taken over certain $H_2(W)^+$ structures on σ.

Case I This is the case where we contract a loop in σ to obtain τ. Here there is only one possible $H_2(W)^+$ structure on σ compatible with that on τ. So $\coprod \overline{M}(W,\sigma) = \overline{M}(W,\sigma)$ and the two horizontal maps in (4) are obtained by glueing two marked points (or sections), as described above. The two vertical maps are given by forgetting the map part of a triple and then stabilizing.

Case II Here we contract a nonlooping edge of σ, i.e., an edge with two vertices. Let v be the edge of τ onto which this edge is being contracted and v_1, v_2 the two vertices of this edge in σ. For an ordered pair $\beta_1, \beta_2 \in H_2(W)^+$ such that $\beta_1 + \beta_2 = \beta(v)$, define a marking on σ by setting $\beta(v_1) = \beta_1$, $\beta(v_2) = \beta_2$ and for the other vertices of σ take the marking induced from τ. Then take the disjoint union over all such pairs (β_1, β_2) of the associated stack of stable maps $\overline{M}(W,\sigma)$. This is $\coprod \overline{M}(W,\sigma)$. The maps in (4) are now defined the same way as in Case I.

Case III This is the case where τ is obtained from σ by forgetting a tail. The $H_2(W)^+$ structure on σ is induced in a unique way from τ, $\coprod \overline{M}(W,\sigma) = \overline{M}(W,\sigma)$ and all the maps in (4) have been explained already.

Case IV In this case τ is obtained from σ by 'relabelling'. In other words there is given an isomorphism between σ and τ. The $H_2(W)^+$ structure on σ is induced via this isomorphism from τ and $\coprod \overline{M}(W,\sigma) = \overline{M}(W,\sigma)$. Moreover, the horizontal maps in (4) are isomorphisms.

Now in each case the commutative diagram (4) induces a morphism

$$h\colon \coprod \overline{M}(W,\sigma) \longrightarrow \overline{M}_\sigma \times_{\overline{M}_\tau} \overline{M}(W,\tau)$$

and our axiom states that

$$h_*(\sum J(W,\sigma)) = \Phi^! J(W,\tau).$$

Note that Φ is a local complete intersection morphism, since both $\overline{M}_\sigma$ and $\overline{M}_\tau$ are smooth. Therefore the Gysin pullback

$$\Phi^!\colon A_*(\overline{M}(W,\tau)) \longrightarrow A_*(\overline{M}_\sigma \times_{\overline{M}_\tau} \overline{M}(W,\tau))$$

exists. Since all the stacks involved are complete, the morphism h is proper and so the proper pushforward exists.

Proposition 3.4 *The five axioms for $J(W,\tau)$ imply the eight axioms for $I_\tau(\beta)$.*

Proof The grading axiom follows from the fact that the $J(W,\tau)$ have the correct degree. The product, glueing tails and mapping to point axiom for I follow from the axioms for J with the same name. The forgetting tails axiom for J implies the divisor axiom for I. Finally, the isomorphisms, contractions and fundamental class axioms for I all follow from the isogenies axiom (with the same proof). $\square$

Remark The part of the isogenies axiom dealing with omitting tails (Case III) follows from the forgetting tails axiom (as can be seen, for example, by examining the proof of the isogenies axiom in [2]). So technically, the axioms for the virtual fundamental classes $J(W,\tau)$ contain some redundancy.

The reason why the isogenies axiom is stated in this slightly redundant form is that in this formulation it characterizes an aspect of the operad nature of $J(W,\cdot)$. (By this, we mean its description as a natural transformation between functors from a category of graphs to a category of vector spaces.) The forgetting tails axiom does not feature in the operad picture, but it is still needed for the divisor axiom (which does not fit naturally into the operad framework).

3.4 The unobstructed case

In the unobstructed case there is no need for a virtual fundamental class. The usual fundamental class of the moduli stack will do the job.

Definition 3.5 We say that a stable map $f\colon (C,x) \to W$ is *trivially unobstructed*, if $H^1(C, f^*T_W) = 0$.

Definition 3.6 A smooth and projective variety W is *convex* if for every morphism $f\colon \mathbb{P}^1 \to W$ we have $H^1(\mathbb{P}^1, f^*T_W) = 0$.

Examples of convex varieties are projective spaces $\mathbb{P}^n$, generalized flag varieties G/P (were G is a reductive algebraic group and P a parabolic subgroup), and in fact any variety for which the tangent bundle is generated by global sections.

The following proposition is not difficult to prove.

Proposition 3.7 *If W is convex, then all stable maps of genus 0 to W are trivially unobstructed.*

Because of this, the 'tree level' system of Gromov–Witten invariants for convex varieties may be constructed without recourse to virtual fundamental classes. By the tree level system we mean all the invariants $I_\tau(\beta)$, where the graph τ is a forest.

Theorem 3.8 *Let W be convex. Then for every forest τ the stack $\overline{M}(W,\tau)$ is smooth of dimension*

$$\dim(W,\tau) = \chi(\tau)(\dim W - 3) - \beta(\tau)(\omega_W) + \# S_\tau - \# E_\tau.$$

Moreover, the system of fundamental classes (where τ runs over all stable forests with $H_2(W)^+$ marking)

$$J(W,\tau) = [\overline{M}(W,\tau)]$$

satisfies the above five axioms.[11]

Proof Details of the proof can be found in [4]. Essentially, what is going on is that the definition of trivially unobstructed is of course precisely the condition needed to assure that the obstructions vanish,[12] which implies that the moduli stack is smooth. (More about obstruction theory in a later section.) The first four axioms for J follow from basic properties of Chern classes and Gysin pullbacks. For the last axiom one has to note also that h is an isomorphism generically. $\square$

It is explained in [11] and [7] how to construct the quantum cohomology algebra of W from the tree level system of Gromov–Witten invariants.

If one wants to count rational curves through a number of points in general position on a convex variety, then the cycles $\Gamma_1, \ldots, \Gamma_n$ are all just points, and

[11] Of course only those instances of the axioms for which all graphs involved are forests.

[12] The obstructions may also vanish if $H^1(C, f^*T_W) \neq 0$, but $H^1(C, f^*T_W) = 0$ is the only 'general' condition that always assures vanishing of the obstructions.

it follows from generic smoothness, that (at least in characteristic zero) the points can be put into general position. Therefore these special Gromov–Witten invariants actually solve numerical geometry problems (i.e., they are *enumerative*).

For the case of generalized flag varieties G/P all cohomology is algebraic and so all Gromov–Witten invariants can be defined in terms of algebraic cycles $\Gamma_1, \ldots, \Gamma_n$. Using results of Kleiman one can then prove that it is possible to move $\Gamma_1, \ldots, \Gamma_n$ into general position. Therefore these tree level Gromov–Witten invariants are enumerative. For more details see [7].

Let us now give a few examples of stable maps that are not trivially unobstructed.

1. Consider stable maps to $W = \mathbb{P}^r$. If $f\colon (C, x) \to \mathbb{P}^r$ is such a map, then we may pull back the exact sequence

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{O}(1)^{r+1} \longrightarrow T_{\mathbb{P}^r} \longrightarrow 0$$

to C to get the surjection

$$f^*\mathcal{O}(1)^{r+1} \longrightarrow f^*T_{\mathbb{P}^r} \longrightarrow 0$$

and the surjection

$$H^1(C, f^*\mathcal{O}(1))^{r+1} \longrightarrow H^1(C, f^*T_{\mathbb{P}^r}) \longrightarrow 0.$$

So if C is irreducible and $\deg f = \deg f^*\mathcal{O}(1) > 2g(C) - 2$, then f is trivially unobstructed.

Thus the 'good' elements of $\overline{M}_{g,n}(\mathbb{P}^r, d)$ (i.e., those corresponding to irreducible C) are trivially unobstructed for sufficiently high degree $d > 2g - 2$. If we knew that $\overline{M}_{g,n}(\mathbb{P}^r, d)$ is irreducible, then its generic element would be trivially unobstructed, and the virtual fundamental class would be equal to the usual one. But whether $\overline{M}_{g,n}(\mathbb{P}^r, d)$ is irreducible is far from clear. Anyway, the Gromov–Witten axioms involve the boundary of $\overline{M}_{g,n}(\mathbb{P}^r, d)$ in an essential way, and so this unobstructedness result is not of much help.

2. For $g > 0$ already the constant maps are not trivially unobstructed. As we already saw in the two mapping to point axioms, the moduli stack $\overline{M}_{g,n}(W, 0)$ has higher dimension than expected. On the other hand, $\overline{M}_{g,n}(W, 0) = \overline{M}_{g,n} \times W$ is smooth, so there are no obstructions. Constant maps are unobstructed, but not trivially so.

The fact that $\overline{M}_{g,n}(W, 0)$ has higher dimension than expected, leads to boundary components of $\overline{M}_{g,n}(W, \beta)$ with $\beta \neq 0$ having higher dimension than expected. For example, consider $W = \mathbb{P}^r$ and the graph τ with two vertices v_0 and v_1, one edge connecting v_0 and v_1, and $g(v_0) = 0$, $g(v_1) = g$. Let $d(v_0) = d \neq 0$ and $d(v_1) = 0$. Then

$$\overline{M}(\mathbb{P}^r, \tau) = \overline{M}_{0,1}(\mathbb{P}^r, d) \times \overline{M}_{g,1}$$

and so

$$\dim \overline{M}(\mathbb{P}^r, \tau) = r + d(r+1) + 3g - 4$$

whereas the expected dimension is $r + d(r+1) + (3-r)g - 4$. The stack $\overline{M}(\mathbb{P}^r, \tau)$ is a boundary component in $\overline{M}_{g,0}(\mathbb{P}^r, d)$, whose 'good' component attains the expected dimension $r+d(r+1)+(3-r)g-3$ in the range $d > 2g-2$. So if $d > 2g-2$ and $rg > 1$ this boundary component has larger dimension than the 'good' component.

3. Let W be a surface and $E \subset W$ a rational curve with negative self-intersection $E^2 = -n$. Let $f \colon \mathbb{P}^1 \to E \subset W$ be a morphism of degree $d \neq 0$. Pulling back the sequence

$$0 \longrightarrow T_E \longrightarrow T_W \longrightarrow N_{E/W} \longrightarrow 0$$

via f, we get the sequence

$$0 \longrightarrow f^*T_E \longrightarrow f^*T_W \longrightarrow f^*N_{E/W} \longrightarrow 0. \tag{5}$$

Now $\deg(N_{E/W}) = E^2 = -n$ and so $f^*N_{E/W} = \mathcal{O}(-dn)$. Moreover, $T_{\mathbb{P}^1} = \mathcal{O}(2)$ and so $f^*T_E = \mathcal{O}(2d)$. Therefore, we get from the long exact cohomology sequence associated to (5) that

$$\begin{aligned} \dim H^1(\mathbb{P}^1, f^*T_W) &= \dim H^1(\mathbb{P}^1, \mathcal{O}(-dn)) \\ &= \dim H^0(\mathbb{P}^1, \mathcal{O}(dn-2)) \\ &= dn - 1. \end{aligned}$$

So if $d > 1$ or $n > 1$, then f is not trivially unobstructed. Since the 'boundary' of the moduli space usually contains maps of degree larger than 1, we must deal with maps that are not trivially unobstructed as soon as the surface W has -1 curves.

4 Virtual fundamental classes

4.1 Construction of $J(W, \tau)$, an overview

We will give an overview of how to construct the virtual fundamental classes $J(W, \tau)$. Many points are discussed in greater detail in the following sections.

This is where Artin stacks appear for the first time, and so we have to stop pretending that stacks are just spaces. Otherwise many facts would seem counterintuitive. But the only Artin stacks involved are of a particularly simple type, namely quotient stacks associated to the action of a vector bundle on a scheme of cones.

As usual, let τ be a stable modular graph with an $H_2(W)^+$ marking on the vertices. Consider the universal stable map of type τ

$$\begin{array}{ccc} C & \xrightarrow{f} & W \\ {\scriptstyle\pi}\downarrow & & \\ \overline{M}(W,\tau) & & \end{array}$$

From this diagram we get the complex $R\pi_* f^* T_W$, which is an object of $D(\overline{M}(W,\tau))$, the derived category of $\mathcal{O}$-modules on $\overline{M}(W,\tau)$. In fact we may realize $R\pi_* f^* T_W$ as a two-term complex $[E_0 \to E_1]$ of vector bundles on $\overline{M}(W,\tau)$. Then we have $\ker(E_0 \to E_1) = \pi_* f^* T_W$ and $\operatorname{coker}(E_0 \to E_1) = R^1\pi_* f^* T_W$.

In this context, the basic facts of obstruction theory are that for a morphism $f\colon C \to W$, the vector space $H^0(C, f^*T_W)$ classifies the infinitesimal deformations of f and $H^1(f^*T_W)$ contains the obstructions to deformations of f. Thus the complex $R\pi_* f^* T_W$ is intimately related to the obstruction theory of $\overline{M}(W,\tau)$.

To $R\pi_* f^* T_W$ we get an associated *vector bundle stack* $\mathfrak{E}$, which is simply given by the stack quotient $\mathfrak{E} = [E_1/E_0]$ (but is an invariant of the isomorphism class of $R\pi_* f^* T_W$ in the derived category).

The next ingredient is the *intrinsic normal cone.* If X is any scheme (or algebraic space, or Deligne–Mumford stack), it has an associated intrinsic normal cone, which, as the name indicates, is an intrinsic invariant of X, but is constructed from the normal cones coming from various local embeddings of X. The intrinsic normal cone is denoted $\mathfrak{C}_X$ and it is a *cone stack*, i.e., a stack that is étale locally over X of the form $[C/E]$, where $E \to C$ is a vector bundle over X operating on a cone over X. (As above, $[C/E]$ denotes the associated stack quotient.)

The intrinsic normal cone $\mathfrak{C}_X$ is constructed as follows. We choose a local embedding $i\colon X \hookrightarrow M$, where M is smooth. Then we get an action of the vector bundle i^*T_M on the normal cone $C_{X/M}$ and the essential observation is that the associated stack quotient $[C_{X/M}/i^*T_M]$ is independent of the choice of the local embedding $i\colon X \to M$. Thus the various $[C_{X/M}/i^*T_M]$ coming from local embeddings of X glue together to give the cone stack $\mathfrak{C}_X$ over X. A basic fact about $\mathfrak{C}_X$ is that it is always purely of dimension zero.

For our application we will use the *relative* intrinsic normal cone. This is an intrinsic invariant of a morphism $X \to Y$ and is denoted $\mathfrak{C}_{X/Y}$. It has the property that for every local embedding

$$\begin{array}{ccc} X & \xrightarrow{i} & M \\ & \searrow & \downarrow \\ & & Y \end{array}$$

of X into a scheme which is smooth over Y, it is canonically isomorphic to

$$\mathfrak{C}_{X/Y} = [C_{X/M}/i^* T_{M/Y}].$$

In our case we consider the morphism $\overline{M}(W,\tau) \to \mathfrak{M}_\tau$, where $\mathfrak{M}_\tau$ has the same definition as $\overline{M}_\tau$, except that the stability requirement is waived. So $\overline{M}_\tau$ is an open substack of $\mathfrak{M}_\tau$, and $\mathfrak{M}_\tau$ is not of finite type, or Deligne–Mumford, or even separated, but still smooth. The map $\overline{M}(W,\tau) \to \mathfrak{M}_\tau$ is given by forgetting f in a triple (C, x, f), but not stabilizing. We define

$$\mathfrak{C} = \mathfrak{C}_{\overline{M}(W,\tau)/\mathfrak{M}_\tau}.$$

Finally, we remark that there is a natural closed immersion of the cone stack $\mathfrak{C}$ into the vector bundle stack $\mathfrak{E}$ over $\overline{M}(W,\tau)$. This is because $R\pi_* f^* T_W$ is what is called a *(relative) obstruction theory* for $\overline{M}(W,\tau)$ over $\mathfrak{M}_\tau$.

We now consider the pullback diagram

$$\begin{array}{ccc} \overline{M}(W,\tau) & \longrightarrow & \mathfrak{C} \\ \downarrow & & \downarrow \\ \overline{M}(W,\tau) & \xrightarrow{0} & \mathfrak{E} \end{array}$$

where 0 is the zero section of the vector bundle stack $\mathfrak{E}$. We obtain the virtual fundamental class as the intersection of the cone stack $\mathfrak{C}$ with the zero section of $\mathfrak{E}$.

$$J(W,\tau) = [\overline{M}(W,\tau)]^{\mathrm{virt}} = 0^![\mathfrak{C}].$$

We should point out, though, that lacking an intersection theory for Artin stacks, we cannot apply this construction directly. Therefore we choose as above a two-term complex of vector bundles $[E_0 \to E_1]$ representing $R\pi_* f^* T_W$. Then $\mathfrak{C} \subset \mathfrak{E}$ induces a cone $C \subset E_1$ and we define

$$0^!_{\mathfrak{E}}[\mathfrak{C}] = 0^!_{E_1}[C].$$

4.2 Cones and cone stacks

We explain the basics of the theory of cones and cone stacks. For proofs see [3]. Let X be a Deligne–Mumford stack (or algebraic space, or scheme) over k, where k is our ground field. Later, X will be our moduli stack.

Cones

Let us recall the definition of a cone over X.

Consider a graded quasicoherent sheaf of $\mathcal{O}_X$-algebras

$$S = \bigoplus_{i \geq 0} S^i,$$

such that $S^0 = \mathcal{O}_X$, S^1 is coherent and S is locally generated by S^1. Then the affine X-scheme[13] $C = \operatorname{Spec} S$ is a *cone* over X.

The augmentation $S \to S^0$ defines a section $0\colon X \to C$, the *vertex* of the cone $C \to X$. The morphism of $\mathcal{O}_X$-algebras $S \to S[x]$ mapping a homogeneous element $s \in S^i$ of degree i to sx^i defines a morphism $\mathbb{A}^1 \times C \to C$, which we call the $\mathbb{A}^1$*-action* on C. It is an action in the sense that $(\lambda\mu) \cdot c = \lambda(\mu \cdot c)$, $1 \cdot c = c$ and $0 \cdot c = 0$. Another, longer, but more descriptive name for this map could be the 'multiplicative contraction onto the vertex'.

Example (Abelian cones) Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Then we get an associated cone by

$$C(\mathcal{F}) = \operatorname{Spec} \operatorname{Sym} \mathcal{F}.$$

Note that for a k-scheme T we have $C(\mathcal{F})(T) = \operatorname{Hom}(\mathcal{F}_T, \mathcal{O}_T)$, so that $C(\mathcal{F})$ is a group scheme over X. A cone obtained in this way is called an *Abelian cone.*

If C is any cone, then $\operatorname{Sym} S^1 \to \bigoplus S^i$ defines a closed immersion $C \hookrightarrow C(S^1)$. We denote $C(S^1)$ by $A(C)$ and call it the *Abelian hull* of C. It contains C as a closed subcone and is the smallest Abelian cone with this property.

Example (Vector bundles) Let $E \to X$ be a vector bundle and $\mathcal{E}$ the corresponding $\mathcal{O}_X$-module of sections. Then $E \cong C(\mathcal{E}^\vee)$ is an Abelian cone. Note that a cone $C \to X$ is smooth if and only if it is a vector bundle.

Example (Normal cones) Let $i\colon X \to Y$ be a closed immersion (or more generally a local immersion), with ideal sheaf I. Then

$$C_{X/Y} = \operatorname{Spec}_X\Big(\bigoplus_{n \geq 0} I^n/I^{n+1}\Big)$$

is the *normal cone* of X in Y. Its Abelian hull,

$$N_{X/Y} = C(I/I^2)$$

is the *normal sheaf* of X in Y. Note that i is a regular immersion if and only if $C_{X/Y}$ is Abelian (i.e., $C_{X/Y} = N_{X/Y}$) which in turn is equivalent to $C_{X/Y}$ being a vector bundle.

[13]It should be noted that whenever we talk of X-schemes or schemes over a stack X, we actually mean stacks over X that are *relative* schemes over X.

Vector bundle cones

Now consider the following situation. Let E be a vector bundle and C a cone over X, and let $d\colon E \to C$ be a morphism of cones (i.e., an X-morphism that respects the vertices and the $\mathbb{A}^1$-actions). Passing to the Abelian hulls we get a morphism $E \to A(C)$ of cones over X, which is necessarily a homomorphism of group schemes over X, so that E acts on $A(C)$. If C is invariant under the action of E on $A(C)$, so that we get an induced action of E on C, then we say that C is an *E-cone.*

Example Let $i\colon X \to M$ be a closed immersion, where M is smooth (over k). Then $C_{X/M}$ is automatically an i^*T_M-cone.

We now come to a construction that may seem intimidating, if one is not familiar with the language of stacks. We will try to explain why it shouldn't be.

Whenever we have an E-cone C, we associate to it the stack quotient $[C/E]$. At this point it is not very important to know what $[C/E]$ *is*, only to understand the main property, in fact the defining property of $[C/E]$, namely that the diagram of stacks over X

$$\begin{array}{ccc} E \times C & \xrightarrow{\sigma} & C \\ {\scriptstyle p}\downarrow & & \downarrow \\ C & \longrightarrow & [C/E] \end{array} \tag{6}$$

is Cartesian and co-Cartesian.[14] Here σ and p are the action and projection, respectively.

Recall that for an action of a group (like E) on a space (like C), the quotient C/E is defined to be the object (if it exists) which makes the diagram (6) co-Cartesian, i.e., the pushout. (This applies to many categories, not just (schemes/X).) If it turns out that (6) is also Cartesian, then C/E is the best possible kind of quotient, since (6) Cartesian means that the quotient map $C \to C/E$ is a principal E-bundle (or torsor, in different terminology).

The construction of stacks like $[C/E]$ should be viewed as a purely formal process which supplies such ideal quotients if they do not exist. On a certain level, this is analogous to the construction of the rational numbers from the integers. If a certain division 'doesn't go', one formally adjoins a quotient.

Applying this viewpoint to our situation, where we are trying to divide cones by vector bundles, we may say that the division C/E 'goes' (or that

[14] Note that everything is happening *over* X; E is a relative group over X (so its fibers over X are groups), C is a relative cone over X (so its fibers over X are 'usual' cones), the action of E on C is relative to X and so in particular, the product $E \times C$ is a product over X.

E divides C) if there exists a cone such that when inserted for $[C/E]$ in (6) it makes (6) Cartesian and co-Cartesian. If E does not divide C, then we formally adjoin the quotient. Of course, one has to introduce an equivalence relation on these formal quotients. So if C is an E-cone and C' an E'-cone, and there exists a Cartesian diagram

$$\begin{array}{ccc} E' & \longrightarrow & C' \\ \downarrow & & \downarrow \\ E & \longrightarrow & C \end{array} \tag{7}$$

where $C' \to C$ is a smooth epimorphism, then we call the quotients $[C/E]$ and $[C'/E']$ isomorphic. This may be motivated by noting that if we have a diagram (7) then there exists a vector bundle F such that $C = C'/F$ and $E = E'/F$ and thus we should have

$$[C/E] = \frac{[C'/F]}{[E'/F]} = [C'/E'].$$

By this process one enlarges the category of cones over X, and obtains a category where quotients of cones by vector bundles always exist. The one convenience one has to give up in the process is that of having a *category* of objects. The stacks that we obtain in this way form a 2-category, where there are objects, morphisms, and isomorphisms of morphisms. Butfor the most part we ignore that effect, to keep things simple.

Of course one has to do some work to prove that one can still do geometry with these new objects $[C/E]$. If one does this, then the quotient map $C \to [C/E]$ turns out to be an honest principal E-bundle. So over each point x of X the fiber $[C/E]_x$ is the quotient $[C_x/E_x]$. This is a usual cone divided by a usual vector bundle, but the quotient map $C_x \to [C_x/E_x]$ is a principal E_x-bundle, which means that the fibers of $C_x \to [C_x/E_x]$ are all copies of E_x (but not canonically).

It also makes sense to speak of the dimension of $[C/E]$. Since the fibers of the morphism $C \to [C/E]$ are vector spaces of dimension $\operatorname{rank} E$ we have $\dim[C/E] = \dim C - \operatorname{rank} E$.

Two extreme cases might be worth pointing out: if $E = X$, then $[C/E] = C$. If $C = X$, then $[C/E] = [X/E]$ is the stack over X whose fiber over $x \in X$ is $[\{x\}/E_x]$, a point divided by a vector space. One also uses the notation $BE_x = [\mathrm{pt}/E_x]$ and $BE = [X/E]$. So in the naive picture of a stack as a collection of points with groups attached, BE has points $\{x \mid x \in X\}$ and groups $(E_x)_{x\in X}$. Note that $\dim BE = \dim X - \operatorname{rank} E$.[15]

[15]The appearance of negative dimensions for Artin stacks is completely analogous to the appearance of fractions when counting points of finite Deligne–Mumford stacks.

Cone stacks

We have to take one more step to get the category of cone stacks. We need to localize, meaning that we want to call objects cone stacks if they locally look like the $[C/E]$ we just constructed. For the applications we have in mind, this step is not really necessary, since in the end all cone stacks we use will turn out to be of the form $[C/E]$. But for the general theory of the intrinsic normal cone it would be an awkward restriction to require cone stacks to be global quotients. So we make the following definition.

Definition 4.1 Let $\mathfrak{C} \to X$ be an algebraic stack with vertex $0\colon X \to \mathfrak{C}$ and $\mathbb{A}^1$-action $\gamma\colon \mathbb{A}^1 \times \mathfrak{C} \to \mathfrak{C}$. Then $\mathfrak{C}$ is a *cone stack* if, étale locally on X, there exists a vector bundle E over X and an E-cone C over X such that $\mathfrak{C} \cong [C/E]$ as stacks over X with $\mathbb{A}^1$-action and vertex.

Every such C is called a *local presentation* of $\mathfrak{C}$. If one can find local presentations C which are vector bundles (so that locally $\mathfrak{C} \cong [E_1/E_0]$, for a homomorphism of vector bundles $E_0 \to E_1$, then $\mathfrak{C}$ is called a *vector bundle stack*.

4.3 The intrinsic normal cone

As before, let X be a Deligne–Mumford stack over k.

A *local embedding* of X is a diagram

$$\begin{array}{ccc} U & \xrightarrow{f} & M \\ {\scriptstyle i}\downarrow & & \\ X & & \end{array}$$

where i is étale, f a closed immersion and M is smooth.

A *morphism of local embeddings* is a commutative diagram

$$\begin{array}{ccc} U' & \xrightarrow{f'} & M' \\ \downarrow & & \downarrow \\ U & \xrightarrow{f} & M \end{array}$$

where $U' \to U$ is an étale X-morphism and $M' \to M$ is smooth.

Given such a morphism of local embeddings we get a commutative diagram

$$\begin{array}{ccccc} f'^*T_{M'/M} & \longrightarrow & f'^*T_{M'} & \longrightarrow & f^*T_M|U' \\ \downarrow & & \downarrow & & \downarrow \\ f'^*T_{M'/M} & \longrightarrow & C_{U'/M'} & \longrightarrow & C_{U/M}|U' \end{array}$$

The rows are exact sequences of cones. The square on the right is Cartesian and $C_{U'/M'} \to C_{U/M}|U'$ is a smooth epimorphism. All these are basic properties of normal cones and tangent bundles.

As explained above, in this situation the quotients $[C_{U/M}/f^*T_M]|U'$ and $[C_{U'/M'}/f'^*T_{M'}]$ are canonically isomorphic. Thus all these locally defined cone stacks (one for each local embedding) glue together to give rise to a globally defined cone stack on X (note that X can be covered by étale $U \to X$ that are embeddable into smooth varieties). This cone stack is called *the intrinsic normal cone* of X and is denoted by $\mathfrak{C}_X$.

Proposition 4.2 *The stack $\mathfrak{C}_X$ is a cone stack of pure dimension*[16] *zero. For any local embedding we have*

$$\mathfrak{C}_X|U = [C_{U/M}/f^*T_M].$$

Proof It is a general property of normal cones that they always have the dimension of the ambient variety. So we have $\dim C_{U/M} = \dim M$ and therefore

$$\dim[C_{U/M}/f^*T_M] = \dim C_{U/M} - \operatorname{rank} f^*T_M = \dim M - \dim M = 0. \quad \square$$

Remark One can do the same construction with normal sheaves $N_{U/M}$ instead of normal cones $C_{U/M}$. Then one gets the *intrinsic normal sheaf* $\mathfrak{N}_X$ of X. Moreover, $\mathfrak{C}_X \subset \mathfrak{N}_X$ is a closed substack and $\mathfrak{N}_X$ is the Abelian hull of $\mathfrak{C}_X$ (this notion also makes sense for cone stacks).

Let L_X be the cotangent complex of X and $\tau_{\geq -1}L_X$ its truncation at -1. Again, this is nothing deep, over a local embedding it is simply given by the two term complex

$$(\tau_{\geq -1}L_X)|U = [I/I^2 \to f^*\Omega_M],$$

where I is the ideal sheaf, and the map is the map appearing in the second fundamental exact sequence of Kähler differentials.

One can prove that the stack $\mathfrak{N}_X$ only depends on the quasi-isomorphism class of $\tau_{\geq -1}L_X$, in other words it is an invariant of the object $\tau_{\geq -1}L_X \in \operatorname{ob} D^{[-1,0]}_{\mathrm{coh}}(\mathcal{O}_X)$. In fact, one can define for every $M^\bullet \in \operatorname{ob} D^{[-1,0]}_{\mathrm{coh}}(\mathcal{O}_X)$ an associated Abelian cone stack $\mathfrak{C}(M^\bullet)$. To do this, write (locally over X) $M^\bullet = [M^{-1} \to M^0]$ with M^0 free. Then pass to $C(M^0) \to C(M^{-1})$, the associated Abelian cones, and let $\mathfrak{C}(M^\bullet)$ be the stack quotient $\mathfrak{C}(M^\bullet) = [C(M^{-1})/C(M^0)]$. This construction globalizes and is functorial. Alternative notations are $\mathfrak{C}(M^\bullet) = h^1/h^0(M^{\bullet\vee})$, used in [3] or $\mathfrak{C}(M^\bullet) = \mathrm{ch}(M^{\bullet\vee})$, in [1], Exposé XVII.

[16] Absolute dimension over k, not dimension over X

The following are a few basic results on the intrinsic normal cone. None of them are deep or difficult to prove, they just reformulate known results about normal cones and tangent bundles.

Proposition 4.3 *The following are equivalent.*

1. *X is a local complete intersection,*
2. *$\mathfrak{C}_X$ is a vector bundle stack,*
3. *$\mathfrak{C}_X = \mathfrak{N}_X$.*

If X is smooth then $\mathfrak{C}_X = \mathfrak{N}_X = BT_X$.

Proposition 4.4 $\mathfrak{C}_{X\times Y} = \mathfrak{C}_X \times \mathfrak{C}_Y$ *(absolute product, over k).*

Proposition 4.5 *Let $f\colon X \to Y$ be a local complete intersection morphism. Then there is a short exact sequence of cone stacks*

$$\mathfrak{N}_{X/Y} \longrightarrow \mathfrak{C}_X \longrightarrow f^*\mathfrak{C}_Y.$$

Here $\mathfrak{N}_{X/Y} = \mathfrak{C}(L^\bullet_{X/Y})$, which is a vector bundle stack. The notion of short exact sequence of cone stacks is a straightforward generalization of the notion of short exact sequence of cones. What it means is that the cone stack on the right may be viewed as the quotient of the cone stack in the middle by the action of the vector bundle stack on the left.

For example, if f is smooth we have an exact sequence

$$BT_{X/Y} \longrightarrow \mathfrak{C}_X \longrightarrow f^*\mathfrak{C}_Y,$$

and if f is a regular immersion we have

$$N_{X/Y} \longrightarrow \mathfrak{C}_X \longrightarrow f^*\mathfrak{C}_Y.$$

4.4 The intrinsic normal cone and obstructions

We will now look at the 'fiber' of the intrinsic normal cone over a point of X. So let $p\colon \operatorname{Spec} k \to X$ be a geometric point of X (which just means that k is an algebraically closed field, not necessarily equal to the ground field, by abuse of notation). Pulling back the intrinsic normal cone $\mathfrak{C}_X$ via p, we get a cone stack over $\operatorname{Spec} k$.

If we look at cone stacks over an algebraically closed field, they are necessarily given as the quotient $[C/E]$ associated to an E-cone C, where E is just a vector space. In this case the quotient of C by the image of $d\colon E \to C$ exists, and choosing a complementary subspace for $\ker d$ in E, we get a Cartesian diagram

$$\begin{array}{ccc} E & \xrightarrow{d} & C \\ \downarrow & & \downarrow \\ \ker d & \xrightarrow{0} & C/\operatorname{im} d \end{array}$$

showing that, as cone stacks, $[C/E]$ is isomorphic to the the quotient of $C' = C/\operatorname{im} d$ by $\ker d$ acting trivially. So for studying this cone stack, we may as well replace $d\colon E \to C$ by $0\colon \ker d \to C'$ and assume from the start that E acts trivially on C, i.e., that the map $d\colon E \to C$ is the zero map. Then we have that $[C/E] \cong BE \times C$, where BE is the quotient of the point $\operatorname{Spec} k$ by the vector space E.

Considering such a cone stack $BE \times C$ over $\operatorname{Spec} k$, we may interpret the cone C as the 'coarse moduli space' of $BE \times C$. Any stack has a coarse moduli space associated to it; it *is* the set of isomorphism classes of whatever the objects are that the stack classifies. The vector space E is the common automorphism group of all the objects that the stack classifies.

Now let us determine what these objects and automorphisms are, for the case of $p^*\mathfrak{C}_X$. Before dealing with the intrinsic normal cone, though, let us consider the intrinsic normal sheaf. We have $p^*\mathfrak{N}_X = p^*\mathfrak{C}(\tau_{\geq -1} L_X) = \mathfrak{C}(p^*\tau_{\geq -1} L_X)$.

Recall the 'higher tangent spaces'

$$T^i_{X,p} = \operatorname{Ext}^i(p^*L_X, k) = h^i(p^*L_X)^\vee$$

of X at p. For example, $T^0_{X,p} = \operatorname{Hom}(\Omega_X, k)$ is the usual Zariski tangent space. It classifies first order deformations of p, i.e., (isomorphism classes of) diagrams

$$\begin{array}{ccc} \operatorname{Spec} k & \longrightarrow & \operatorname{Spec} k[\epsilon] \\ & {\scriptstyle p} \searrow & \downarrow {\scriptstyle p'} \\ & & X \end{array}$$

where $k[\epsilon]$ is the ring of dual numbers (meaning that $\epsilon^2 = 0$). The first higher tangent space $T^1_{X,p}$ is the *obstruction space*, and classifies obstructions.

Now

$$p^*\tau_{\geq -1} L_X \cong [h^{-1}(p^*L_X) \xrightarrow{0} h^0(p^*L_X)]$$

and so

$$p^*\mathfrak{N}_X = BT^0_{X,p} \times T^1_{X,p}.$$

Thus the intrinsic normal sheaf classifies obstructions, with deformations as automorphism group. Since the intrinsic normal cone is a closed substack of $\mathfrak{N}_X$, we get that

$$p^*\mathfrak{C}_X \cong BT^0_{X,p} \times C_{X,p},$$

where $C_{X,p} \subset T^1_{X,p}$ is some cone of obstructions.

To describe what kind of obstructions the intrinsic normal cone classifies, let us recall what an obstruction is. Let $A' \to A$ be an epimorphism of local Artinian k-algebras with kernel k (i.e., a *small extension*). Let $T = \operatorname{Spec} A$ and $T' = \operatorname{Spec} A'$ and assume given an extension x of p to A, i.e., a diagram

$$\begin{tikzcd} \operatorname{Spec} k \arrow[r] \arrow[dr, "p"'] & T \arrow[d, "x"] \\ & X \end{tikzcd}$$

In this situation we get a canonical morphism $x^*L_X \to L_T$, by the contravariant nature of the cotangent complex. From the morphism $T \to T'$ we get a morphism $L_T \xrightarrow{+1} k$ of degree 1. It is essentially the morphism from L_T to $L_{T/T'}$. Composing, we get a morphism $x^*L_X \to k$ of degree 1, in other words an element of

$$\operatorname{Ext}^1(x^*L_X, k) \xrightarrow{\sim} \operatorname{Ext}^1(p^*L_X, k) = T^1_{X,p},$$

which is called the *obstruction* of $(A' \to A, x)$. The justification for this terminology is that it vanishes if and only if x extends to A', i.e., if and only if there exists $x' \colon T' \to X$ making the diagram

$$\begin{tikzcd} T \arrow[r] \arrow[dr, "x"'] & T' \arrow[d, "x'"] \\ & X \end{tikzcd}$$

commute.

In more concrete terms the obstruction of $(A' \to A, x)$ can be described as follows. Choose a local embedding $f \colon U \to M$ of X at p, where U and M are affine. Let I be the corresponding sheaf of ideals, which we identify with an ideal in the affine coordinate ring of M. Then we have

$$T^1_{X,p} = \operatorname{coker}(p^*f^*T_M \to (p^*I/I^2)^\vee).$$

Now given $x \colon T \to X$ it is possible to choose $x'' \colon T' \to M$ such that

$$\begin{tikzcd} T \arrow[r] \arrow[d, "x"'] & T' \arrow[d, "x''"] \\ U \arrow[r, "f"] & M \end{tikzcd}$$

commutes, since M is smooth. This diagram of two closed immersions induces a morphism on the level of ideals, namely $I \to \ker(A' \to A) = k$. This element of $I^\vee$ induces the obstruction in $T^1_{X,p}$ (which is independent of the choice of f and M and x'').

The small extension $A' \to A$ is *curvilinear* if it is isomorphic to $k[t]/t^{s+1} \to k[t]/t^s$ for some $s \geq 1$. This notion gives the answer to our question what the obstructions are that the intrinsic normal cone classifies:

Proposition 4.6 *Every element of $T^1_{X,p}$ obstructs some small extension. It obstructs a small curvilinear extension if and only if it is in $C_{X,p} \subset T^1_{X,p}$.*

Proof See [3], Proposition 4.7. □

4.5 Obstruction theory

Let us start with an example. Consider the Cartesian diagram

$$\begin{array}{ccc} X & \xrightarrow{v} & V \\ \downarrow & & \downarrow f \\ \operatorname{Spec} k & \xrightarrow{w} & W \end{array} \tag{8}$$

where f is a morphism between smooth varieties. Thus X is a fiber of f. This is a typical intersection theory situation. One defines a cycle class on X by $[X]^{\mathrm{virt}} = w^![V]$. This class is called the *specialization* of $[V]$ at w. The class $[X]^{\mathrm{virt}}$ is first of all in the expected degree, namely $\dim V - \dim W$, even if X actually has larger dimension. Moreover, it leads to numerical data which is independent of the parameter $w \in W$. In the case that $\dim V = \dim W$ this means that the degree of the zero cycle $[X]^{\mathrm{virt}}$ is independent of w. For an explanation of what it means if $\dim V > \dim W$, see [6], Chapter 10.

Because of this invariance of numerical data defined in terms of $[X]^{\mathrm{virt}}$, this class is a sensible one to use for questions in enumerative geometry. Let us recall the construction of $w^![V]$ (or at least how the definition is reduced to the linear case). One replaces Diagram (8) by the following:

$$\begin{array}{ccc} X & \longrightarrow & C_{X/V} \\ \downarrow & & \downarrow \\ X & \xrightarrow{0} & T_W(w) \times X \end{array} \tag{9}$$

Here $C_{X/V}$ is the normal cone and the normal bundle of $w\colon \operatorname{Spec} k \to W$ pulled back to X is $T_W(w) \times X$, the tangent space to W at w times X. Then $[X]^{\mathrm{virt}} = 0^![C_{X/V}]$.

Now we also have a Cartesian diagram

$$\begin{array}{ccc} X & \longrightarrow & [C_{X/V}/v^*T_V] \\ \downarrow & & \downarrow \\ X & \xrightarrow{0} & [T_W(w)_X/v^*T_V] \end{array} \tag{10}$$

which is obtained from (9) simply by dividing through (in the stack sense) by v^*T_V. Now note that $[C_{X/V}/v^*T_V] = \mathfrak{C}_X$ is just the intrinsic normal cone of X and $\mathfrak{E} = [T_W(w)_X/v^*T_V]$ is a vector bundle stack on X into which $\mathfrak{C}_X$ is embedded.

If there was an intersection theory for Artin stacks (or just cone stacks), then certainly $0^![C_{X/V}] = 0^!_{\mathfrak{E}}[\mathfrak{C}_X]$, where the first 0 is the zero section from (9). So we can characterize the virtual fundamental class of X in terms of the intrinsic normal cone of X, which is completely intrinsic to X and the vector bundle stack $\mathfrak{E}$, which is, of course, not intrinsic to X, but has to do with the obstruction theory of X.

In fact, $\mathfrak{E} = v^*[f^*T_W/T_V]$, and because $[T_V \to f^*T_W] = T^\bullet_{V/W}$ is the tangent complex of f, we have that $\mathfrak{E} = v^*\mathfrak{C}(L_{V/W})$. So $\mathfrak{E}$ can be thought of as the linearization of f. Moreover, $h^0(v^*T_{V/W}) = T_X$ classifies the first order deformations of X, and $h^1(v^*T_{V/W})$ contains the obstructions to deforming X.

So we have replaced the ambient morphism f, which defined a virtual fundamental class on X, by this vector bundle stack $\mathfrak{E}$, which serves the same purpose. Now if X is a moduli space (or stack), then it might be hopeless to try to embed X globally into a smooth space (or stack) but such an $\mathfrak{E}$ can sometimes still be found; in fact, it comes naturally from the moduli problem that X solves.

The two essential properties of $\mathfrak{E}$ are

1. $\mathfrak{E}$ is a vector bundle stack, i.e., it is locally defined in term of a complex of two vector bundles $[E^{-1} \to E^0]$. Such a complex is referred to as being *perfect of amplitude contained in* $[-1,0]$. In other words, it is an object of $D^{[-1,0]}_{\text{parf}}(\mathcal{O}_X)$.[17]

2. The intrinsic normal cone $\mathfrak{C}_X$ is embedded as a closed subcone stack into $\mathfrak{E}$.

This motivates the following definition.

[17]Note that the superscript $[-1,0]$ does not refer to the object of the derived category having cohomology in the interval $[-1,0]$, but to its perfect amplitude being in that interval. The latter is stronger than the former.

Definition 4.7 Let E be an object of $D^{[-1,0]}_{\text{parf}}(\mathcal{O}_X)$. A homomorphism $\varphi\colon E \to L_X$ (and by abuse of language also E itself) is called a *perfect obstruction theory* for X if

1. $h^0(\varphi)$ is an isomorphism,

2. $h^{-1}(\varphi)$ is surjective.

It is not difficult to prove that the two conditions on φ are equivalent to the morphism $\mathfrak{N}_X \to \mathfrak{E}$ (where $\mathfrak{E} = \mathfrak{C}(E)$) induced by φ being a closed immersion. Moreover, if $p\colon \operatorname{Spec} k \to X$ is a geometric point of X, then an obstruction theory induces an isomorphism

$$T^0_{X,p} \xrightarrow{\sim} h^0(p^*E^\vee)$$

and a monomorphism

$$T^1_{X,p} \hookrightarrow h^1(p^*E^\vee),$$

so, in a sense, E reflects the deformation theory of X and contains the obstructions of X.

As an example, let C be a pre-stable curve, W a smooth projective variety and $f\colon C \to W$ a morphism. Then $H^0(C, f^*T_W)$ classifies the infinitesimal deformations of f. The obstructions are contained in $H^1(C, f^*T_W)$. To see this, let U_α be an affine open cover of C. By the infinitesimal lifting property the morphism f can be extended over each U_α. Over the overlaps $U_{\alpha\beta}$ two extensions differ by an infinitesimal deformation, i.e., a section of f^*T_W over $U_{\alpha\beta}$. The vanishing of this Čech 1-cocycle with values in f^*T_W means extendability of f.

These observations can be translated into the following statement. Let $X = \operatorname{Mor}(C, W)$ be the scheme of morphisms from C to W. Then there is a perfect obstruction theory on X given by $(R\pi_* f^*T_W)^\vee \to L_X$, where

$$\begin{array}{ccc} C \times X & \xrightarrow{f} & W \\ {\scriptstyle \pi}\downarrow & & \\ X & & \end{array}$$

is the universal map. Note that since $\pi\colon C \times X \to X$ has one dimensional fibers, the complex $(R\pi_* f^*T_W)^\vee$ is indeed perfect of amplitude 1.

This is in fact the obstruction theory we want to use to construct the virtual fundamental class on $\overline{M}(W, \tau)$. But a deformation of a stable map may deform the curve C as well as the map $f\colon C \to W$. So we note that the morphism $\overline{M}(W, \tau) \to \mathfrak{M}_\tau$ that forgets the map (and does not stabilize) has fibers of the form $\mathrm{Mor}(C, W)$. So we would like to adapt the above theory to this relative situation.

Working in the relative rather than the absolute setting has the advantage that the obstruction theory is much simpler. Also, many of the axioms we will have to check involve the relative setting of $\overline{M}(W, \tau)$ over $\overline{M}_\tau$. So the relative obstruction theory is better suited for proving the axioms of Gromov–Witten theory. (Note, however, the difference between $\overline{M}_\tau$ and $\mathfrak{M}_\tau$. It is the main difficulty in proving the axioms.)

The reason why the relative obstruction theory works, is that the base $\mathfrak{M}_\tau$ is smooth.

So we replace the base $\operatorname{Spec} k$ by Y, where Y is any smooth algebraic k-stack of constant dimension n. It does not even have to be of Deligne–Mumford type. Let $X \to Y$ be a morphism which makes X a *relative* Deligne–Mumford stack over Y. This just means that any base change to a base Y', where Y' is a scheme, makes the fibered product X' a Deligne–Mumford stack.

Embedding X locally into stacks that are smooth and relative schemes over Y, one defines just as in the absolute case the intrinsic normal cone $\mathfrak{C}_{X/Y}$ and its Abelian hull $\mathfrak{N}_{X/Y}$. A complex of $\mathcal{O}_X$-modules E that is locally quasi-isomorphic to a two term complex of vector bundles, together with a map in the derived category $E \to L_{X/Y}$ is called a perfect relative obstruction theory, if it induces a closed immersion of cone stacks $\mathfrak{C}_{X/Y} \to \mathfrak{C}(E)$.

It follows from [3], Proposition 2.7 that the relative intrinsic normal cone $\mathfrak{C}_{X/Y}$ is 'just' the quotient of the absolute intrinsic normal cone $\mathfrak{C}_X$ by the natural action of the tangent vector bundle stack $\mathfrak{T}_Y$ of Y. The same is true for the intrinsic normal sheaves. Moreover, in our application, the relationship between the vector bundle stacks given by the relative and absolute obstruction theories, respectively, is also the same. This implies that the virtual fundamental class defined in the relative setting is the same as that defined in the absolute setting.

Let us make more precise the sense in which a relative obstruction theory $E \to L_{X/Y}$ governs the obstructions of X over Y.

Let

$$\begin{array}{ccc} T & \xrightarrow{x} & X \\ \downarrow & & \downarrow \\ T' & \longrightarrow & Y \end{array} \tag{11}$$

be a commutative diagram, where $T \to T'$ is a square zero extension of (affine) schemes, with ideal N (i.e., a closed immersion with ideal sheaf N such that $N^2 = 0$). Such a diagram induces an obstruction $o \in \operatorname{Ext}^1(x^*E, N)$, which vanishes if and only if a map $T' \to X$ completing Diagram (11) exists. Moreover, if the obstruction o vanishes, then all arrows $T' \to X$ completing (11) form a torsor under $\mathrm{Ext}^0(x^*E, N)$, i.e., there exists a natural action of $\mathrm{Ext}^0(x^*E, N)$ on the set of such arrows $T' \to X$, which is simply transitive.

The obstruction o is obtained as follows. A fundamental fact about the cotangent complex is that is classifies extensions of algebras, i.e., that

$$\mathrm{Extalg}_Y(T, N) = \mathrm{Ext}^1(L_{T/Y}, N).$$

Thus $T \to T'$ gives a morphism of degree one $L_{T/Y} \to N$. Composing with the natural maps $x^*L_{X/Y} \to L_{T/Y}$ and $x^*E \to x^*L_{X/Y}$ we get a morphism of degree one $x^*E \to N$, in other words an element $o \in Ext^1(x^*E, N)$.

Note In the case that $X \to Y$ is a morphism of smooth schemes, we can take the identity $L^\bullet_{X/Y} \to L^\bullet_{X/Y}$ as relative obstruction theory for X over Y. Pulling this relative obstruction theory back to a fiber of $X \to Y$, we get the absolute obstruction theory of the fiber described earlier.

4.6 Fundamental classes

Since the relative case is no more difficult than the absolute one, we assume from the start that we have a perfect relative obstruction theory E for X over Y. To define the associated virtual fundamental class we need to assume that E has *global resolutions*, i.e., that E is globally quasi-isomorphic to a two term complex $[E^{-1} \to E^0]$ of vector bundles over X. This condition is satisfied for the relative obstruction theory of $\overline{M}(W, \tau)$ (see [2], Proposition 5). Then the stack $\mathfrak{E} = \mathfrak{C}(E)$ associated to E is isomorphic to $[E_1/E_0]$, where E_i denotes the dual bundle of E^{-i}. Since $\mathfrak{C}_X$ is a closed subcone stack of $\mathfrak{E}$, it induces a closed subcone C of E_1 and we define

$$[X]^{\mathrm{virt}} = [X, E] = 0^!_{E_1}[C] \in A_{\dim Y + \mathrm{rank}\, E}(X),$$

which is a class in Vistoli's Chow group with rational coefficients. This class is independent of the global resolution chosen to define it. The fact that this class is in the expected degree $\dim Y + \mathrm{rank}\, E$ follows immediately from the fact that the relative intrinsic normal cone has pure dimension $\dim Y$ (which corresponds to the fact that the absolute intrinsic normal cone has pure dimension zero).

Example 4.8 If X is smooth over Y, then $h^0(E) \cong h^0(L_{X/Y}) \cong \Omega_{X/Y}$ is locally free. Hence $E^{-1} \to E^0$ has locally free h^0 and h^{-1}, and so the same holds for the dual $\varphi \colon E_0 \to E_1$. Also, $\mathfrak{C}_{X/Y} = BT_{X/Y} \hookrightarrow [E_1/E_0]$ identifies $BT_{X/Y}$ with $B \ker \varphi$, which is isomorphic to $[\mathrm{im}\, \varphi / E_0]$, and so the cone induced by $\mathfrak{C}_X$ in E_1 is equal to $\mathrm{im}\, \varphi$. Hence

$$\begin{aligned} [X]^{\mathrm{virt}} &= 0^!_{E_1}[\mathrm{im}\, \varphi] \\ &= c_{\mathrm{top}}(\mathrm{coker}\, \varphi) \cap [X] \\ &= c_{\mathrm{top}}(h^1(E^\vee)) \cap [X]. \end{aligned}$$

In the above example of $\mathrm{Mor}(C,W)$ we have

$$[X]^{\mathrm{virt}} = c_{\mathrm{top}}(R^1\pi_* T_W) \cap [X].$$

Proposition 4.9 *If X has the expected dimension $\dim Y + \operatorname{rank} E$, then X is a local complete intersection and $[C,E] = [X]$, the usual fundamental class.*

Proof For simplicity, let us explain the absolute case $Y = \mathrm{pt}$. Let k be algebraically closed and A a localization of a finite type k-algebra at a maximal ideal. Write $A = (k[x_1,\ldots,x_n]/(f_1,\ldots,f_r))_{(x_1,\ldots,x_n)}$, let $\mathfrak{m}$ be the maximal ideal of $k[x_1,\ldots,x_n]_{(x_1,\ldots,x_n)}$ and $I = (f_1,\ldots,f_r) \subset k[x_1,\ldots,x_n]_{(x_1,\ldots,x_n)}$. Then the truncation at -1 of the cotangent complex of A is $I/I^2 \to \Omega_{k[x_1,\ldots,x_n]} \otimes A$ and if we tensor it over A with k we get $I/\mathfrak{m} \to \mathfrak{m}/\mathfrak{m}^2$ and so there is an exact sequence

$$0 \longrightarrow T^1(A)^\vee \longrightarrow I/\mathfrak{m}I \longrightarrow \mathfrak{m}/\mathfrak{m}^2 \longrightarrow T^0(A)^\vee \longrightarrow 0,$$

where $T^i(A)$ is the i-th tangent space of A at the maximal ideal.

After projecting $\operatorname{Spec} A$ into its tangent space at the origin, which only changes A by an étale map, we may assume that $I \subset \mathfrak{m}^2$. This entails that $I/\mathfrak{m}I \to \mathfrak{m}/\mathfrak{m}^2$ is the zero map and hence $T^0(A)^\vee = \mathfrak{m}/\mathfrak{m}^2$ and $T^1(A)^\vee = I/\mathfrak{m}I$. Clearly, $\overline{x}_1,\ldots,\overline{x}_n$ is a basis of $\mathfrak{m}/\mathfrak{m}^2$. By Nakayama's lemma we may also assume that $\overline{f}_1,\ldots,\overline{f}_r$ form a basis of $I/\mathfrak{m}I$. Hence $n = \dim T^0(A)$ and $r = T^1(A)$.

Now clearly, $\dim A \geq n - r$. If equality holds, then $f_1,\ldots,f_r$ is a regular sequence for $k[x_1,\ldots,x_r]_{(x_1,\ldots,x_r)}$ and so A is Cohen–Macaulay and a local complete intersection.

Now assume given a perfect obstruction theory $E^\bullet$ for A. Then $T^0(A) = h^0(E^{\bullet\vee} \otimes k)$ and $T^1(A) \hookrightarrow h^1(E^{\bullet\vee} \otimes k)$. Hence $n - r \geq \operatorname{rank} E^\bullet$ and so $\dim A \geq \operatorname{rank} E^\bullet$. By the previous argument $\dim A = \operatorname{rank} E^\bullet$ implies that A is a local complete intersection. Moreover, $\dim A = \operatorname{rank} E^\bullet$ implies $n - r = \operatorname{rank} E^\bullet$ and $T^1(A) = h^1(E^{\bullet\vee} \otimes k)$. This, in turn, implies that $E^\bullet \to L^\bullet_A$ is an isomorphism and so $[X, E^\bullet] = [X]$. $\square$

Corollary 4.10 *If the expected and actual dimension are both zero, then $[X]^{\mathrm{virt}}$ counts the number of points of X with their scheme (or stack) theoretic multiplicity.*

Remark If X can be embedded into a smooth scheme M and $E^\bullet$ is an absolute obstruction theory for X then we have (in the notation above)

$$[X, E^\bullet] = (c(E_1) \cap s(C))_{\operatorname{rank} E^\bullet},$$

where $s(C)$ is the Segre class of C and c denotes the total Chern class. The subscript denotes the component of degree $\operatorname{rank} E^\bullet$. (See [6], Chapter 6.) Now we have

$$\begin{aligned} c(E_1) \cap s(C) &= c(E_1)c(E_0)^{-1}c(E_0) \cap s(C) \\ &= c(E_\bullet)^{-1}c(E_0) \cap s(C) \\ &= c(E_\bullet)^{-1}c(i^*T_M) \cap s(C_{X/M}) \\ &= c(E^{\bullet\vee})^{-1}c_*(X), \end{aligned}$$

where $c_*(X)$ is the canonical class of X (see [ibid.]). Hence

$$[X, E^\bullet] = (c(E^{\bullet\vee})^{-1}c_*(X))_{\operatorname{rank} E^\bullet}.$$

Thus the intrinsic normal cone may be viewed as the geometric object underlying the canonical class. As such it glues (which cycle classes usually do not) and is thus also defined for nonembeddable X.

4.7 Gromov–Witten invariants

As always, let W be a smooth projective k-variety and τ a stable modular graph with an $H_2(W)^+$ marking β.

As indicated, we use the Artin stacks

$$\mathfrak{M}_\tau = \prod_{v \in V_\tau} \mathfrak{M}_{g(v), F_\tau(v)},$$

where $\mathfrak{M}_{g,S}$ is the stack of S-marked pre-stable curves of genus g. Pre-stable means that the singularities are at worst nodes and all marks avoid the nodes. Note that $\mathfrak{M}_\tau$ is smooth of dimension

$$\dim(\tau) = \# S_\tau - \# E_\tau - 3\chi(\tau).$$

We consider the morphism

$$\begin{aligned} \overline{M}(W,\tau) &\longrightarrow \mathfrak{M}_\tau \\ (C,x,f) &\longmapsto (C,x), \end{aligned}$$

where no stabilization takes place. Note that the fiber of this morphism over a point of $\mathfrak{M}_\tau$ corresponding to a curve C is an open subscheme of the scheme of morphisms $\operatorname{Mor}(C,W)$.

As before, let $\pi\colon C \to \overline{M}(W,\tau)$ be the universal curve and $f\colon C \to W$ the universal map. Then we have a perfect relative obstruction theory (even though this was only explained in the absolute case)

$$(R\pi_* f^* T_W)^\vee \longrightarrow L_{\overline{M}(W,\tau)/\mathfrak{M}_\tau},$$

and hence a virtual fundamental class

$$J(W,\tau) = [\overline{M}(W,\tau)]^{\text{virt}} = [\overline{M}(W,\tau),(R\pi_* f^* T_W)^\vee]$$

in $A_*(\overline{M}(W,\tau))$ of degree $\dim(\tau) + \operatorname{rank} R\pi_* f^* T_W$.

Let us check that this is the degree we claimed $[\overline{M}(W,\tau)]^{\text{virt}}$ to have:

$$\begin{aligned}
&\dim(\tau) + \operatorname{rank} R\pi_* f^* T_W \\
&\quad = \# S_\tau - \# E_\tau - 3\chi(\tau) + \chi(f^* T_W) \\
&\quad = \# S_\tau - \# E_\tau - 3\chi(\tau) + \deg f^* T_W + \dim W \chi(\mathcal{O}_C) \\
&\quad = \# S_\tau - \# E_\tau - 3\chi(\tau) - \beta(\omega_W) + \dim W \chi(\mathcal{O}_C) \\
&\quad = \chi(\tau)(\dim W - 3) - \beta(\omega_W) + \# S_\tau - \# E_\tau \\
&\quad = \dim(W,\tau).
\end{aligned}$$

This calculation justifies the grading axiom for Gromov–Witten invariants.

Theorem 4.11 *The classes $J(W,\tau)$ satisfy all five axioms required.*

Proof The mapping to point axiom follows from the Example 4.8. For the proofs of the other axioms see [2]. One has to prove various compatibilities of virtual fundamental classes. These follow from the properties of normal cones proved by Vistoli [13]. □

Corollary 4.12 *The Gromov–Witten invariants $I_\tau(\beta)$ defined in terms of $J(W,\tau)$ satisfy all eight axioms required.*

4.8 Complete intersections

These ideas can easily be adapted to construct the tree level system of Gromov–Witten invariants for possibly singular complete intersections.

So let $W \in \mathbb{P}^n$ be a complete intersection, $i\colon W \to \mathbb{P}^n$ the inclusion morphism. Then $[i^* T_{\mathbb{P}^n} \to N_{W/\mathbb{P}^n}]$ is the tangent complex of W. So as obstruction theory for $\overline{M}(W,\tau) \to \mathfrak{M}_\tau$ we may take $(R\pi_* f^*[i^* T_{\mathbb{P}^n} \to N_{W/\mathbb{P}^n}])^\vee$. This will be a perfect obstruction theory if we restrict to the case where τ is a forest, because then the higher direct images under π of $f^* i^* T_{\mathbb{P}^n}$ and $f^* N_{W/\mathbb{P}^n}$ vanish. So we get the tree level system of Gromov–Witten invariants of W.

As an example, consider a cone over a plane cubic, which is a degenerate cubic surface in $\mathbb{P}^3$. There is a one dimensional family of lines on this cubic, namely the ruling of the cone. On the other hand, the expected dimension of the space of lines on a cubic in $\mathbb{P}^3$ is zero. Therefore the Gromov–Witten invariant $I_{0,0}(1)$ is a number, which turns out to be 27. So the 'ideal' number of lines on a cubic is 27, even in degenerate cases.

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