A Calabi–Yau threefold with non-Abelian fundamental group

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Abstract

This note, written in 1994, answers a question of Dolgachev by constructing a Calabi–Yau threefold whose fundamental group is the quaternion group H_8. The construction is reminiscent of Reid's unpublished construction of a surface with $p_g = 0$, $K^2 = 2$ and $\pi_1 = H_8$; I explain below the link between the two problems.

1 The example

Let $H_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ be the quaternion group of order 8, and V its regular representation. We denote by $\widehat{H}_8$ the group of characters $\chi\colon H_8 \to \mathbb{C}^*$, which is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$. The group H_8 acts on $\mathbb{P}(V)$ and[1] on $\mathrm{S}^2 V$; for each $\chi \in \widehat{H}_8$, we denote by $(\mathrm{S}^2 V)_\chi$ the eigensubspace of $\mathrm{S}^2 V$ with respect to χ, that is, the space of quadratic forms Q on $\mathbb{P}(V)$ such that $h \cdot Q = \chi(h) Q$ for all $h \in H_8$.

Theorem 1.1 *For each $\chi \in \widehat{H}_8$, let Q_χ be a general element of $(\mathrm{S}^2 V)_\chi$. The subvariety $\widetilde{X}$ of $\mathbb{P}(V)$ defined by the 4 equations*

$$Q_\chi = 0 \quad \textit{for all } \chi \in \widehat{H}_8$$

is a smooth threefold, on which the group H_8 acts freely. The quotient $X := \widetilde{X}/H_8$ is a Calabi–Yau threefold with $\pi_1(X) = H_8$.

Let me observe first that the last assertion is an immediate consequence of the others. Indeed, since $\widetilde{X}$ is a Calabi–Yau threefold, we have $h^{1,0}(\widetilde{X}) = h^{2,0}(\widetilde{X}) = \chi(\mathcal{O}_{\widetilde{X}}) = 0$, hence $h^{1,0}(X) = h^{2,0}(X) = \chi(\mathcal{O}_X) = 0$. This implies

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[1] I use Grothendieck's notation, that is, $\mathbb{P}(V)$ is the space of hyperplanes in V.

$h^{3,0}(X)=1$, so there exists a nonzero holomorphic 3-form ω on X; since its pullback to $\widetilde{X}$ is everywhere nonzero, ω has the same property, hence X is a Calabi–Yau threefold. Finally $\widetilde{X}$ is a complete intersection in $\mathbb{P}(V)$, hence simply connected by the Lefschetz theorem, so the fundamental group of X is isomorphic to H_8.

So the problem is to prove that H_8 acts freely and $\widetilde{X}$ is smooth. To do this, we will need to write down explicit elements of $(\mathrm{S}^2 V)_\chi$. As an H_8-module, V is the direct sum of the 4 one-dimensional representations of H_8 and twice the irreducible two-dimensional representation ρ. Thus there exists a system of homogeneous coordinates $(X_1, X_\alpha, X_\beta, X_\gamma; Y, Z; Y', Z')$ such that

$$g\cdot(X_1, X_\alpha, X_\beta, X_\gamma; Y, Z; Y', Z') = \\ (X_1, \alpha(g)X_\alpha, \beta(g)X_\beta, \gamma(g)X_\gamma; \rho(g)(Y,Z); \rho(g)(Y',Z')).$$

To be more precise, I denote by α (respectively β, γ) the nontrivial character which is $+1$ on i (respectively j, k), and I take for ρ the usual representation via Pauli matrices:

$$\rho(i)(Y,Z) = (\sqrt{-1}\,Y, -\sqrt{-1}\,Z),\quad \rho(j)(Y,Z) = (-Z, Y),\\ \rho(k)(Y,Z) = (-\sqrt{-1}\,Z, -\sqrt{-1}\,Y).$$

Then the general element Q_χ of $(\mathrm{S}^2 V)_\chi$ can be written

$$\begin{aligned}
Q_1 &= t_1^1X_1^2 + t_2^1X_\alpha^2 + t_3^1X_\beta^2 + t_4^1X_\gamma^2 + t_5^1(YZ' - Y'Z),\\
Q_\alpha &= t_1^\alpha X_1X_\alpha + t_2^\alpha X_\beta X_\gamma + t_3^\alpha YZ + t_4^\alpha Y'Z' + t_5^\alpha(YZ' + ZY'),\\
Q_\beta &= t_1^\beta X_1X_\beta + t_2^\beta X_\alpha X_\gamma + t_3^\beta(Y^2+Z^2) + t_4^\beta(Y'^2+Z'^2) + t_5^\beta(YY'+ZZ'),\\
Q_\gamma &= t_1^\gamma X_1X_\gamma + t_2^\gamma X_\alpha X_\beta + t_3^\gamma(Y^2-Z^2) + t_4^\gamma(Y'^2-Z'^2) + t_5^\gamma(YY'-ZZ').
\end{aligned}$$

For fixed $\mathbf{t} := (t_i^\chi)$, let $\mathcal{X}_\mathbf{t}$ be the subvariety of $\mathbb{P}(V)$ defined by the equations $Q_\chi = 0$. Let us check first that the action of H_8 on $\mathcal{X}_\mathbf{t}$ has no fixed points for $\mathbf{t}$ general enough. Since a point fixed by an element h of H_8 is also fixed by h^2, it is sufficient to check that the element $-1\in H_8$ acts without fixed point, that is, that $\mathcal{X}_\mathbf{t}$ does not meet the linear subspaces L_+ and L_- defined by $Y=Z=Y'=Z'=0$ and $X_1=X_\alpha=X_\beta=X_\gamma=0$ respectively.

Let $x=(0,0,0,0;Y,Z;Y',Z')\in\mathcal{X}_\mathbf{t}\cap L_-$. One of the coordinates, say Z, is nonzero; since $Q_1(x)=0$, there exists $k\in\mathbb{C}$ such that $Y'=kY$, $Z'=kZ$. Substituting in the equations $Q_\alpha(x)=Q_\beta(x)=Q_\gamma(x)=0$ gives

$$(t_3^\alpha + t_5^\alpha k + t_4^\alpha k^2)YZ = (t_3^\beta + t_5^\beta k + t_4^\beta k^2)(Y^2+Z^2) = \\ (t_3^\alpha + t_5^\alpha k + t_4^\alpha k^2)(Y^2-Z^2) = 0$$

which has no nonzero solutions for a generic choice of $\mathbf{t}$.

Now let $x = (X_1, X_\alpha, X_\beta, X_\gamma; 0, 0; 0, 0) \in \mathcal{X}_{\mathbf{t}} \cap L_+$. As soon as the t_i^χ are nonzero, two of the X-coordinates cannot vanish, otherwise all the coordinates would be zero. Expressing that $Q_\beta = Q_\gamma = 0$ has a nontrivial solution in (X_β, X_γ) gives X_α^2 as a multiple of X_1^2, and similarly for X_β^2 and X_γ^2. But then $Q_1(x) = 0$ is impossible for a general choice of $\mathbf{t}$.

Now we want to prove that $\mathcal{X}_{\mathbf{t}}$ is smooth for $\mathbf{t}$ general enough. Let $\mathcal{Q} = \bigoplus_{\chi \in \widehat{H}_8} (\mathrm{S}^2 V)_\chi$; then $\mathbf{t} := (t_i^\chi)$ is a system of coordinates on $\mathcal{Q}$. The equations $Q_\chi = 0$ define a subvariety $\mathcal{X}$ in $\mathcal{Q} \times \mathbb{P}(V)$, whose fibre above a point $\mathbf{t} \in \mathcal{Q}$ is $\mathcal{X}_{\mathbf{t}}$. Consider the second projection $p\colon \mathcal{X} \to \mathbb{P}(V)$. For $x \in \mathbb{P}(V)$, the fibre $p^{-1}(x)$ is the linear subspace of $\mathcal{Q}$ defined by the vanishing of the Q_χ, viewed as linear forms in $\mathbf{t}$. These forms are clearly linearly independent as soon as they do not vanish. In other words, if we denote by B_χ the base locus of the quadrics in $(\mathrm{S}^2 V)_\chi$ and put $B = \bigcup B_\chi$, the map $p\colon \mathcal{X} \to \mathbb{P}(V)$ is a vector bundle fibration above $\mathbb{P}(V) \setminus B$; in particular $\mathcal{X}$ is nonsingular outside $p^{-1}(B)$. Therefore it is enough to prove that $\mathcal{X}_{\mathbf{t}}$ is smooth at the points of $B \cap \mathcal{X}_{\mathbf{t}}$.

Observe that an element x in B has two of its X-coordinates zero. Since the equations are symmetric in the X-coordinates we may assume $X_\beta = X_\gamma = 0$. Then the Jacobian matrix

$$\left(\frac{\partial Q_\chi}{\partial X_\psi}(x)\right) \quad \text{takes the form} \quad \begin{pmatrix} 2t_1^1 X_1 & 2t_2^1 X_\alpha & 0 & 0 \\ t_1^\alpha X_\alpha & t_1^\alpha X_1 & 0 & 0 \\ 0 & 0 & t_1^\beta X_1 & t_2^\beta X_\alpha \\ 0 & 0 & t_2^\gamma X\alpha & t_1^\gamma X_1 \end{pmatrix}.$$

For generic $\mathbf{t}$, this matrix is of rank 4 except when all the X-coordinates of x vanish; but we have seen that this is impossible when $\mathbf{t}$ is general enough. $\square$

2 Some comments

As mentioned in the introduction, the construction is inspired by Reid's example [R] of a surface of general type with $p_g = 0$, $K^2 = 2$, $\pi_1 = H_8$. This is more than a coincidence. In fact, let $\widetilde{S}$ be the hyperplane section $X_1 = 0$ of $\widetilde{X}$. It is stable under the action of H_8 (so that H_8 acts freely on $\widetilde{S}$), and we can prove as above that it is smooth for a generic choice of the parameters. The surface $S := \widetilde{S}/H_8$ is a Reid surface, embedded in X as an ample divisor, with $h^0(X, \mathcal{O}_X(S)) = 1$. In general, let us consider a Calabi–Yau threefold X which contains a *rigid ample surface*, that is, a smooth ample divisor S such that $h^0(\mathcal{O}_X(S)) = 1$. Put $L := \mathcal{O}_X(S)$. Then S is a minimal surface of general type (because $K_S = L_{|S}$ is ample); by the Lefschetz theorem, the natural map $\pi_1(S) \to \pi_1(X)$ is an isomorphism. Because of the exact sequence

$$0 \to \mathcal{O}_X \longrightarrow L \longrightarrow K_S \to 0,$$

the geometric genus $p_g(S) := h^0(K_S)$ is zero.

We have $K_S^2 = L^3$; the Riemann–Roch theorem on X yields

$$1 = h^0(L) = \frac{L^3}{6} + \frac{L \cdot c_2}{12} \cdot$$

Since $L \cdot c_2 > 0$ as a consequence of Yau's theorem (see for instance [B], Cor. 2), we obtain $K_S^2 \leq 5$.

For surfaces with $p_g = 0$ and $K_S^2 = 1$ or 2, we have a great deal of information about the *algebraic* fundamental group, that is the profinite completion of the fundamental group (see [B-P-V] for an overview). In the case $K_S^2 = 1$, the algebraic fundamental group is cyclic of order ≤ 5; if $K_S^2 = 2$, it is of order ≤ 9; moreover the dihedral group D_8 cannot occur. D. Naie [N] has recently proved that the symmetric group $\mathfrak{S}_3$ can also not occur; therefore the quaternion group H_8 is the only non-Abelian group which occurs in this range.

On the other hand, little is known about surfaces with $p_g = 0$ and $K_S^2 = 3, 4$ or 5. Inoue has constructed examples with $\pi_1 = H_8 \times (\mathbb{Z}_2)^n$, with $n = K^2 - 2$ (*loc. cit.*); I do not know if they can appear as rigid ample surfaces in a Calabi–Yau threefold.

Let us denote by $\widetilde{X}$ the universal cover of X, by $\widetilde{L}$ the pullback of L to $\widetilde{X}$, and by ρ the representation of G on $H^0(\widetilde{X}, \widetilde{L})$. We have $\operatorname{Tr}\rho(g) = 0$ for $g \neq 1$ by the holomorphic Lefschetz formula, and $\operatorname{Tr}\rho(1) = \chi(\widetilde{L}) = |G|\,\chi(L) = |G|$. Therefore *$\rho$ is isomorphic to the regular representation.* Looking at the list in *loc. cit.* we get a few examples of this situation, for instance:

- $G = \mathbb{Z}_5$, $\widetilde{X}$ = a quintic hypersurface in $\mathbb{P}^4$;
- $G = (\mathbb{Z}_2)^3$ or $\mathbb{Z}_4 \times \mathbb{Z}_2$, $\widetilde{X}$ = an intersection of 4 quadrics in $\mathbb{P}^7$ as above;
- $G = \mathbb{Z}_3 \times \mathbb{Z}_3$, $\widetilde{X}$ = a hypersurface of bidegree $(3,3)$ in $\mathbb{P}^2 \times \mathbb{P}^2$.

Of course, when looking for Calabi–Yau threefolds with interesting π_1, there is no reason to assume that it contains an ample rigid surface. Observe however that if we want to use the preceding method, in other words, to find a projective space $\mathbb{P}(V)$ with an action of G and a smooth invariant linearly normal Calabi–Yau threefold $\widetilde{X} \subset \mathbb{P}(V)$, then the line bundle $\mathcal{O}_{\widetilde{X}}(1)$ will be the pullback of an ample line bundle L on X, and by the above argument the representation of G on V will be $h^0(L)$ times the regular representation. This leaves little hope to find an invariant Calabi–Yau threefold when the product $h^0(L)|G|$ becomes large.

References

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