Birational Calabi–Yau n-folds have equal Betti numbers

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Abstract

Let X and Y be two smooth projective n-dimensional algebraic varieties X and Y over $\mathbb{C}$ with trivial canonical line bundles. We use methods of p-adic analysis on algebraic varieties over local number fields to prove that if X and Y are birational, they have the same Betti numbers.

1 Introduction

The purpose of this note is to show how to use the elementary theory of p-adic integrals on algebraic varieties to prove cohomological properties of birational algebraic varieties over $\mathbb{C}$. We prove the following theorem, which was used by Beauville in his recent explanation of a Yau–Zaslow formula for the number of rational curves on a K3 surface [1] (see also [3, 12]):

Theorem 1.1 *Let X and Y be smooth n-dimensional irreducible projective algebraic varieties over $\mathbb{C}$. Assume that the canonical line bundles Ω^n_X and Ω^n_Y are trivial and that X and Y are birational. Then X and Y have the same Betti numbers, that is,*

$$H^i(X, \mathbb{C}) \cong H^i(Y, \mathbb{C}) \quad \textit{for all } i \geq 0.$$

Note that Theorem 1.1 is obvious for $n = 1$, and for $n = 2$, it follows from the uniqueness of minimal models of surfaces with $\kappa \geq 0$, that is, from the property that any birational map between two such minimal models extends to an isomorphism [5]. Although n-folds with $\kappa \geq 0$ no longer have a unique minimal models for $n \geq 3$, Theorem 1.1 can be proved for $n = 3$ using a result of Kawamata ([6], §6): he showed that any two birational minimal models of 3-folds can be connected by a sequence of flops (see also [7]), and simple topological arguments show that if two projective 3-folds with at worst $\mathbb{Q}$-factorial terminal singularities are birational via a flop, then their singular Betti numbers are equal. Since one still knows very little about flops in dimension $n \geq 4$, it seems unlikely that a consideration of flops could help to prove Theorem 1.1 in dimension $n \geq 4$. Moreover, Theorem 1.1 is false in general for projective algebraic varieties with at worst $\mathbb{Q}$-factorial Gorenstein terminal singularities of dimension $n \geq 4$. For this reason, the condition in Theorem 1.1 that X and Y are smooth becomes very important in the case $n \geq 4$. We remark that in the case of holomorphic symplectic manifolds some stronger result is obtained in [4].

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2 Gauge forms and p-adic measures

Let F be a local number field, that is, a finite extension of the p-adic field $\mathbb{Q}_p$ for some prime $p \in \mathbb{Z}$. Let $R \subset F$ be the maximal compact subring, $\mathfrak{q} \subset R$ the maximal ideal, $F_\mathfrak{q} = R/\mathfrak{q}$ the residue field with $|F_\mathfrak{q}| = q = p^r$. We write

$$N_{F/\mathbb{Q}_p} \colon F \to \mathbb{Q}_p$$

for the standard norm, and $\|\cdot\| \colon F \to \mathbb{R}_{\geq 0}$ for the multiplicative p-adic norm

$$a \mapsto \|a\| = p^{-\operatorname{Ord}(N_{F/\mathbb{Q}_p}(a))}.$$

Here Ord is the p-adic valuation.

Definition 2.1 Let $\mathcal{X}$ be an arbitrary flat reduced algebraic S-scheme, where $S = \operatorname{Spec} R$. We denote by $\mathcal{X}(R)$ the set of S-morphisms $S \to \mathcal{X}$ (or sections of $\mathcal{X} \to S$). We call $\mathcal{X}(R)$ the set of *R-integral points* in $\mathcal{X}$. The set of sections of the morphism $\mathcal{X} \times_S \operatorname{Spec} F \to \operatorname{Spec} F$ is denoted by $\mathcal{X}(F)$ and called the set of *F-rational points* in $\mathcal{X}$.

Remark 2.2 (i) If $\mathcal{X}$ is an affine S-scheme, then one can identify $\mathcal{X}(R)$ with the subset

$$\big\{x \in \mathcal{X}(F) \bigm| f(x) \in R \text{ for all } f \in \Gamma(\mathcal{X}, \mathcal{O}_\mathcal{X})\big\} \subset \mathcal{X}(F).$$

(ii) If $\mathcal{X}$ is a projective (or proper) S-scheme, then $\mathcal{X}(R) = \mathcal{X}(F)$.

Now let X be a smooth n-dimensional algebraic variety over F. We assume that X admits an extension $\mathcal{X}$ to a regular S-scheme. Denote by Ω^n_X the canonical line bundle of X and by $\Omega^n_{\mathcal{X}/S}$ the relative dualizing sheaf on $\mathcal{X}$.

Recall the following definition introduced by Weil [11]:

Definition 2.3 A global section $\omega \in \Gamma(\mathcal{X}, \Omega^n_{\mathcal{X}/S})$ is called a *gauge form* if it has no zeros in $\mathcal{X}$. By definition, a gauge form ω defines an isomorphism $\mathcal{O}_{\mathcal{X}} \cong \Omega^n_{\mathcal{X}/S}$, sending 1 to ω. Clearly, a gauge form exists if and only if the line bundle $\Omega^n_{\mathcal{X}/S}$ is trivial.

Weil observed that a gauge form ω determines a canonical p-adic measure $\mathrm{d}\mu_\omega$ on the locally compact p-adic topological space $\mathcal{X}(F)$ of F-rational points in $\mathcal{X}$. The p-adic measure $\mathrm{d}\mu_\omega$ is defined as follows:

Let $x \in \mathcal{X}(F)$ be an F-point, $t_1, \ldots, t_n$ local p-adic analytic parameters at x. Then $t_1, \ldots, t_n$ define a p-adic homeomorphism $\theta \colon U \to \mathbb{A}^n(F)$ of an open subset $\mathcal{U} \subset \mathcal{X}(F)$ containing x with an open subset $\theta(\mathcal{U}) \subset \mathbb{A}^n(F)$. We stress that the subsets $\mathcal{U} \subset \mathcal{X}(F)$ and $\theta(\mathcal{U}) \subset \mathbb{A}^n(F)$ are considered to be open in the p-adic topology, not in the Zariski topology. We write

$$\omega = \theta^* \left(g \mathrm{d}t_1 \wedge \cdots \wedge \mathrm{d}t_n \right),$$

where $g = g(t)$ is a p-adic analytic function on $\theta(\mathcal{U})$ having no zeros. Then the p-adic measure $\mathrm{d}\mu_\omega$ on $\mathcal{U}$ is defined to be the pullback with respect to θ of the p-adic measure $\|g(t)\|\mathbf{dt}$ on $\theta(\mathcal{U})$, where $\mathbf{dt}$ is the standard p-adic Haar measure on $\mathbb{A}^n(F)$ normalized by the condition

$$\int_{\mathbb{A}^n(R)} \mathbf{dt} = 1.$$

It is a standard exercise using the Jacobian to check that two p-adic measures $\mathrm{d}\mu'_\omega, \mathrm{d}\mu''_\omega$ constructed as above on any two open subsets $\mathcal{U}', \mathcal{U}'' \subset \mathcal{X}(F)$ coincide on the intersection $\mathcal{U}' \cap \mathcal{U}''$.

Definition 2.4 The measure $\mathrm{d}\mu_\omega$ on $\mathcal{X}(F)$ constructed above is called the *Weil p-adic measure* associated with the gauge form ω.

Theorem 2.5 ([11], Theorem 2.2.5) *Let $\mathcal{X}$ be a regular S-scheme, ω a gauge form on $\mathcal{X}$, and $\mathrm{d}\mu_\omega$ the corresponding Weil p-adic measure on $\mathcal{X}(F)$. Then*

$$\int_{\mathcal{X}(R)} \mathrm{d}\mu_\omega = \frac{|\mathcal{X}(F_{\mathfrak{q}})|}{q^n},$$

where $\mathcal{X}(F_{\mathfrak{q}})$ is the set of closed points of $\mathcal{X}$ over the finite residue field $F_{\mathfrak{q}}$.

Proof Let

$$\varphi \colon \mathcal{X}(R) \to \mathcal{X}(F_{\mathfrak{q}}) \quad \text{given by} \quad x \mapsto \overline{x} \in \mathcal{X}(F_{\mathfrak{q}})$$

be the natural surjective mapping. The proof is based on the idea that if $\overline{x} \in \mathcal{X}(F_{\mathfrak{q}})$ is a closed $F_{\mathfrak{q}}$-point of $\mathcal{X}$ and $g_1, \ldots, g_n$ are generators of the maximal ideal of $\overline{x}$ in $\mathcal{O}_{\mathcal{X},\overline{x}}$ modulo the ideal $\mathfrak{q}$, then the elements $g_1, \ldots, g_n$ define a p-adic analytic homeomorphism

$$\gamma\colon \varphi^{-1}(\overline{x}) \to \mathbb{A}^n(\mathfrak{q}),$$

where $\varphi^{-1}(\overline{x})$ is the fiber of φ over $\overline{x}$ and $\mathbb{A}^n(\mathfrak{q})$ is the set of all R-integral points of $\mathbb{A}^n$ whose coordinates belong to the ideal $\mathfrak{q} \subset R$. Moreover, the p-adic norm of the Jacobian of γ is identically equal to 1 on the whole fiber $\varphi^{-1}(\overline{x})$. In order to see the latter we remark that the elements define an étale morphism $g\colon V \to \mathbb{A}^n$ of some Zariski open neighbourhood V of $\overline{x} \in \mathcal{X}$. Since $\varphi^{-1}(\overline{x}) \subset V(R)$ and $g^*(\mathrm{d}t_1, \wedge \cdots \wedge \mathrm{d}t_n) = h\omega$, where h is invertible in V, we obtain that h has p-adic norm 1 on $\varphi^{-1}(\overline{x})$. So, using the p-adic analytic homeomorphism γ, we obtain

$$\int_{\varphi^{-1}(\overline{x})} \mathrm{d}\mu_\omega = \int_{\mathbb{A}^n(\mathfrak{q})} \mathbf{dt} = \frac{1}{q^n}$$

for each $\overline{x} \in \mathcal{X}(F_{\mathfrak{q}})$. $\square$

Now we consider a slightly more general situation. We assume only that $\mathcal{X}$ is a regular scheme over S, but do not assume the existence of a gauge form on $\mathcal{X}$ (that is, of an isomorphism $\mathcal{O}_{\mathcal{X}} \cong \Omega^n_{\mathcal{X}/S}$). Nevertheless under these weaker assumptions we can define a unique natural p-adic measure $\mathrm{d}\mu$ at least on the compact $\mathcal{X}(R) \subset \mathcal{X}(F)$ – although possibly not on the whole p-adic topological space $\mathcal{X}(F)$!

Let $\mathcal{U}_1, \ldots, \mathcal{U}_k$ be a finite covering of $\mathcal{X}$ by Zariski open S-subschemes such that the restriction of $\Omega^n_{\mathcal{X}/S}$ on each $\mathcal{U}_i$ is isomorphic to $\mathcal{O}_{\mathcal{U}_i}$. Then each $\mathcal{U}_i$ admits a gauge form ω_i and we define a p-adic measure $\mathrm{d}\mu_i$ on each compact $\mathcal{U}_i(R)$ as the restriction of the Weil p-adic measure $\mathrm{d}\mu_{\omega_i}$ associated with ω_i on $\mathcal{U}_i(F)$. We note that the gauge forms ω_i are defined uniquely up to elements $s_i \in \Gamma(\mathcal{U}_i, \mathcal{O}^*_{\mathcal{X}})$. On the other hand, the p-adic norm $\|s_i(x)\|$ equals 1 for any element $s_i \in \Gamma(\mathcal{U}_i, \mathcal{O}^*_{\mathcal{X}})$ and any R-rational point $x \in \mathcal{U}_i(R)$. Therefore, the p-adic measure on $\mathcal{U}_i(R)$ that we constructed does not depend on the choice of a gauge form ω_i. Moreover, the p-adic measures $\mathrm{d}\mu_i$ on $\mathcal{U}_i(R)$ glue together to a p-adic measure $\mathrm{d}\mu$ on the whole compact $\mathcal{X}(R)$, since one has

$$\mathcal{U}_i(R) \cap \mathcal{U}_j(R) = (\mathcal{U}_i \cap \mathcal{U}_j)(R) \quad \text{for } i, j = 1, \ldots, k$$

and

$$\mathcal{U}_1(R) \cup \cdots \cup \mathcal{U}_k(R) = (\mathcal{U}_1 \cup \cdots \cup \mathcal{U}_k)(R) = \mathcal{X}(R).$$

Definition 2.6 The p-adic measure constructed above defined on the set $\mathcal{X}(R)$ of R-integral points of a S-scheme $\mathcal{X}$ is called the *canonical p-adic measure.*

For the canonical p-adic measure $\mathrm{d}\mu$, we obtain the same property as for the Weil p-adic measure $\mathrm{d}\mu_\omega$:

Theorem 2.7

$$\int_{\mathcal{X}(R)} \mathrm{d}\mu = \frac{|\mathcal{X}(F_\mathfrak{q})|}{q^n}.$$

Proof Using a covering of $\mathcal{X}$ by some Zariski open subsets $\mathcal{U}_1, \ldots, \mathcal{U}_k$, we obtain

$$\int\limits_{\mathcal{X}(R)} \mathrm{d}\mu = \sum_{i_1} \int\limits_{\mathcal{U}_{i_1}(R)} \mathrm{d}\mu - \sum_{i_1<i_2} \int\limits_{(\mathcal{U}_{i_1}\cap\mathcal{U}_{i_2})(R)} \mathrm{d}\mu + \cdots + (-1)^k \int\limits_{(\mathcal{U}_1\cap\cdots\cap\mathcal{U}_k)(R)} \mathrm{d}\mu$$

and

$$\left|\mathcal{X}(F_\mathfrak{q})\right| = \sum_{i_1} \left|\mathcal{U}_{i_1}(F_\mathfrak{q})\right| - \sum_{i_1<i_2} \left|(\mathcal{U}_{i_1} \cap \mathcal{U}_{i_2})(F_\mathfrak{q})\right| + \cdots + (-1)^k \left|(\mathcal{U}_1 \cap \cdots \cap \mathcal{U}_k)(F_\mathfrak{q})\right|.$$

It remains to apply Theorem 2.5 to every intersection $\mathcal{U}_{i_1} \cap \cdots \cap \mathcal{U}_{i_s}$. □

Theorem 2.8 *Let $\mathcal{X}$ be a regular integral S-scheme and $\mathcal{Z} \subset \mathcal{X}$ a closed reduced subscheme of codimension ≥ 1. Then the subset $\mathcal{Z}(R) \subset \mathcal{X}(R)$ has zero measure with respect to the canonical p-adic measure* $\mathrm{d}\mu$ *on $\mathcal{X}(R)$.*

Proof Using a covering of $\mathcal{X}$ by Zariski open affine subsets $\mathcal{U}_1, \ldots, \mathcal{U}_k$, we can always reduce to the case when $\mathcal{X}$ is an affine regular integral S-scheme and $\mathcal{Z} \subset \mathcal{X}$ an irreducible principal divisor defined by an equation $f = 0$, where f is a prime element of $A = \Gamma(\mathcal{X}, \mathcal{O}_\mathcal{X})$.

Consider the special case $\mathcal{X} = \mathbb{A}^n_S = \operatorname{Spec} R[X_1, \ldots, X_n]$ and $\mathcal{Z} = \mathbb{A}^{n-1}_S = \operatorname{Spec} R[X_2, \ldots, X_n]$, that is, $f = X_1$. For every positive integer m, we denote by $\mathcal{Z}_m(R)$ the subset in $\mathbb{A}^n(R)$ consisting of all points $x = (x_1, \ldots, x_d) \in R^n$ such that $x_1 \in \mathfrak{q}^m$. One computes the p-adic integral in the straightforward way:

$$\int\limits_{\mathcal{Z}_m(R)} \mathbf{dx} = \int\limits_{\mathbb{A}^1(\mathfrak{q}^m)} \mathrm{d}x_1 \prod_{i=2}^{n} \left(\int_{\mathbb{A}^1(R)} \mathrm{d}x_i \right) = \frac{1}{q^m}.$$

On the other hand, we have

$$\mathcal{Z}(R) = \bigcap_{m=1}^{\infty} \mathcal{Z}_m(R).$$

Hence

$$\int_{\mathcal{Z}(R)} \mathbf{dx} = \lim_{m\to\infty} \int_{\mathcal{Z}_m(R)} \mathbf{dx} = 0,$$

and in this case the statement is proved. Using the Noether normalization theorem reduces the more general case to the above special one. □

3 The Betti numbers

Proposition 3.1 *Let X and Y be birational smooth projective n-dimensional algebraic varieties over $\mathbb{C}$ having trivial canonical line bundles. Then there exist Zariski open dense subsets $U \subset X$ and $V \subset Y$ such that U is isomorphic to V and $\operatorname{codim}_X(X \setminus U), \operatorname{codim}_Y(Y \setminus V) \geq 2$.*

Proof Consider a birational rational map $\varphi\colon X \dashrightarrow Y$. Since X is smooth and Y is projective, φ is regular at the general point of any prime divisor of X, so that there exists a maximal Zariski open dense subset $U \subset X$ with $\operatorname{codim}_X(X \setminus U) \geq 2$ such that φ extends to a regular morphism $\varphi_0\colon U \to Y$. Since $\varphi^*\omega_Y$ is proportional to ω_X, the morphism φ_0 is étale, that is, φ_0 is an open embedding of U into the maximal open subset $V \subset Y$ where φ^{-1} is defined. Similarly φ^{-1} induces an open embedding of V into U, so we conclude that φ_0 is an isomorphism of U onto V. □

Proof of Theorem 1.1 Let X and Y be smooth projective birational varieties of dimension n over $\mathbb{C}$ with trivial canonical bundles. By Proposition 3.1, there exist Zariski open dense subsets $U \subset X$ and $V \subset Y$ with $\operatorname{codim}_X(X \setminus U) \geq 2$ and $\operatorname{codim}_Y(Y \setminus V) \geq 2$ and an isomorphism $\varphi\colon U \to V$.

By standard arguments, one can choose a finitely generated $\mathbb{Z}$-subalgebra $\mathcal{R} \subset \mathbb{C}$ such that the projective varieties X and Y and the Zariski open subsets $U \subset X$ and $V \subset Y$ are obtained by base change $* \times_{\mathcal{S}} \operatorname{Spec}\mathbb{C}$ from regular projective schemes $\mathcal{X}$ and $\mathcal{Y}$ over $\mathcal{S} := \operatorname{Spec}\mathcal{R}$ together with Zariski open subschemes $\mathcal{U} \subset \mathcal{X}$ and $\mathcal{V} \subset \mathcal{Y}$ over $\mathcal{S}$. Moreover, one can choose $\mathcal{R}$ in such a way that both relative canonical line bundles $\Omega^n_{\mathcal{X}/\mathcal{S}}$ and $\Omega^n_{\mathcal{Y}/\mathcal{S}}$ are trivial, both codimensions $\operatorname{codim}_{\mathcal{X}}(\mathcal{X} \setminus \mathcal{U})$ and $\operatorname{codim}_{\mathcal{Y}}(\mathcal{Y} \setminus \mathcal{V})$ are ≥ 2, and the isomorphism $\varphi\colon U \to V$ is obtained by base change from an isomorphism $\Phi\colon \mathcal{U} \to \mathcal{V}$ over $\mathcal{S}$.

For almost all prime numbers $p \in \mathbb{N}$, there exist a regular R-integral point $\pi \in \mathcal{S} \times_{\operatorname{Spec}\mathbb{Z}} \operatorname{Spec}\mathbb{Z}_p$, where R is the maximal compact subring in a local p-adic field F; let $\mathfrak{q}$ be the maximal ideal of R. By an appropriate choice of

$\pi \in \mathcal{S} \times_{\operatorname{Spec} \mathbb{Z}} \operatorname{Spec} \mathbb{Z}_p$, we can ensure that both $\mathcal{X}$ and $\mathcal{Y}$ have good reduction modulo $\mathfrak{q}$. Moreover, we can assume that the maximal ideal $I(\overline{\pi})$ of the unique closed point in

$$S := \operatorname{Spec} R \stackrel{\pi}{\hookrightarrow} \mathcal{S} \times_{\operatorname{Spec} \mathbb{Z}} \operatorname{Spec} \mathbb{Z}_p$$

is obtained by base change from some maximal ideal $J(\overline{\pi}) \subset \mathcal{R}$ lying over the prime ideal $(p) \subset \mathbb{Z}$.

Let $\omega_{\mathcal{X}}$ and $\omega_{\mathcal{Y}}$ be gauge forms on $\mathcal{X}$ and $\mathcal{Y}$ respectively and $\omega_{\mathcal{U}}$ and $\omega_{\mathcal{V}}$ their restriction to $\mathcal{U}$ (respectively $\mathcal{V}$). Since Φ^* is an isomorphism over $\mathcal{S}$, $\Phi^*\omega_{\mathcal{Y}}$ is another gauge form on $\mathcal{U}$. Hence there exists a nowhere vanishing regular function $h \in \Gamma(\mathcal{U}, \mathcal{O}^*_{\mathcal{X}})$ such that

$$\Phi^*\omega_{\mathcal{V}} = h\omega_{\mathcal{U}}.$$

The property $\operatorname{codim}_{\mathcal{X}}(\mathcal{X} \setminus \mathcal{U}) \geq 2$ implies that h is an element of $\Gamma(\mathcal{X}, \mathcal{O}^*_{\mathcal{X}}) = \mathcal{R}^*$. Hence, one has $\|h(x)\| = 1$ for all $x \in \mathcal{X}(F)$, that is, the Weil p-adic measures on $\mathcal{U}(F)$ associated with $\Phi^*\omega_{\mathcal{V}}$ and $\omega_{\mathcal{U}}$ are the same. The latter implies the following equality of the p-adic integrals

$$\int_{\mathcal{U}(F)} \mathrm{d}\mu_{\mathcal{X}} = \int_{\mathcal{V}(F)} \mathrm{d}\mu_{\mathcal{Y}}.$$

By Theorem 2.8 and Remark 2.2, (ii), we obtain

$$\int_{\mathcal{U}(F)} \mathrm{d}\mu_{\mathcal{X}} = \int_{\mathcal{X}(F)} \mathrm{d}\mu_{\mathcal{X}} = \int_{\mathcal{X}(R)} \mathrm{d}\mu_{\mathcal{X}}$$

and

$$\int_{\mathcal{V}(\mathcal{F})} \mathrm{d}\mu_{\mathcal{Y}} = \int_{\mathcal{Y}(\mathcal{F})} \mathrm{d}\mu_{\mathcal{Y}} = \int_{\mathcal{Y}(R)} \mathrm{d}\mu_{\mathcal{Y}}.$$

Now, applying the formula in Theorem 2.7, we come to the equality

$$\frac{|\mathcal{X}(F_{\mathfrak{q}})|}{q^n} = \frac{|\mathcal{Y}(F_{\mathfrak{q}})|}{q^n}.$$

This shows that the numbers of $F_{\mathfrak{q}}$-rational points in $\mathcal{X}$ and $\mathcal{Y}$ modulo the ideal $J(\overline{\pi}) \subset \mathcal{R}$ are the same. We now repeat the same argument, replacing R by its cyclotomic extension $\mathcal{R}^{(r)} \subset \mathbb{C}$ obtained by adjoining all complex $(q^r - 1)$th roots of unity; we deduce that the projective schemes $\mathcal{X}$ and $\mathcal{Y}$ have the same number of rational points over $F_{\mathfrak{q}}^{(r)}$, where $F_{\mathfrak{q}}^{(r)}$ is the extension of the finite field $F_{\mathfrak{q}}$ of degree r. We deduce in particular that the Weil zeta functions

$$Z(\mathcal{X}, p, t) = \exp\left(\sum_{r=1}^{\infty} |\mathcal{X}(F_{\mathfrak{q}}^{(r)})| \frac{t^r}{r}\right)$$

and

$$Z(\mathcal{Y}, p, t) = \exp\left(\sum_{r=1}^{\infty} |\mathcal{Y}(F_{\mathfrak{q}}^{(r)})| \frac{t^r}{r}\right)$$

are the same. Using the Weil conjectures proved by Deligne [9] and the comparison theorem between the étale and singular cohomology, we obtain

$$Z(\mathcal{X}, p, t) = \frac{P_1(t)P_3(t)\cdots P_{2n-1}(t)}{P_0(t)P_2(t)\cdots P_{2n}(t)} \tag{1}$$

and

$$Z(\mathcal{Y}, p, t) = \frac{Q_1(t)Q_3(t)\cdots Q_{2n-1}(t)}{Q_0(t)Q_2(t)\cdots Q_{2n}(t)},$$

where $P_i(t)$ and $Q_i(t)$ are polynomials with integer coefficients having the properties

$$\deg P_i(t) = \dim H^i(X, \mathbb{C}), \quad \deg Q_i(t) = \dim H^i(Y, \mathbb{C}) \quad \text{for all } i \geq 0. \tag{2}$$

Since the standard Archimedean absolute value of each root of polynomials $P_i(t)$ and $Q_i(t)$ must be $q^{-i/2}$ and $P_i(0) = Q_i(0) = 1$ for all $i \geq 0$, the equality $Z(\mathcal{X}, p, t) = Z(\mathcal{Y}, p, t)$ implies $P_i(t) = Q_i(t)$ for all $i \geq 0$. Therefore, we have $\dim H^i(X, \mathbb{C}) = \dim H^i(Y, \mathbb{C})$ for all $i \geq 0$. $\square$

4 Further results

Definition 4.1 Let $\varphi\colon X \dashrightarrow Y$ be a birational map between smooth algebraic varieties X and Y. We say that φ *does not change the canonical class*, if for some Hironaka resolution $\alpha\colon Z \to X$ of the indeterminacies of φ the composite $\alpha \circ \varphi$ extends to a morphism $\beta\colon Z \to Y$ such that $\beta^*\Omega^n_Y \cong \alpha^*\Omega^n_X$.

The statement of Theorem 1.1 can be generalized to the case of birational smooth projective algebraic varieties which do not necessary have trivial canonical classes as follows:

Theorem 4.2 *Let X and Y be irreducible birational smooth n-dimensional projective algebraic varieties over $\mathbb{C}$. Assume that the exists a birational rational map $\varphi\colon X \dashrightarrow Y$ which does not change the canonical class. Then X and Y have the same Betti numbers.*

Proof We repeat the same arguments as in the proof of Theorem 1.1 with the only difference that instead of the Weil p-adic measures associated with gauge forms we consider the canonical p-adic measures (see Definition 2.6). Using the birational morphisms $\alpha\colon \mathcal{Z} \to \mathcal{X}$ and $\beta\colon \mathcal{Z} \to \mathcal{Y}$ having the property

$$\beta^* \Omega^n_{\mathcal{Y}/S} \cong \alpha^* \Omega^n_{\mathcal{X}/S},$$

we conclude that for some prime $p \in \mathbb{N}$, the integrals of the canonical p-adic measures $\mu_{\mathcal{X}}$ and $\mu_{\mathcal{Y}}$ over $\mathcal{X}(R)$ and $\mathcal{Y}(R)$ are equal, since there exists a dense Zariski open subset $\mathcal{U} \subset \mathcal{Z}$ on which we have $\alpha^*\mu_{\mathcal{X}} = \beta^*\mu_{\mathcal{Y}}$. By Theorem 2.7, the zeta functions of $\mathcal{X}$ and $\mathcal{Y}$ must be the same. □

Another immediate application of our method is related to the McKay correspondence [10].

Theorem 4.3 *Let $G \subset \mathrm{SL}(n, \mathbb{C})$ be a finite subgroup. Assume that there exist two different resolutions of singularities on $W := \mathbb{C}^n/G$:*

$$f\colon X \to W, \quad g\colon Y \to W$$

such that both canonical line bundles Ω^n_X and Ω^n_Y are trivial. Then the Euler numbers of X and Y are the same.

Proof We extend the varieties X and Y to regular schemes over a scheme $\mathcal{S}$ of finite type over $\operatorname{Spec}\mathbb{Z}$. Moreover, one can choose $\mathcal{S}$ in such a way that the birational morphisms f and g extend to birational $\mathcal{S}$-morphisms

$$F\colon \mathcal{X} \to \mathcal{W}, \quad G\colon \mathcal{Y} \to \mathcal{W},$$

where $\mathcal{W}$ is a scheme over $\mathcal{S}$ extending W. Using the same arguments as in the proof of Theorem 1.1, one obtains that there exists a prime $p \in \mathbb{N}$ such that $Z(\mathcal{X}, p, t) = Z(\mathcal{Y}, p, t)$. On the other hand, in view of (2), the Euler number is determined by the Weil zeta function (1) as the degree of the numerator minus the degree of the denominator. Hence $e(X) = e(Y)$. □

With a little bit more work one can prove even more precise statement:

Theorem 4.4 *Let $G \subset \mathrm{SL}(n, \mathbb{C})$ be a finite subgroup and $W := \mathbb{C}^n/G$. Assume that there exists a resolution*

$$f\colon X \to W$$

with trivial canonical line bundle Ω^n_X. Then the Euler number of X equals the number of conjugacy classes in G.

Remark 4.5 As we saw in the proof of Theorem 3.1, the Weil zeta functions of $Z(\mathcal{X}, p, t)$ and $Z(\mathcal{Y}, p, t)$ are equal for almost all primes $p \in \operatorname{Spec} \mathbb{Z}$. This fact being expressed in terms of the associated L-functions indicates that the isomorphism $H^i(X, \mathbb{C}) \cong H^i(Y, \mathbb{C})$ for all $i \geq 0$ we have established must have some more deep motivic nature. Recently Kontsevich suggested an idea of a motivic integration [8], developed by Denef and Loeser [2]. In particular, this technique allows to prove that not only the Betti numbers, but also the Hodge numbers of X and Y in 1.1 must be the same.

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